Pierce instability and diocotron instability of a hollow electron beam

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Pierce instability

Maximum current that can be transported through the vacuum chamber with radius $a$. In case current exceeds critical value, part of the beam is reflected, i.e. beam potential sags to zero.

Two modes were chosen:

$I = 3A, \ U = 12 \ kV$

$I = 5A, \ U = 15 \ kV$
Estimation of potential sagging in chosen modes

- In the vacuum chamber
  - $U_0 = 12$ keV, $I = 3$ A
  - $U_0 = 15$ keV, $I = 5$ A

- In the bending
  - $U_0 = 12$ keV, $I = 3$ A
  - $U_0 = 15$ keV, $I = 5$ A
Different angular velocities for different radii provide relative motion of layers. Small initial asymmetry may lead to the significant density equilibrium violation and cluster origin.

\[ \omega_{\text{rot}}(r) = \frac{E_r(r)}{r B_z} \]

Angular velocity for the given radius \( r \) (arises in crossed electric and magnetic fields, beam field \( E_r(r) \) and external magnetic field \( B_z \)).

Diocotron instability
Theoretical consideration (Davidson, Physics of nonneutral plasma)

Figure 6.2. Annular electron density profile $n_e^0(r)$ assumed in Eq. (6.30). The inner conductor at $r = a$ carries a charge $Q$ per unit length.
Theoretical consideration

To investigate stability properties, we assume small-amplitude perturbations of an azimuthally symmetric initial equilibrium (characterized by the density profile). These perturbations can be found from the Poisson’s equation:

\[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \phi_0(r) = 4\pi en_e^0(r) \]

We shall find perturbed density and potential as Fourier series:

\[ n_e(r, \theta, t) = n_e^0(r) + \sum_{l=-\infty}^{\infty} \delta n_e^l(r) \exp(il\theta - i\omega t) \]

\[ \phi(r, \theta, t) = \phi_0(r) + \sum_{l=-\infty}^{\infty} \delta \phi^l(r) \exp(il\theta - i\omega t) \]

\( L \) here means number of the eigenfunction; in terms of diococotron instability \( L \) is the number of clusters to be formed.
One obtains the dispersion relation:

\[(\omega/\omega_D)^2 - b_\ell (\omega/\omega_D) + c_\ell = 0\]

\[b_\ell = \left( \ell \left\{ \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^2 \right] + \frac{\omega_q}{\omega_D} \left[ 1 + \left( \frac{r_b^-}{r_b^+} \right)^2 \right] \right\} \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right] \\
+ \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \right] \left[ \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} - \left( \frac{a}{r_b^-} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right] \right)^{-1}\]

\[c_\ell = \left( \ell^2 \frac{\omega_q}{\omega_D} \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^2 \right] + \frac{\omega_q}{\omega_D} \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right] \right) \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right] \]

\[\left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^2 \right] \left[ 1 - \left( \frac{a}{r_b^-} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{r_b^-} \right)^{2\ell} \right] \]

\[+ \ell \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^2 \right] + \frac{\omega_q}{\omega_D} \left( \frac{r_b^-}{r_b^+} \right)^2 \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{r_b^-} \right)^{2\ell} \right] \]

\[\left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{r_b^-} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{r_b^-} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right] \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right]^{-1}\]

\[\omega_q = -\frac{2Qc}{B_0(r_b^-)^2} \quad \omega_D = \frac{2\pi \hat{n}_e ee}{B_0}\]
Dispersion relation solution

\[
(\omega/\omega_D)^2 - b_\ell (\omega/\omega_D) + c_\ell = 0
\]

\[
\omega = \frac{1}{2} \omega_D \left[ b_\ell \pm (b_\ell^2 - 4c_\ell)^{1/2} \right]
\]

\[
b_\ell^2 \geq 4c_\ell
\]

Complex frequencies
\[
\text{Re} \omega = \frac{1}{2} b_\ell \omega_D,
\]
\[
\text{Im} \omega = \pm \frac{1}{2} (4c_\ell - b_\ell^2)^{1/2} \omega_D
\]

\[
4c_\ell > b_\ell
\]

Real frequencies, stable state

Stability condition

\[
T = 1/\text{Im} \omega \quad \text{– characteristic time of instability growth}
\]
Substituting $b_i$ and $c_i$ into stability condition, we obtain:

\[
\left\{ -\ell \left( 1 - \frac{\omega_q}{\omega_D} \right) \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^2 \right] \left[ 1 - \left( \frac{a}{b} \right)^{2\ell} \right] \\
+ 2 \left[ 1 + \left( \frac{a}{b} \right)^{2\ell} \right] - \left[ 1 + \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \right] \left[ \left( \frac{a}{r_b^-} \right)^{2\ell} + \left( \frac{r_b^+}{b} \right)^{2\ell} \right] \right\}^2
\geq 4 \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \left[ 1 - \left( \frac{r_b^+}{b} \right)^{2\ell} \right]^2 \left[ 1 - \left( \frac{a}{r_b^-} \right)^{2\ell} \right]^2
\]

Figure 10: An illustrative density profile $\hat{n}(r)$.
Diocotron instability

Current $I = 3 \, A$
Voltage $U = 12 \, kV$
Magnetic field $B = 0.3 \, T$
Inner radius $r_1 = 0.8 \, mm$
Outer radius $r_2 = 1 \, mm$
Tube radius $b = 40 \, mm$
Distance $L = 0.6 \, m$

Beam shape is round at the beginning of motion in crossed fields
Proposed beam profiles: beam with a uniform density distribution
Proposed beam profiles: beam with a density peak

Beam density profile from the Fermilab’s gun

Simplified beam density profile for further qualitative estimations
Study of two beam profiles with the stability condition

1) Uniform density beam

2) Peak density beam

The same parameters:

- $U = 15$ kV, $I = 5$ A, $B = 4$ T
- Inner radius $r_1 = 0.9$ mm
- Outer radius $r_2 = 1.8$ mm
- Tube radius $a = 30$ mm

What beam is more stable?
Beam with a uniform density distribution

Current density, A/cm²

U = 15 kV, I = 5 A, B = 4 T
Tube radius a = 30 mm

r1 = 0.9 mm  r2 = 1.8 mm
Stability condition for the uniform density beam


\[
\left\{ -\ell \left[ 1 - \left( \frac{r_b^-}{r_b^+} \right)^2 \right] + 2 - \left[ \left( \frac{r_b^+}{b} \right)^{2\ell} + \left( \frac{r_b^-}{b} \right)^{2\ell} \right] \right\}^2 \\
\geq 4 \left( \frac{r_b^-}{r_b^+} \right)^{2\ell} \left[ 1 - \left( \frac{r_b^+}{b} \right)^{2\ell} \right]^2
\]

\( r_b^- \) – inner radius (r1)
\( r_b^+ \) – outerradius (r2)
\( b \) – tube radius (we use 30 mm)

**Note**: for the uniform density beam stability condition depends on geometry only. Beam density and magnetic field affect rate only the instability growth
Stability charts for uniform density beam distribution

Shaded region correspond to the beam stable state. Red lines correspond to $r_1 = 0.9$ mm, $r_2 = 1.8$ mm. I.e., if line intersection lies in the shaded region, beam is stable.

- $l = 1$: 
  Stability edge: If $r_1 = 1$ mm, $T = 100$ ns

- $l = 2$: 
  $T = 5300$ ns

- $l = 3$

- $l = 4$

- $l = 5$

- $l = 6$
Beam with a peak density distribution

Current density, A/cm²

U = 15 kV, I = 5 A
Tube radius a = 30 mm

r1 = 0.9 mm  r12 = 1 mm  r2 = 1.8 mm
Dispersion relation for the beam with a peak density

Similar derivation for uniform beam is given at [R. C. Davidson, *Physics of Nonneutral Plasmas*]. Solving Poisson’s equation, we obtain system of equations:

\[
\begin{align*}
(\omega - l\omega_{\text{rot}}(r_1))(r_1 \varphi'_2(r_1) - r_1 \varphi'_1(r_1)) &= \frac{-lj_1 \varphi'_1(r_1)}{B_z \varepsilon_0 v_z} \\
(\omega - l\omega_{\text{rot}}(r_{12}))(r_{12} \varphi'_3(r_{12}) - r_{12} \varphi'_2(r_{12})) &= \frac{l(j_1 - j_2) \varphi'_2(r_{12})}{B_z \varepsilon_0 v_z} \\
(\omega - l\omega_{\text{rot}}(r_2))(r_2 \varphi'_4(r_2) - r_2 \varphi'_3(r_2)) &= \frac{lj_2 \varphi'_3(r_2)}{B_z \varepsilon_0 v_z}
\end{align*}
\]

System is linear by B, C, D

To have non-zero solution determinant is zero

Dispersion relation, cubic equation by \( \omega \).

Stability condition – all three roots are real

(Full condition is too long and so not cited here)
Stability condition for the beam with a density peak

*Note*: unlike uniform beam, stability depends not only on the beam dimensions but also on density in both beam regions.

We shall **fixe beam parameters**: total current, current densities in the peak and other beam part, peak thickness relative to the beam size.

Like we did it in case of uniform beam, **consider stability charts** in variables $r_1$ and $r_2$ (beam inner and outer radii, respectively).
Stability charts for the beam with a density peak (1/9 of the beam)

Thickness of the peak is 1/9 of the beam thickness
Red lines correspond to r1 = 0.9 mm, r2 = 1.8 mm.
If line intersection lies in the shaded region, beam is stable
Stability charts for the beam with a density peak (1/5 of the beam)

Thickness of the peak is 1/5 of the beam thickness
Red lines correspond to $r_1 = 0.9$ mm, $r_2 = 1.8$ mm.
If line intersection lies in the shaded region, beam is stable

$T = 34$ ns
Why peak density seems to be stable in some cases

Difference in frequencies of layers in most part of beam is less in case of peak density

Closer frequencies for different radii

\[ \downarrow \]

Slower relative motion of layers

\[ \downarrow \]

More stable state

The result is unambiguous. Sometimes peak density beam is more stable (for the same radii). However, due to higher density in the peak some diocotron instability modes arise where uniform beam is stable.
Comparison of stability of two beams

Peak in the beam density “stabilizes” some diocotron instability modes. However, some higher modes are instable while there are stable in the uniform beam.

In peak density beam there is instable region. Unless it is smaller compare to this of the uniform beam, it is very sensitive to the beam dimensions and current densities. We need very precisely choose peak density profile (current densities, dimensions). Otherwise, we risk to fall into non-stable region.

Position of the instability region depends significantly, for example, on the peak thickness (two cases of thickness were compared).

Uniform density profile is easier to create. Asymmetric perturbations (which in fact may lead to the diocotron instabilities) are more likely to appear in the experiment with peak density profile.

Behavior of the uniform density beam is better predictable. Uniform density beam is not so sensitive to changing of beam parameters (and actually it have less parameters to change).