

Quadrupolar Techniques for Understanding Space Charge

Adrian Oeftiger

Acknowledgements:

Simon Albright, Marcel Coly, Heiko Damerau, Vincenzo Forte, Matthew Fraser, Marek Gasior,
Tom Levens, Elias Métral, Guido Sterbini, Malte Titze, Panagiotis Zisopoulos



Extracting Information from Electro-Magnetic Monitors in Hadron Accelerators
Workshop 2018, CERN, Switzerland

14. May 2018

1st order



rigid dipolar centroid oscillation:

- Newton's third law,
actio = reactio
- no influence from direct
space charge (SC)

1st order



rigid dipolar centroid oscillation:

- Newton's third law, actio = reactio
- no influence from direct space charge (SC)

2nd order



quadrupolar envelope oscillation:

- defocused by transverse space charge
- frequency of envelope oscillation decreases with SC

⇒ **measure direct space charge** through frequency shift of beam size oscillations about matched $\sigma_{x,y}$

Content of this talk:

① Introduction

- spectrum of a quadrupolar pick-up

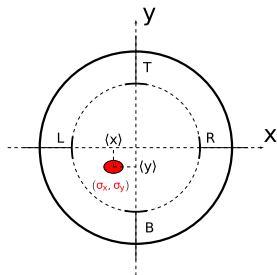
② Equipment in Injector Rings: Status and Plans

③ Applications and Ongoing Studies

- quadrupolar frequency response to characterise high-brightness PS beams
 - influence of chromaticity
 - coherent dispersive mode
- PS injection: transfer line mismatch

1. Introduction

Schematic Quadrupolar Pick-up



modified image taken from [1]

Evaluating the four pick-up signals as

$$(L + R) - (T + B)$$

results in the time signal

$$S_{\text{QPU}}(t) \propto \langle x^2 \rangle - \langle y^2 \rangle = \sigma_x^2(t) - \sigma_y^2(t) + \langle x \rangle^2(t) - \langle y \rangle^2(t) \quad . \quad (1)$$

Some Historical Perspective

QPU in **time domain** for emittance measurements:

- 1983, R. H. Miller et al. at SLAC [2]
- 2002, A. Jansson at CERN in PS [3]

QPU in **frequency domain** for emittance measurements:

- 2007, C.Y. Tang at Fermilab [4]

QPU in **frequency domain** for space charge measurements:

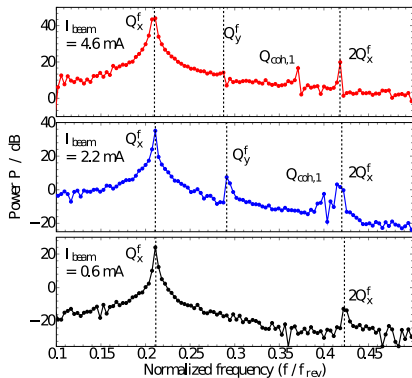
- 1996, M. Chanel at CERN in LEAR [5]
- 1999, T. Uesugi et al. at NIRS in HIMAC [6]
- 2000, R. Bär at GSI in SIS-18 [7]
- 2014, R. Sing et al. at GSI in SIS-18 [1]

⇒ all **far away from coupling** and **coasting** beams

CERN's proton synchrotrons peculiar:

- ① close to coupling ⇒ quadrupolar mode frequencies change
- ② bunched beam

QPU measurements at GSI by R. Singh, M. Gasior et al. [1]



particle type	N^{7+}
E_{kin} (MeV/u)	11.45
I_{beam} (mA)	0.6 – 6
ϵ_x, ϵ_y (mm-mrad)	8, 12.75
Q_{x0}, Q_{y0}	4.21, 3.3

$$Q_x^f \hat{=} Q_x$$

$$Q_y^f \hat{=} Q_y$$

$$Q_{coh} \hat{=} Q_{\pm}$$

Figure 6: Shift of coherent quadrupole mode $Q_{coh,1}$ with beam current.

→ far away from coupling resonance

→ coasting beam \Rightarrow sharp envelope peak

Far Away vs. On the Coupling Resonance

2 eigenmodes for coherent quadrupolar betatron oscillation:

far away from coupling



(a) horizontal mode (b) vertical mode

Relation of mode frequencies to incoh. KV space charge tune shift:

$$Q_{\pm} = 2Q_{x,y} - \left| \Delta Q_{x,y}^{KV} \right| \left(3 - \frac{\sigma_{x,y}}{\sigma_x + \sigma_y} \right) / 2 \quad (2)$$

Far Away vs. On the Coupling Resonance

2 eigenmodes for coherent quadrupolar betatron oscillation:

far away from coupling



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full coupling



(a) breathing mode (b) antisym. mode

Relation of mode frequencies to incoh. KV space charge tune shift:

$$Q_+ = 2Q_0 - \left| \Delta Q_{x,y}^{KV} \right| \quad (3a)$$

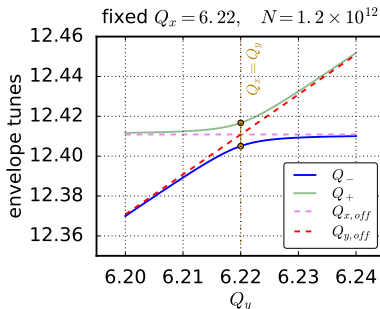
$$Q_- = 2Q_0 - \frac{3}{2} \left| \Delta Q_{x,y}^{KV} \right| \quad (3b)$$

(assuming round beams, $Q_{x,y} \equiv Q_0$)

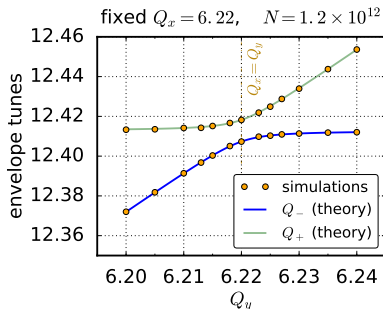
Peculiarity 1: Near Coupling Resonance

At vanishing lattice coupling, keep constant incoherent SC tune shift and fixed Q_x . Vary Q_y for a coasting round beam:

analytic expression



simulation results



Peculiarity 2: Bunched Beam Envelope Signal

Assumption:

- synchrotron motion much slower than betatron motion, $Q_s \ll Q_{x0,y0}$
 - 3D RMS envelope equation (Sacherer) decouples to 2D + 1D
 - ⇒ for a given longitudinal bunch slice, the coherent transverse quadrupolar oscillation depends on local line charge density $\lambda(z)$, longitudinal motion is quasi-stationary and independent (see e.g. [6])

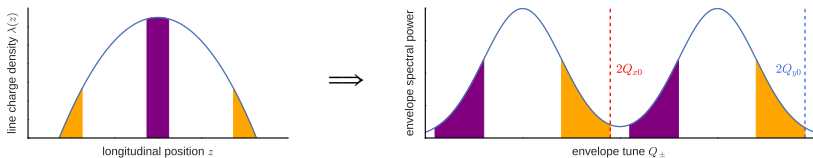
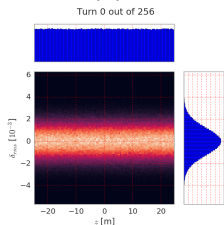
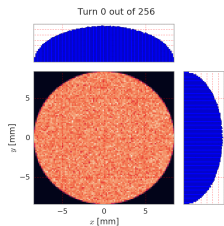


Figure: sketch of envelope detuning scaling with local line charge density

1. Introduction: QPU Spectrum Simulations

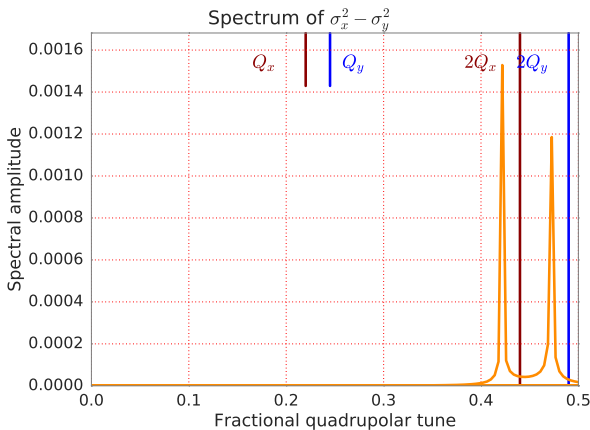
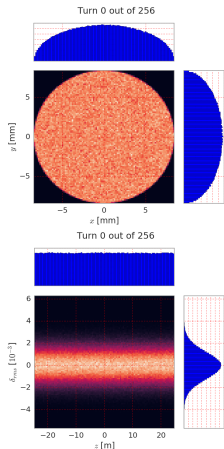
Coasting: KV Beam

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
no	KV (uniform)	no	no	no



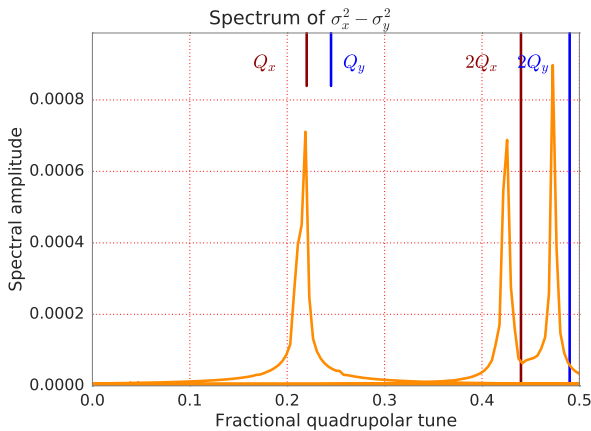
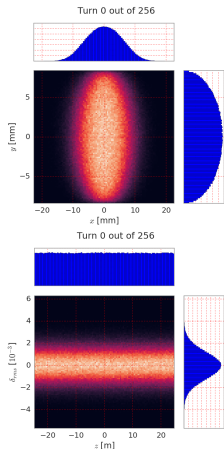
Coasting: KV Beam

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no	KV (uniform)	no	no	no



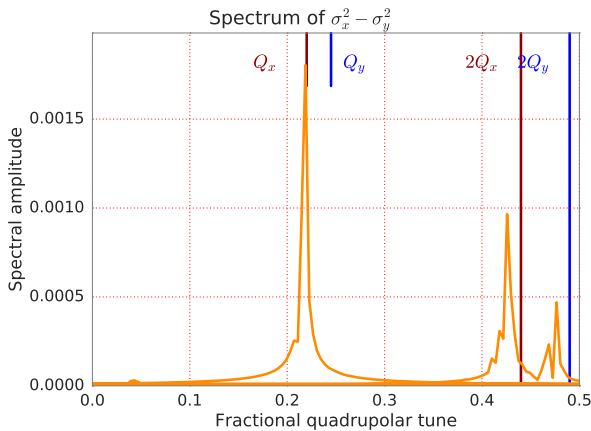
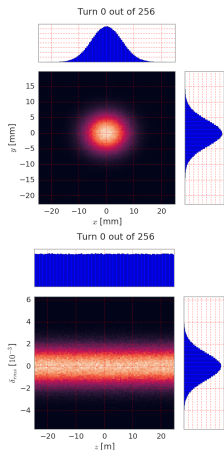
Coasting: KV Beam with Dispersion

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
no	KV (uniform)	no	yes	no



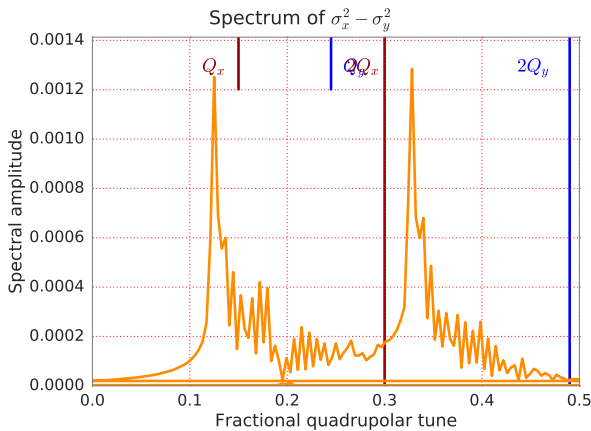
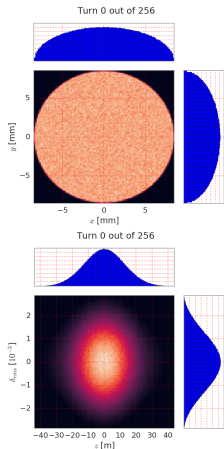
Coasting: RMS-equiv. Gaussian Beam with Dispersion

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
no	Gaussian	no	yes	no



Bunched KV Beam

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	no	no	no



2. Equipment in LHC Injectors: Status and Plans

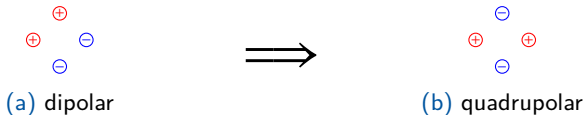
PS: Quadrupolar Pick-up

Stripline pick-up PR.BQL72 is part of the BBQ system:



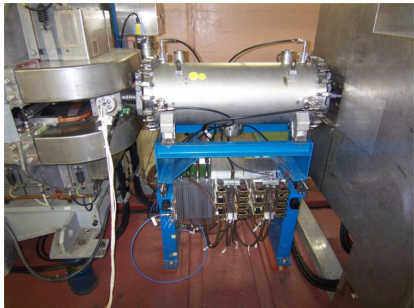
courtesy Tom Levens

Currently recabled to quadrupolar mode (first time in 2016):



PS: Transverse Feedback (TFB) as Quadrupolar Kicker

Kicker in section 97 is part of the new PS transverse feedback system:



courtesy Guido Sterbini

Since last week:

- source signal for quadrupolar excitation comes from BBQ system
 - new dedicated card BQL72_Q to control excitation parameters separately from dipolar tune measurements

Pick-up side:

- in PS, upgrade BQL72 with *3 channel frontend*, simultaneously extract
 - dipolar signals $\langle x \rangle$ and $\langle y \rangle$,
 - quadrupolar signal Q

→ *technical stop in June 2018*
- in PSB, make use of brand-new (2018) stripline pick-ups
 - install 3 channel frontend to include Q channel
 - *during 2018*
- in SPS, upgrade existing BBQ system with Q channel
 - install new 3 channel frontend *during LS2*

Kicker side:

- separate quadrupolar excitation signal path from rest of system
 - possibility to operate dipolar feedback system in closed loop + simultaneous quadrupolar excitation (*these coming weeks*)

3. Application: quadrupolar beam transfer function (2017)

Motivation

In the context of **strong space charge regime** with **LHC Injectors Upgrade** beam parameters: determine beam brightness (or incoherent space charge (SC) tune shift) **directly** via coherent quadrupolar modes

Starting from nominal LHC beam-type set-up:

- (large) natural chromaticity: $Q'_x = -0.83Q_x$ and $Q'_y = -1.12Q_y$
- lattice is usually strongly coupled via skew quadrupoles to stabilise slow horizontal head-tail instabilities
 - decouple lattice during envelope measurements
 - ⇒ only space charge coupling in envelope tunes
- measure *quadrupolar* beam transfer function to learn about space charge

Experimental Set-up

Ingredients:

- small time window of 15 ms with decoupled optics
- chirped **quadrupolar excitation** of beam via transverse feedback:
external waveform generator connected to kicker plates
 - 12 ms long frequency sweep with 1 ms return
 - harmonic $h = 5$ with frequency range 2.19 MHz to 2.4 MHz

Experimental Set-up

Ingredients:

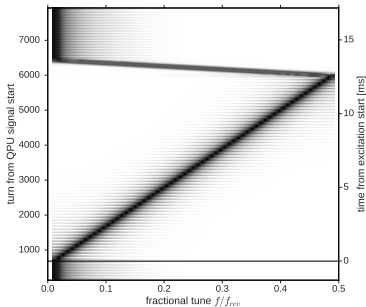
- small time window of 15 ms with decoupled optics
- chirped **quadrupolar excitation** of beam via transverse feedback: external waveform generator connected to kicker plates
 - 12 ms long frequency sweep with 1 ms return
 - harmonic $h = 5$ with frequency range 2.19 MHz to 2.4 MHz
- single bunch in PS with a factor 5 smaller incoherent SC tune shift compared to currently operational LHC beams, **off coupling**

intensity	$N \approx 0.3 - 0.4 \times 10^{12}$ ppb
transverse emittance	$\epsilon_{x,y} \approx 2.3$ mm mrad
average betatron function	$\beta_x \approx \beta_y \approx 16$ m
average dispersion	$D_x \approx 3$ m
momentum deviation spread	$\sigma_\delta \approx 1 \times 10^{-3}$
bunch length	$B_L \approx 180$ ns
synchrotron tune	$Q_s = 1/600 = 1.67 \times 10^{-3}$
KV space charge tune shift	$\Delta Q_{x,y}^{KV} \approx 0.02$

Quadrupolar Excitation: Chirp

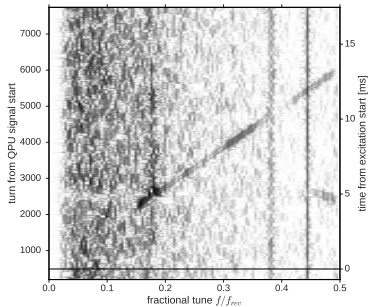
excitation signal

spectrogram for quadrupolar excitation frequency



beam response (via QPU)

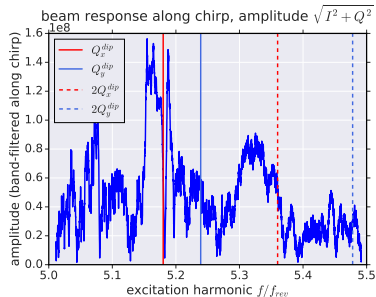
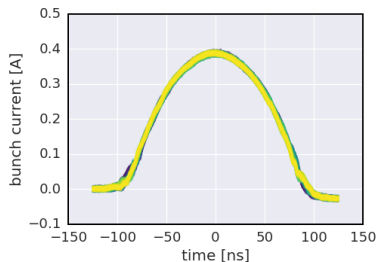
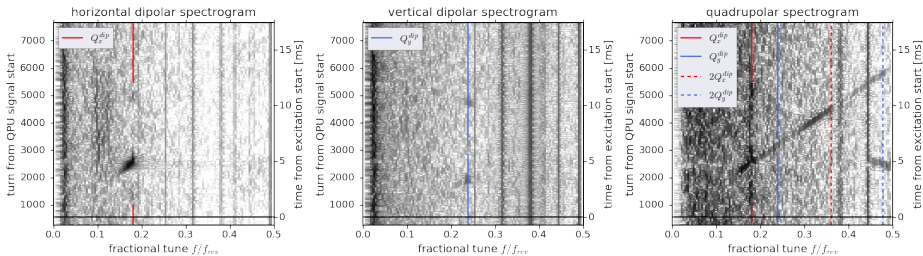
quadrupolar spectrogram for $N = 0.291 \times 10^{12}$



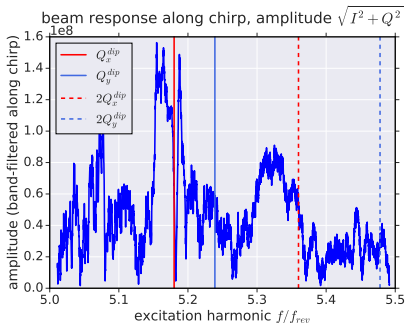
- distinct peaks around machine tunes $f < 0.25 f_{rev}$
- frequency bands around twice the machine tunes
- (disregard the constant frequencies, due to instrumentation)

Measured Quadrupolar Beam Transfer Function

$$N = 0.307 \times 10^{12}$$



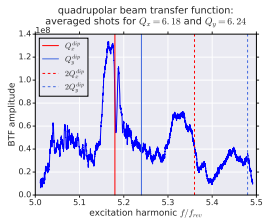
Observations in Spectrum



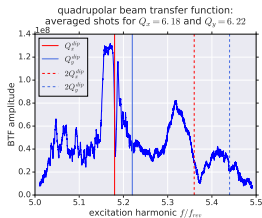
Observations:

- significant peaks around Q_x
 - dispersive coherent mode?
 - influence of chromaticity?
- envelope band below $2Q_x$ clearly visible
 - ⚠ would infer $\Delta Q_{x,y}^{KV} \approx 0.04$ (factor 2 too large!)
 - difficult to extract maximum shift, always many peaks (chromaticity?)

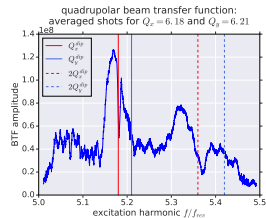
Tune Scan: BTFs Averaged over Shots



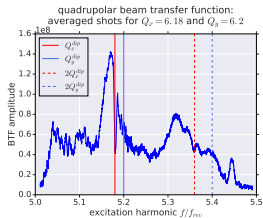
(a) $Q_x = 6.18$, $Q_y = 6.24$



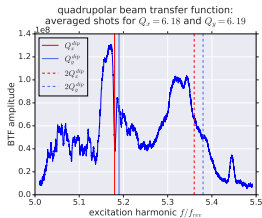
(b) $Q_x = 6.18$, $Q_y = 6.22$



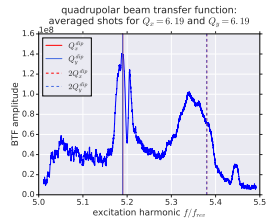
(c) $Q_x = 6.18$, $Q_y = 6.21$



(d) $Q_x = 6.18$, $Q_y = 6.20$



(e) $Q_x = 6.18$, $Q_y = 6.19$

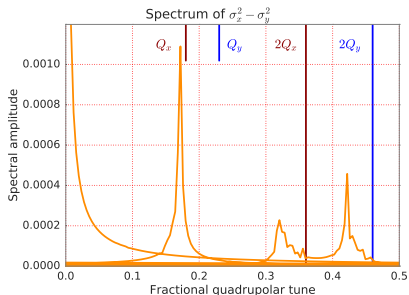


(f) $Q_x = 6.19$, $Q_y = 6.19$

... simulations?

Dispersive Coherent Mode

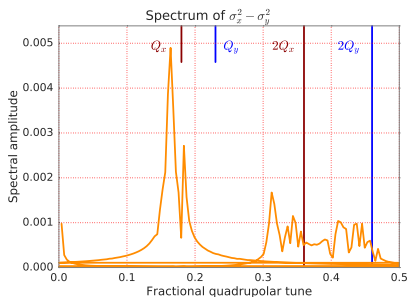
bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	yes	yes	no



- experimental parameters (here $N = 4 \times 10^{11}$ ppb)
 - evident quadrupolar betatron bands below $2Q_{x,y}$
 - coherent dispersive mode slightly below Q_x (shifted by space charge!)
- however, only one peak is seen as opposed to experiment...

Including Chromaticity

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
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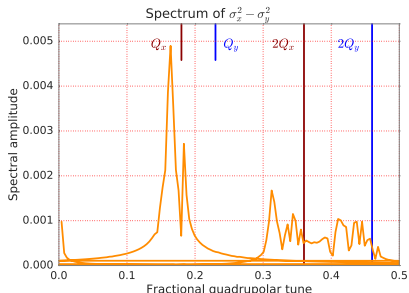
→ however, only one peak is seen as opposed to experiment...

→ including natural chromaticity ($Q'_x = -0.83Q_x$ and $Q'_y = -1.12Q_y$):

- broadens dispersive peak (here FFT undersamples sidebands)
- produces additional peaks, shifted dominant peak

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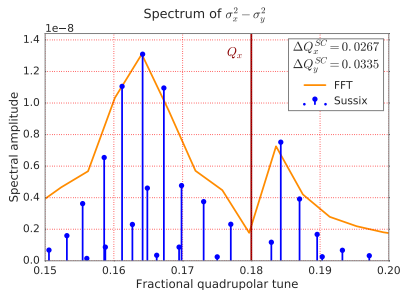
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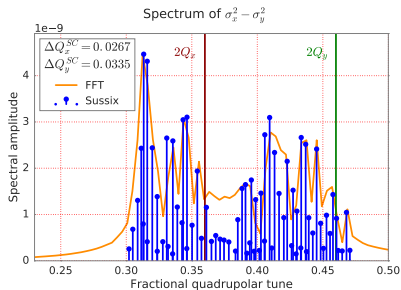
- broadens dispersive peak (here FFT undersamples sidebands)
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NB: simulations ran with 10×10^6 macro-particles on 150 longitudinal slices across the RF bucket (≈ 80 m) where space charge is solved on 128×128 grids (no significant transverse difference between 2.5D / 3D PIC)

Detailed Dispersive and Betatron QPU Spectrum



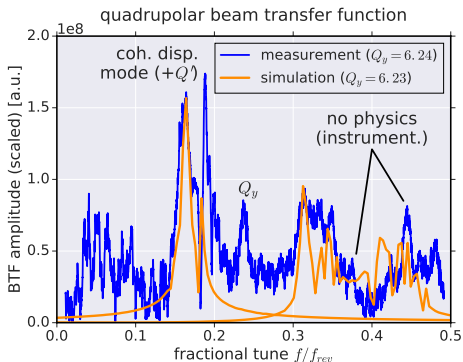
(a) dispersive part



(b) betatron part

- Sussix tune analysis reveals regular sideband structure around dispersive mode (blue peaks)
- chromaticity also affects betatron spectrum, additional peaks distort betatron band (e.g. vertical extending beyond $2Q_y$)
 - with finite chromaticity, measuring betatron band width seems intricate

just giving it a try... Measurement vs. Simulation



- horizontal quadrupolar betatron band below $2Q_x \approx 0.36$: **similar width**
- vertical quadrupolar betatron band (different Q_y in simulation and experiment): **similar width**
- **chromaticity** seems to explain larger width of betatron bands (\sim factor 2) w.r.t. computation from 2D envelope equations (without dispersion)
- coherent dispersive mode peak **at same frequency**
 - width and sidebands closer to measurements (than without chromaticity)

3. Applications: PS injection oscillations (2018)

Motivation:

- PSB transfer line to PS is optically mismatched in terms of D_x
- LHC Injectors Upgrade requires much larger longitudinal emittances ϵ_z (from present 1.3 eVs towards 3 eVs)
 - momentum spread σ_δ increases
 - ⇒ corresponding dispersion mismatch may lead to large **transverse emittance growth** of 50% and more
 - LIU PS beam dynamics meeting presentation ↗

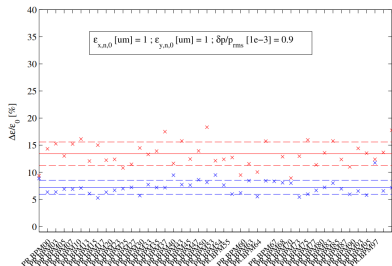
⇒ Quadrupolar spectrum of injection oscillations could be useful?

Two scenarios:

- ① optics A: nominal PS lattice ($\beta_{x,y}$ matched, D_x mismatched)
 - ② optics C: optimised PS lattice ($\beta_{x,y}$ mismatched, D_x matched)
- dipolar injection oscillations corrected! RF off!

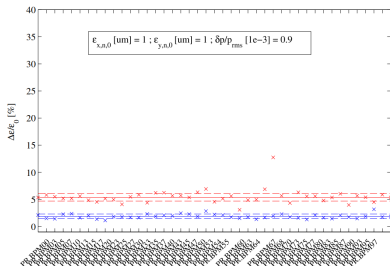
Injection Oscillations: QPU Spectrograms

optics A



courtesy Vincenzo Forte

optics C

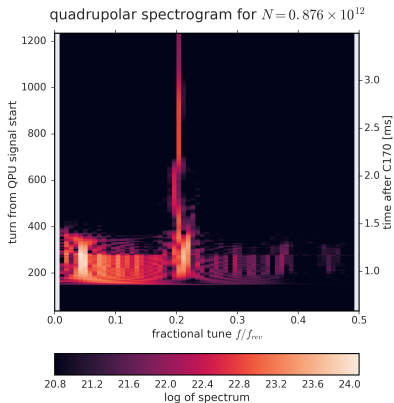


courtesy Vincenzo Forte

→ preliminary data from last week, to be investigated in detail!

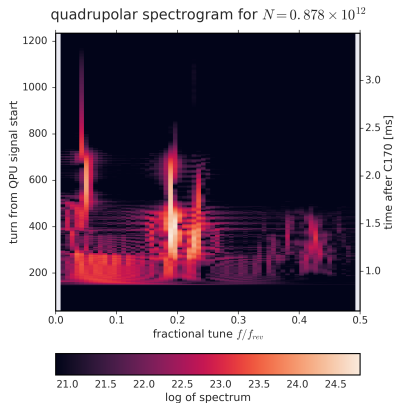
Injection Oscillations: QPU Spectrograms

optics A



128 turns FFT sliding window

optics C

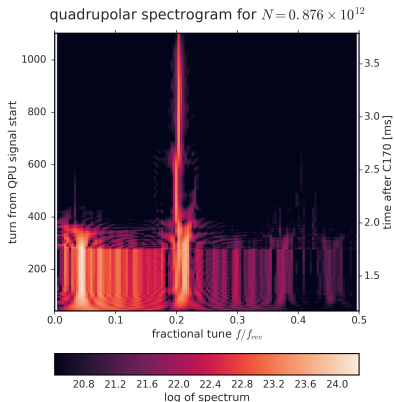


128 turns FFT sliding window

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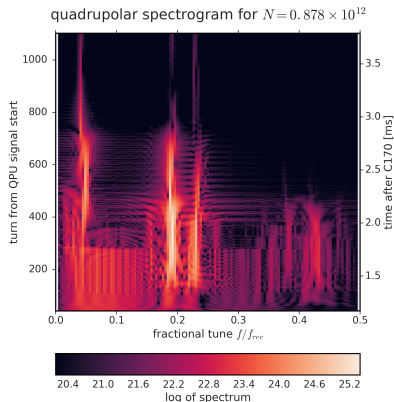
Injection Oscillations: QPU Spectrograms

optics A



256 turns FFT sliding window

optics C



256 turns FFT sliding window

→ preliminary data from last week, to be investigated in detail!

In conclusion:

- coherent **dispersive mode** identified as strong quadrupolar spectral component (especially for injection oscillations)
- **chromaticity** significantly impacts quadrupolar spectrum:
 - broadens betatron bands \implies complicates estimation of $\Delta Q_{x,y}^{KV}$
 - shifts coherent dispersive mode and creates sidebands
- rich information in QPU spectrum \implies very powerful diagnostic tool for space charge and dispersive mismatch at injection

Next steps for ongoing studies:

- better understand the injection oscillations and spectrum for the dispersive mismatch
- repeat quadrupolar BTF with $h = 1$ excitation
- estimate $\Delta Q_{x,y}^{KV}$ with vanishing chromaticity in PS?

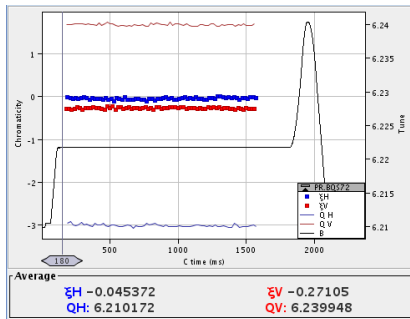


Figure: last week: $Q'_x = -0.045Q_x$ and $Q'_y = -0.271Q_y$ in the PS

Thank you for your attention!

Appendix

- [1] R Singh et al. “Observations of the quadrupolar oscillations at GSI SIS-18”. In: (2014).
- [2] R H Miller et al. *Nonintercepting emittance monitor*. Tech. rep. Stanford Linear Accelerator Center, 1983.
- [3] Andreas Jansson. “Noninvasive single-bunch matching and emittance monitor”. In: *Physical Review Special Topics-Accelerators and Beams* 5.7 (2002), p. 072803.
- [4] Cheng-Yang Tan. *Using the quadrupole moment frequency response of bunched beam to measure its transverse emittance*. Tech. rep. Fermi National Accelerator Laboratory (FNAL), Batavia, IL, 2007.
- [5] Michel Chanel. *Study of beam envelope oscillations by measuring the beam transfer function with quadrupolar pick-up and kicker*. Tech. rep. 1996.
- [6] T Uesugi et al. “Observation Of Quadrupole Mode Frequency And Its Connection With Beam Loss”. In: KEK-99-98 (1999). URL: <http://cds.cern.ch/record/472700>.

- [7] R C Baer. “Untersuchung der quadrupolaren BTF-Methode zur Diagnose intensiver Ionenstrahlen”. [Universitaet Frankfurt, Germany, 2000.](#)

Incoherent KV Tune Shift

The uniform (Kapchinskij-Vladimirskij / KV) beam distribution has all particles at same incoherent space charge tune shift:

$$\Delta Q_{x,y}^{\text{KV}} \doteq -\frac{K^{\text{SC}} R^2}{4\sigma_{x,y}(\sigma_x + \sigma_y)Q_{x,y}} \quad (4a)$$

$$\doteq \frac{1 + \sigma_{x,y}/\sigma_{y,x}}{2Q_{x,y}} \Lambda \quad (4b)$$

Connect Λ quantity to general envelope mode expressions in terms of **observables**:

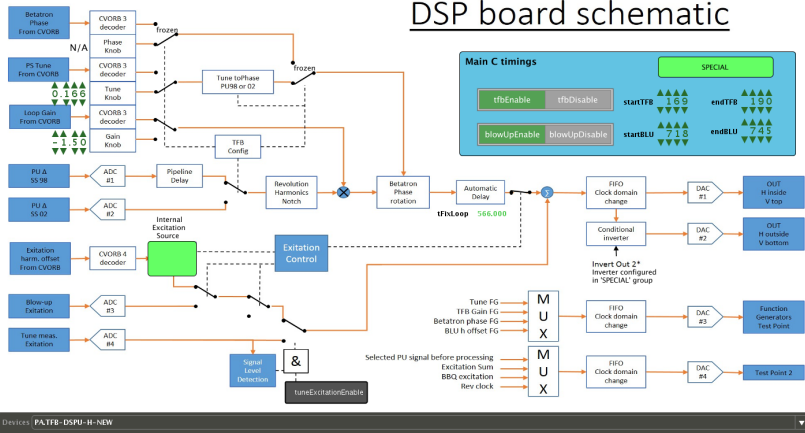
$$\Lambda = \frac{Q_+^2 + Q_-^2 - 4(Q_x^2 + Q_y^2)}{4 + 3(\sigma_x/\sigma_y + \sigma_y/\sigma_x)} \quad (5)$$

(Gaussian tune spread = 2x the RMS-equivalent KV tune shift!)

$$\text{space charge perveance } K^{\text{SC}} \doteq \frac{q\lambda}{2\pi\epsilon_0\beta\gamma^2 p_0 c}$$

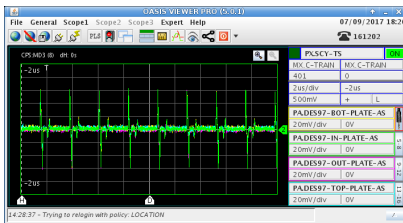
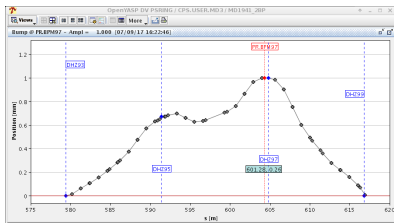
TFB: Schematic Plan

One of the two planes configuration

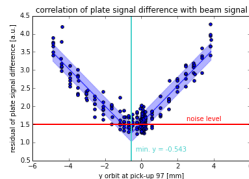
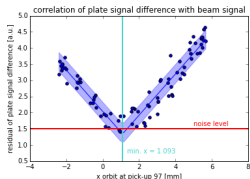
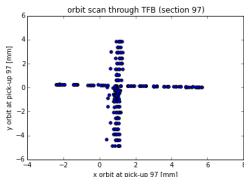


TFB: Impact of Orbit

Set up a local bump through the TFB and measure the induced beam signal on the plates (effectively a BPM):

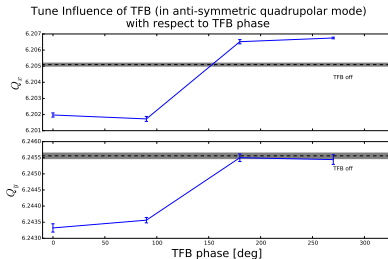
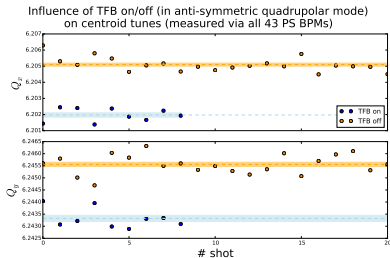


By scanning the orbit location one can minimise the difference signal:

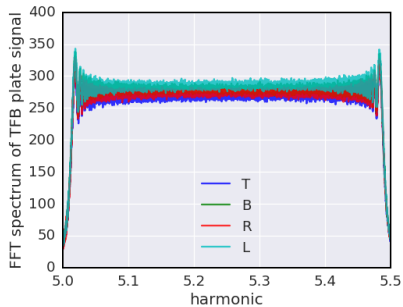
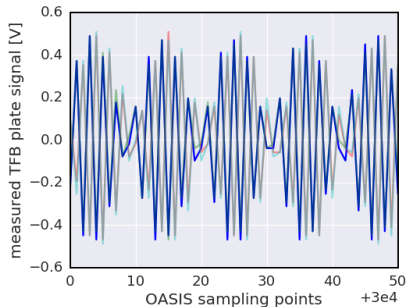


TFB: Static Quadrupole on $h = 1$

- TFB pulsing at f_{rev} becomes a static quadrupole to the beam
- varying the phase of the pulsing RF quadrupole changes the tune impact



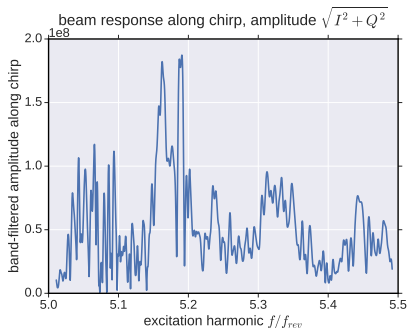
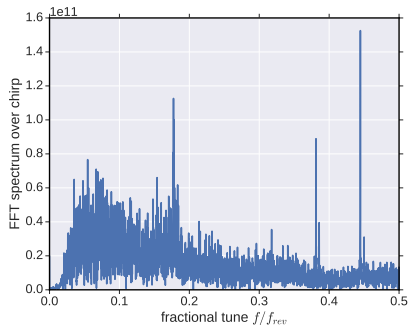
TFB: Quadrupolar Chirp



- top and bottom plates oscillate together in anti-phase to right and left plates of transverse feedback
 - quadrupolar RF excitation (anti-symmetric mode)
- frequency swept during BTF measurement: 2.19 MHz to 2.4 MHz
 - harmonic 5 to 5.5 (PS revolution frequency $f_{\text{rev}} = 437 \text{ kHz}$)

Extracting the Beam Response...

(a) FFT across up-chirp time is not such a useful idea...



(b) ... instead project and band filter along local excitation frequency

Approach: In-phase and Quadrature Components

Take

- QPU time signal $S_{\text{QPU}}(t)$
- excitation signal $S_{\text{exc}}(t)$ (sine wave with increasing frequency)
- 90 deg shifted excitation signal $C_{\text{exc}}(t) = S_{\text{exc}}(t)|_{\phi \rightarrow \phi + 90 \text{ deg}}$

Assume immediate beam response to chirp:

- correlation:** find excitation start in $S_{\text{QPU}}(t)$ by correlation with $S_{\text{exc}}(t)$
- demodulation** of measured QPU time signal into

$$I(t) = S_{\text{QPU}}(t) \cdot S_{\text{exc}}(t) \quad (\text{in-phase component})$$

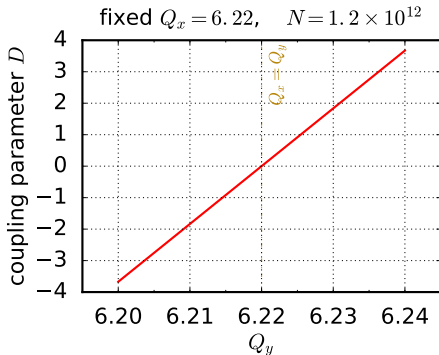
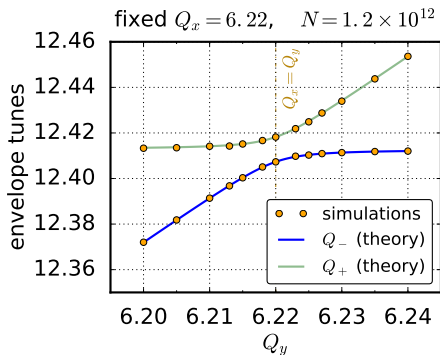
and

$$Q(t) = S_{\text{QPU}}(t) \cdot C_{\text{exc}}(t) \quad (\text{quadrature component})$$

- band filter** original $S_{\text{QPU}}(t)$ around time-varying excitation frequency by low pass filtering $I(t)$ and $Q(t)$
- amplitude** of beam response along chirp amounts to $\sqrt{I^2(t) + Q^2(t)}$

Simulations for Tune Scan

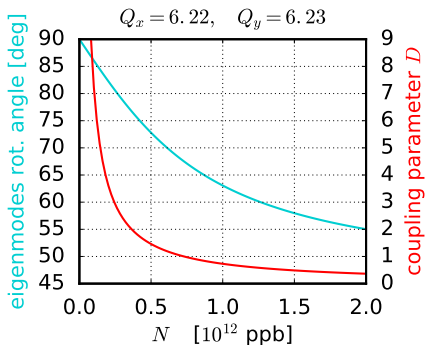
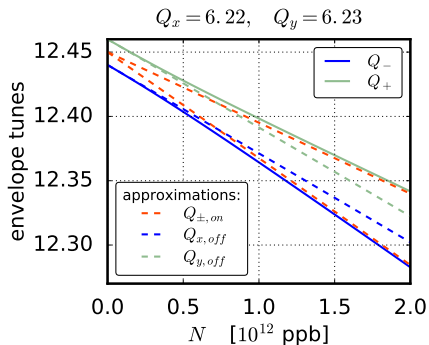
Simulations with KV beams for $N = 1.2 \times 10^{12}$ confirm theory:



- r.m.s. equivalent Gaussian beams (with same $\sigma_{x,y}$ like KV beams) exhibit same quadrupolar tunes as KV
- ⚠ Gaussian spectra broaden quickly

Intensity Scan

With slightly split tunes, approach full coupling by increasing bunch intensity:



⇒ scan space charge tune shift $\Delta Q_{x,y}^{KV}$ and verify theory

Envelope equations of motion (e.o.m.)

$$r_x'' + K_x(s)r_x - \frac{\epsilon_{x,\text{geo}}^2}{r_x^3} - \frac{K^{\text{SC}}}{2(r_x + r_y)} = 0 \quad , \quad (6a)$$

$$r_y'' + K_y(s)r_y - \frac{\epsilon_{y,\text{geo}}^2}{r_y^3} - \frac{K^{\text{SC}}}{2(r_x + r_y)} = 0 \quad (6b)$$

for transverse r.m.s. beam widths $r_{x,y} = \sigma_{x,y}$ have equilibrium

$$\frac{Q_x^2}{R^2} r_{x,m} - \frac{\epsilon_{x,\text{geo}}^2}{r_{x,m}^3} - \frac{K^{\text{SC}}}{2(r_{x,m} + r_{y,m})} = 0 \quad , \quad (7a)$$

$$\frac{Q_y^2}{R^2} r_{y,m} - \frac{\epsilon_{y,\text{geo}}^2}{r_{y,m}^3} - \frac{K^{\text{SC}}}{2(r_{x,m} + r_{y,m})} = 0 \quad (7b)$$

Linear Perturbation in Smooth Approximation

Constant focusing channel

$$K_{x,y} = \frac{1}{\beta_{x,y}^2} = \frac{Q_{x,y}^2}{R^2} = \text{const} \quad (8)$$

gives linearised e.o.m. for perturbation around equilibrium $r = r_m + \delta r$

$$\frac{d^2}{ds^2} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix} = - \underbrace{\begin{pmatrix} \kappa_x & \kappa_{SC} \\ \kappa_{SC} & \kappa_y \end{pmatrix}}_{\doteq (\kappa)} \cdot \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix} \quad (9)$$

$$\text{with } \begin{cases} \kappa_{x,y} = 4 \frac{Q_{x,y}^2}{R^2} - \frac{2\sigma_{x,y} + 3\sigma_{y,x}}{\sigma_{x,y}} \kappa_{SC} \\ \kappa_{SC} \doteq \frac{K^{SC}}{2(\sigma_x + \sigma_y)^2} \end{cases} \quad (10)$$

Coupling Parameter

$$D \doteq \frac{\kappa_y - \kappa_x}{2\kappa_{SC}} = 4 \frac{Q_y^2 - Q_x^2}{K^{SC} R^2} (\sigma_x + \sigma_y)^2 + \frac{3}{2} \left(\frac{\sigma_y}{\sigma_x} - \frac{\sigma_x}{\sigma_y} \right) \quad (11)$$

Rotation Into Decoupled Eigensystem

$$\begin{aligned} \tan(\alpha) &= \frac{1}{2\kappa_{SC}} \left[\kappa_y - \kappa_x + \sqrt{4\kappa_{SC}^2 + (\kappa_y - \kappa_x)^2} \right] \\ &= D + \sqrt{1 + D^2} \end{aligned} \quad (12)$$

KV Space Charge Tune Shift

$$\Delta Q_{x,y}^{\text{KV}} = -\frac{K^{\text{SC}} R^2}{4\sigma_{x,y}(\sigma_x + \sigma_y)Q_{x,y}} \quad (13)$$

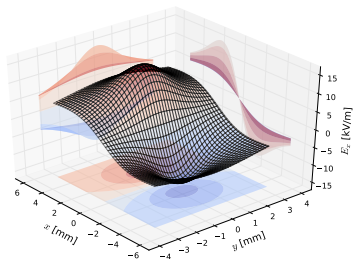
$$\text{with } K^{\text{SC}} \doteq \frac{q\lambda}{2\pi\epsilon_0\beta\gamma^2 p_0 c} \quad (14)$$

R.m.s. Equivalent Gaussian Space Charge Tune Spread

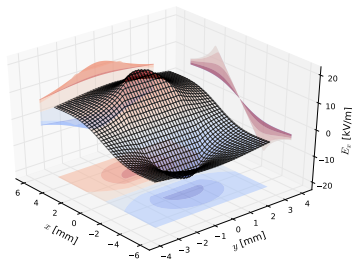
linearised Gaussian e-field = twice r.m.s. equivalent KV e-field

$$\Rightarrow \max \left\{ \Delta Q_{x,y}^{\text{spread}} \right\} = 2 \Delta Q_{x,y}^{\text{KV}} \quad (15)$$

Gaussian vs. R.m.s. Equivalent KV



(a) Gaussian beam

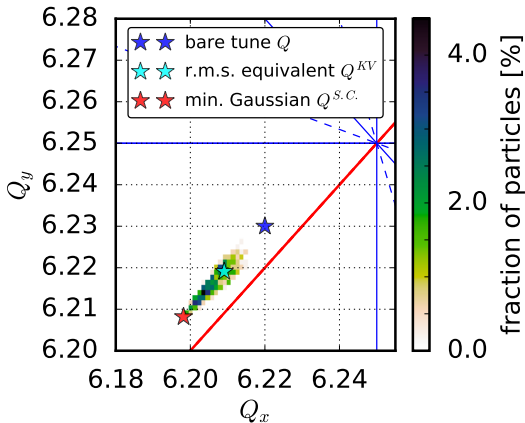


(b) r.m.s. equivalent KV beam

Figure: Electric fields in r.m.s. equivalent distributions with same $\sigma_{x,y}$

Incoherent Tunes and R.m.s. Equivalence

Incoherent tune spread of a coasting, transversely Gaussian distribution:



Quadrupolar Mode Tunes (General Formula)

$$Q_{\pm}^2 = \frac{R^2}{2} \left[\kappa_x + \kappa_y \pm \sqrt{4\kappa_{SC}^2 + (\kappa_y - \kappa_x)^2} \right] \quad (16)$$
$$= 2(Q_x^2 + Q_y^2) - \frac{K^{SC} R^2}{(\sigma_x + \sigma_y)^2} \left[1 + \frac{3}{4} \left(\frac{\sigma_y}{\sigma_x} + \frac{\sigma_x}{\sigma_y} \right) \mp \frac{\sqrt{1 + D^2}}{2} \right]$$

Off-resonance $D \gg 1$ With Round Beam

$$Q_+ = 2Q_y - \frac{5}{4} |\Delta Q_y^{\text{KV}}| \quad , \quad (17a)$$

$$Q_- = 2Q_x - \frac{5}{4} |\Delta Q_x^{\text{KV}}| \quad . \quad (17b)$$

for $Q_y > Q_x$ otherwise exchange $x \leftrightarrow y$

On-resonance $D \approx 0$ With Round Beam

$$Q_+ = 2Q_0 - |\Delta Q^{KV}| \quad , \quad (18a)$$

$$Q_- = 2Q_0 - \frac{3}{2} |\Delta Q^{KV}| \quad . \quad (18b)$$

for $Q_0 \doteq Q_x = Q_y$ and $\Delta Q^{KV} \doteq \Delta Q_x^{KV} = \Delta Q_y^{KV}$

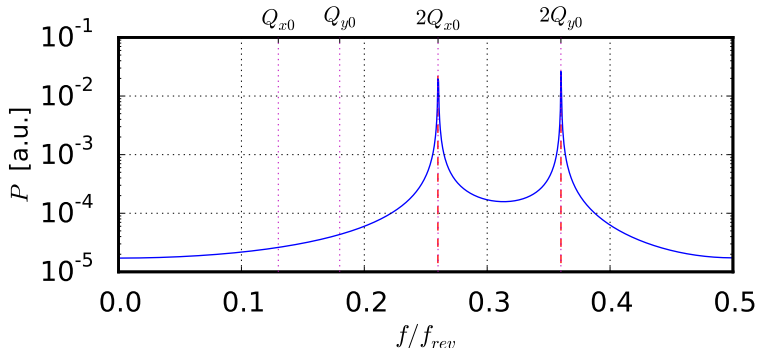
simulation parameters:

- machine: SPS at injection
- $\gamma = 27.7$
- $\epsilon_x = \epsilon_y = 2.5 \text{ mm} - \text{mrad}$
- $N_b = 1.25 \times 10^{11}$
- 512 – 2048 turns
- 2.6×10^5 macro-particles
- longitudinally matched Gaussian-type distribution
- betatron mismatch by 10% in both x, y

⇒ injection oscillations

QPU Spectrum: Only Betatron Mismatch

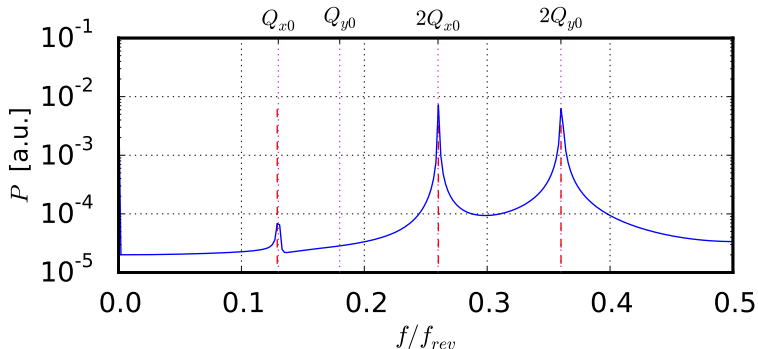
need beam mismatched to both β_x, β_y to see clear peaks



- ⇒ $2Q_{x0}, 2Q_{y0}$ from undepressed envelope oscillations
- ⇒ including synchrotron motion: same spectrum (no coupling!)

QPU Spectrum: Include Dispersion

smooth approximation: constant $D_x = 2.96$ around the ring



⇒ peak at Q_{x0} comes from dispersion

Reason for Dispersion Peak

$$\sigma_x(i_{\text{turn}}) = \sqrt{\langle x_i^2 \rangle_{\text{beam}} - \langle x_i \rangle_{\text{beam}}^2}$$

with $x_i(i_{\text{turn}}) = \sqrt{\beta_x \epsilon_{x,i}^{\text{s.p.}}} \cos(2\pi Q_{x0} i_{\text{turn}} + \Psi_0) + D_x \delta_i$

Reason for Dispersion Peak

$$\sigma_x(i_{\text{turn}}) = \sqrt{\langle x_i^2 \rangle_{\text{beam}} - \langle x_i \rangle_{\text{beam}}^2}$$

$$\text{with } x_i(i_{\text{turn}}) = \sqrt{\beta_x \epsilon_{x,i}^{\text{s.p.}}} \cos(2\pi Q_{x0} i_{\text{turn}} + \Psi_0) + D_x \delta_i$$

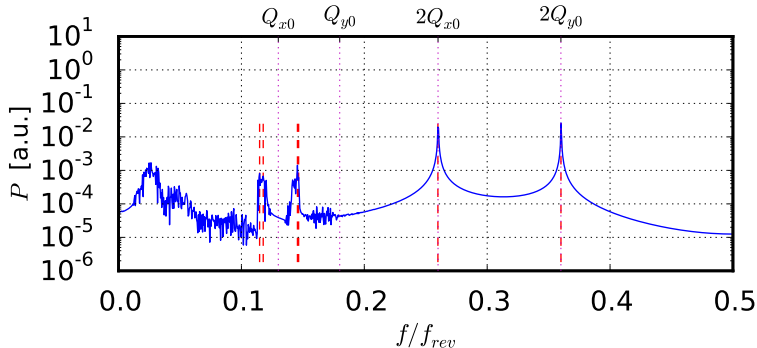
$$\Rightarrow x_i^2 = \dots \underbrace{\cos^2(2\pi Q_{x0} i_{\text{turn}} + \dots)}_{\dots \cos(2\pi 2Q_{x0} i_{\text{turn}} + \dots)} + \dots D_x \delta_i \cdot \cos(2\pi Q_{x0} i_{\text{turn}} + \dots) + \dots$$

$$\text{due to: } 2\cos^2(\alpha) = \cos(2\alpha) + 1$$

i.e. only for $D_x \neq 0 \Rightarrow$ peak at Q_{x0}

QPU Spectrum: Include Synchrotron Motion

synchrotron motion couples to betatron motion through non-zero $D_x = 29.6\text{ m}$ (smooth approximation!)

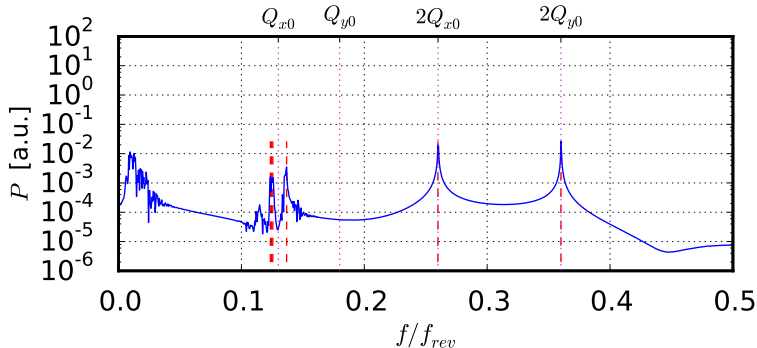


⇒ peak separation at Q_{x0} from synchrobetatron coupling

- $Q_s = 0.017$ at injection for $V = 5.75\text{ MV}$

QPU Spectrum: Slower Synchrotron Motion

synchrotron motion couples to betatron motion through non-zero $D_x = 29.6\text{ m}$ (smooth approximation!)



- $Q_s = 0.007$ changing $\gamma_{tr} = 17.95 \rightarrow 25$ (while $\gamma = 27.7$)
⇒ peak separation shrinks

Reason for Peak Separation with Q_s

$$x_i^2 = \dots + \dots D_x \delta_i \cdot \cos(2\pi Q_{x0} i_{\text{turn}} + \dots) + \dots$$

with $\delta_i(i_{\text{turn}}) = \hat{\delta}_i \cos(2\pi Q_s i_{\text{turn}} + \dots)$

Reason for Peak Separation with Q_s

$$x_i^2 = \dots + \dots D_x \delta_i \cdot \cos(2\pi Q_{x0} i_{\text{turn}} + \dots) + \dots$$

$$\text{with } \delta_i(i_{\text{turn}}) = \hat{\delta}_i \cos(2\pi Q_s i_{\text{turn}} + \dots)$$

$$\Rightarrow \tilde{x}_i^2 = \dots + \dots \underbrace{\cos(2\pi Q_{x0} i_{\text{turn}} + \dots) \cos(2\pi Q_s i_{\text{turn}} + \dots)}_{\cos(2\pi(Q_{x0}-Q_s)i_{\text{turn}}+\dots)+\cos(2\pi(Q_{x0}+Q_s)i_{\text{turn}}+\dots)} + \dots$$

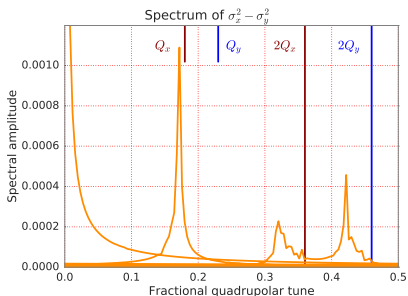
$$\text{due to: } 2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta)$$

i.e. for $D_x \neq 0$ and $Q_s \neq 0$:

one peak at $Q_{x0} \Rightarrow$ two peaks located at $Q_{x0} \pm Q_s$

PS: Bunched Beam with SC (Normal $V_{RF} = 24\text{ kV}$)

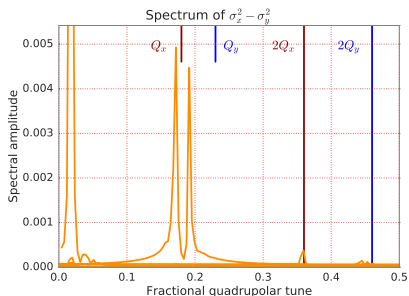
bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	yes	yes	no



- coherent dispersive mode with synchrotron motion splits into two peaks
- at usual $V_{RF} = 24\text{ kV}$ we have $Q_S \approx 1/600$

PS: Bunched Beam with SC (Large $V_{RF} = 1.5 \text{ MV}$)

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	yes	yes	no



- coherent dispersive mode with synchrotron motion splits into two peaks
 - at usual $V_{RF} = 24 \text{ kV}$ we have $Q_S \approx 1/600$
 - at (unrealistic) $V_{RF} = 1.5 \text{ MV}$ we have $Q_S \approx 0.0107$
- ⇒ two peaks are clearly separated in quadrupolar spectrum