Experience at SPring-8 with Beam Position Monitors for Measuring Second-Order Moments of Charged Particle Beams

Kenichi Yanagida, Japan Synchrotron Radiation Research Institute

Contents

- Longitudinal and transverse wall current analyses
- Single and multi-charged-particle systems
- Absolute, centroid and relative moments of multi-particle system
- Contribution of moment to BPM output and effective aperture radius
- Recursive method for calculations of moments (successive iteration)
- Entire calibration (Determination of relative attenuation experimentally)
- Measurement resolution

Introduction

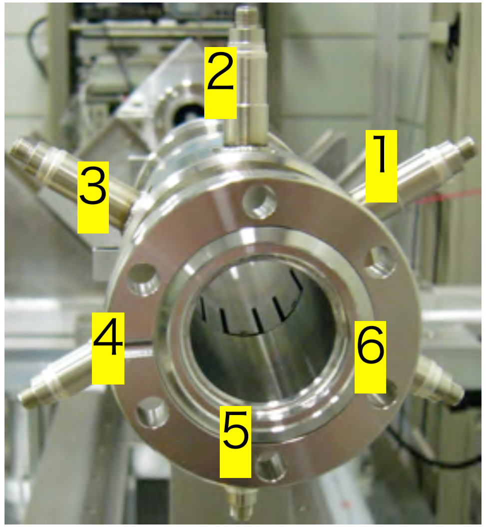
SPring-8 linac BPM was designed so that it could be simply understood with respect to the signal detection. Its transverse cross section keeps cylindrical shape as much as possible.

This features enables us to consider electrostatic interaction only and to calculate moments easily, i.e.,

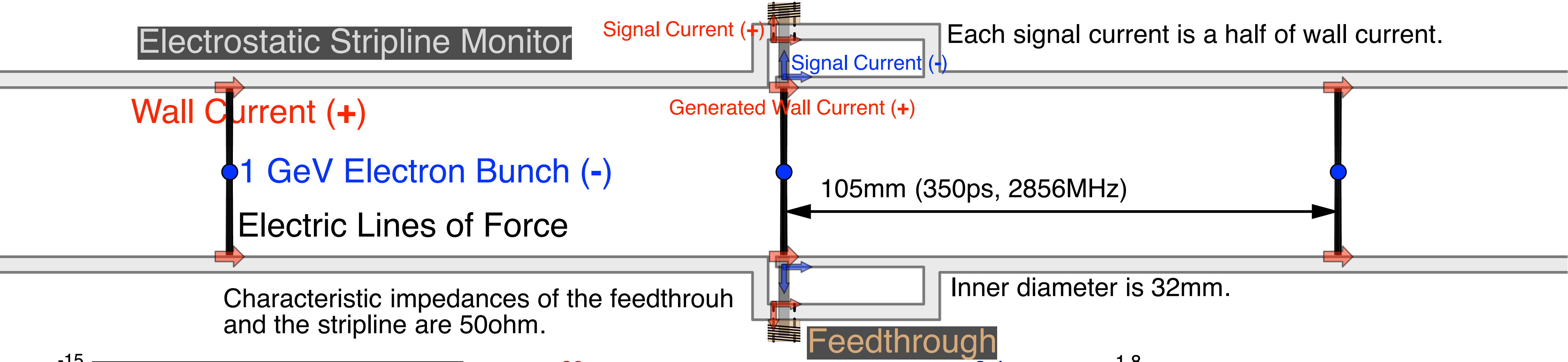
* singal detection is simplified as the detection of **wall current**,
* a field calculation of the BPM with circular cross-section can be done **analytically**, especially for the higher order moment calculatoin.

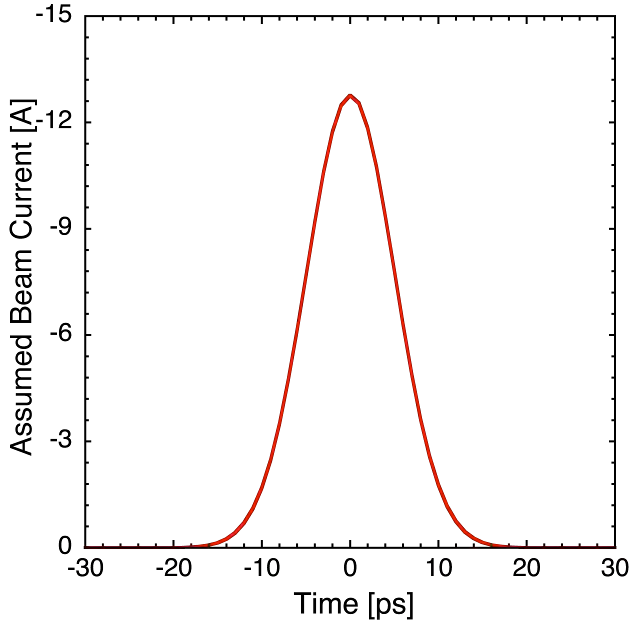
Consider cylindrical coordinate system

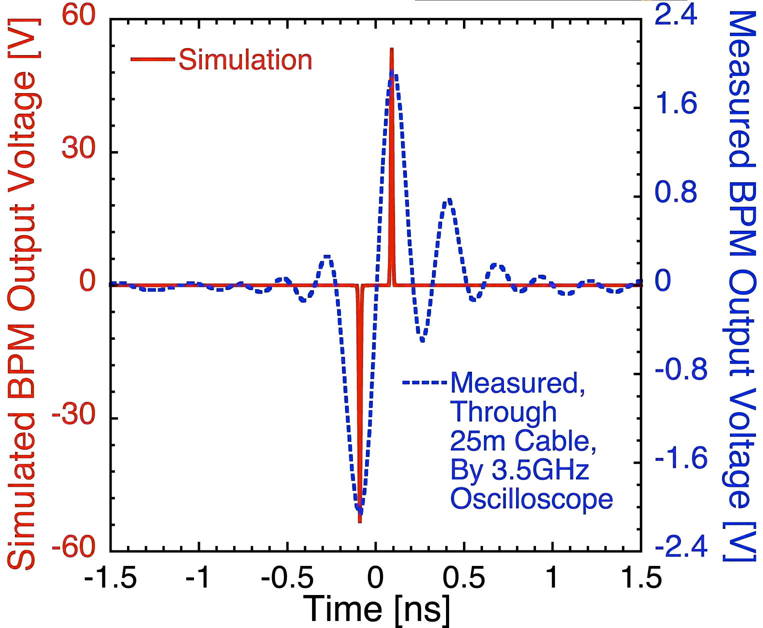
-> Analysis can be separated longitudinally and transversely.

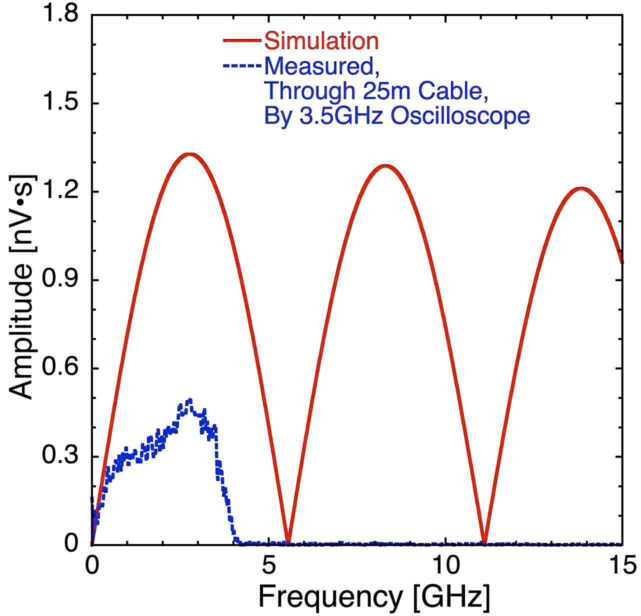


Longitudinal wall current analysis





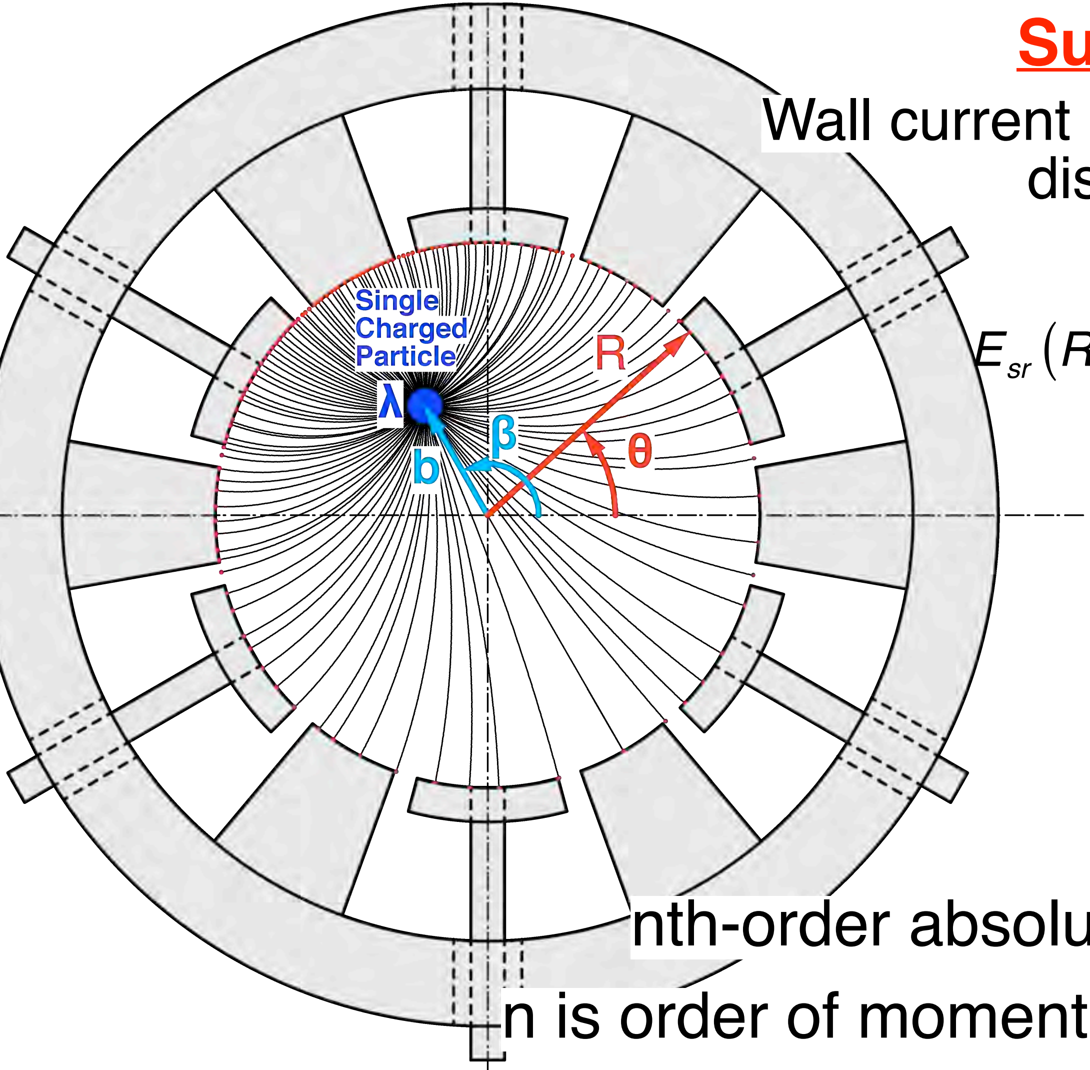




Transverse wall current analysis

Wall current is connected to the charged particle by the electric line of force.

Suppose a single charged particle.



Wall current distribution is propotional to the electric field distribution on the electrode surface Esr(R, θ).

$$E_{sr}(R,\theta) = \frac{\lambda}{2\pi R\varepsilon_0}\left[1+2\sum_{n=1}^{\infty}\frac{b^n}{R^n}\cos\{n(\theta-\beta)\}\right]$$

$$= \frac{\lambda}{2\pi R\varepsilon_0}\left(1+2\sum_{n=1}^{\infty}\frac{b^n\cos n\theta\cos n\beta + b^n\sin n\theta\sin n\beta}{R^n}\right)$$

$$= \frac{\lambda}{2\pi R\varepsilon_0}\left(1+2\sum_{n=1}^{\infty}\frac{p_n\cos n\theta + q_n\sin n\theta}{R^n}\right) \qquad (b \ll R)$$

$$p_n = b^n\cos n\beta\ [m^n]$$

$$q_n = b^n\sin n\beta\ [m^n]$$

nth-order absolute single particle cosine (sine) moment

n is order of moment. There are infinite number of moments.

Multi-particle system (M particles)

Because of the superposition principle, the electric field distribution of multi-paricle system Er(R, θ) is sum of all single particle distributions Esr(R, θ)s, i.e.;

$$E_r(R,\theta) = \sum_{N=1}^{M} E_{sr}(R,\theta)$$

$$= \sum_{N=1}^{M} \frac{\lambda}{2\pi R\varepsilon_0}\left(1+2\sum_{n=1}^{\infty}\frac{p_{Nn}\cos n\theta + q_{Nn}\sin n\theta}{R^n}\right)$$

$$= \frac{\lambda}{2\pi R\varepsilon_0}\left(M + 2\sum_{n=1}^{\infty}\frac{\sum_{N=1}^{M} p_{Nn}\cos n\theta + \sum_{N=1}^{M} q_{Nn}\sin n\theta}{R^n}\right)$$

$$= \frac{M\lambda}{2\pi R\varepsilon_0}\left(1+2\sum_{n=1}^{\infty}\frac{\frac{1}{M}\sum_{N=1}^{M} p_{Nn}\cos n\theta + \frac{1}{M}\sum_{N=1}^{M} q_{Nn}\sin n\theta}{R^n}\right)$$

$$= \frac{\Lambda}{2\pi R\varepsilon_0}\left(1+2\sum_{n=1}^{\infty}\frac{P_n\cos n\theta + Q_n\sin n\theta}{R^n}\right)$$

$$\Lambda = M\lambda$$

$$P_n = \frac{1}{M}\sum_{N=1}^{M} p_{Nn} = \frac{1}{M}\sum_{N=1}^{M} b_N^n \cos n\beta_N \;\left[m^n\right]$$

$$Q_n = \frac{1}{M}\sum_{N=1}^{M} q_{Nn} = \frac{1}{M}\sum_{N=1}^{M} b_N^n \sin n\beta_N \;\left[m^n\right]$$

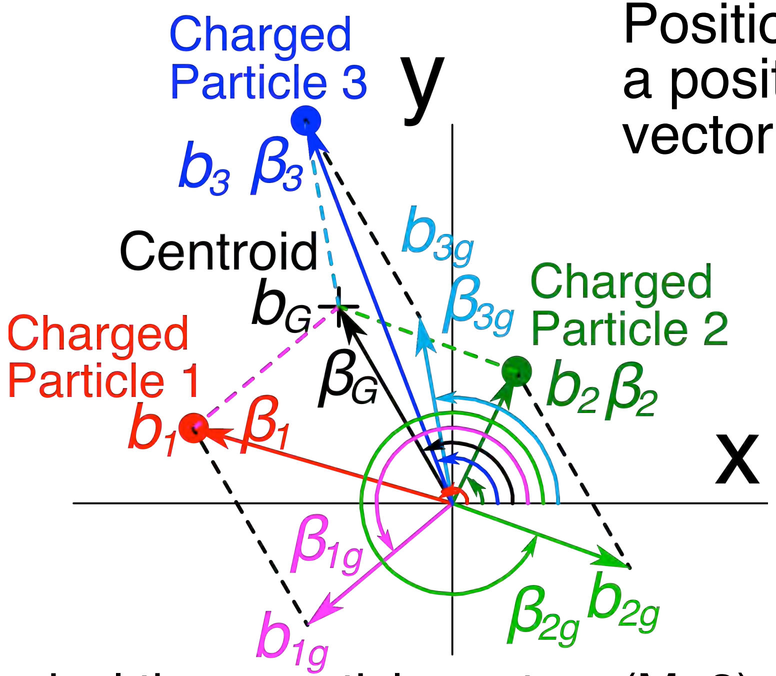
nth-order absolute cosine (sine) moment

NOTE : Absolute moments are measurable physical quantities using BPM.

Separation of centroid and relative moments from absolute moment

We only treat the cosine component here, because sine componet is derived same way.

Multi-particle system (M particles) must have the centroid which is located at (b_G, β_G).



Typical three particle system (M=3)

p_{Gn} : nth-order centroid cosine moment

p_{Ngn} : Nth particle nth-order relative cosine moment

Position vector of Nth particle (b_N, β_N) can be decomposed into a position vector of the centroid (b_G, β_G) and a remaining vector (b_{Ng}, β_{Ng}) so as to satisfy,

$$\begin{cases} b_N \cos\beta_N = b_G \cos\beta_G + b_{Ng} \cos\beta_{Ng} \\ b_N \sin\beta_N = b_G \sin\beta_G + b_{Ng} \sin\beta_{Ng} \end{cases}$$

Therefore nth-order absolute Nth single particle cosine moment p_{Nn} is expressed as,

$$p_{N1} = b_N \cos\beta_N = b_G \cos\beta_G + b_{Ng} \cos\beta_{Ng} = p_{G1} + p_{Ng1}$$

$$p_{N2} = b_N^2 \cos 2\beta_N = b_G^2 \cos 2\beta_G + 2b_G b_{Ng} \cos(\beta_G + \beta_{Ng}) + b_{Ng}^2 \cos 2\beta_{Ng}$$

$$= p_{G2} + 2b_G b_{Ng} \cos(\beta_G + \beta_{Ng}) + p_{Ng2}$$

$$p_{N3} = b_N^3 \cos 3\beta_N$$

$$= b_G^3 \cos 3\beta_G + 3b_G^2 b_{Ng} \cos(2\beta_G + \beta_{Ng}) + 3b_G b_{Ng}^2 \cos(\beta_G + 2\beta_{Ng}) + b_{Ng}^3 \cos 3\beta_{Ng}$$

$$= p_{G3} + 3b_G^2 b_{Ng} \cos(2\beta_G + \beta_{Ng}) + 3b_G b_{Ng}^2 \cos(\beta_G + 2\beta_{Ng}) + p_{Ng3}$$

Absolute, centroid and relative moments of a multi-particle system

nth-order absolute moments

$$P_1 = \frac{1}{M}\sum_{N=1}^{M} p_{N1} = \frac{1}{M}\sum_{N=1}^{M} p_{G1} + \underbrace{\frac{1}{M}\sum_{N=1}^{M} p_{Ng1}}_{\text{zero}} = p_{G1}$$

$$P_2 = \frac{1}{M}\sum_{N=1}^{M} p_{N2} = \frac{1}{M}\sum_{N=1}^{M} p_{G2} + \underbrace{\frac{1}{M}\sum_{N=1}^{M} 2b_G b_{Ng}\cos(\beta_G+\beta_{Ng})}_{\text{zero}} + \frac{1}{M}\sum_{N=1}^{M} p_{Ng2} = p_{G2} + P_{g2}$$

$$P_3 = \frac{1}{M}\sum_{N=1}^{M} p_{N3} = \frac{1}{M}\sum_{N=1}^{M} p_{G3} + \underbrace{\frac{1}{M}\sum_{N=1}^{M} 3b_G^2 b_{Ng}\cos(2\beta_G+\beta_{Ng})}_{\text{zero}}$$

$$+ \frac{1}{M}\sum_{N=1}^{M} 3b_G b_{Ng}^2\cos(\beta_G+2\beta_{Ng}) + \frac{1}{M}\sum_{N=1}^{M} p_{Ng3}$$

$$= p_{G3} + \frac{1}{M}\sum_{N=1}^{M} 3b_G b_{Ng}^2\cos(\beta_G+2\beta_{Ng}) + P_{g3}$$

How do we treat the cross term?

$$Note: \frac{1}{M}\sum_{N=1}^{M} p_{Gn} = p_{Gn}, \qquad \frac{1}{M}\sum_{N=1}^{M} q_{Gn} = q_{Gn}$$

nth-order centroid moments

$$\frac{1}{M}\sum_{N=1}^{M} p_{Ng1} = \frac{1}{M}\sum_{N=1}^{M} b_{Ng}\cos\beta_{Ng} = 0, \qquad \frac{1}{M}\sum_{N=1}^{M} q_{Ng1} = \frac{1}{M}\sum_{N=1}^{M} b_{Ng}\sin\beta_{Ng} = 0$$

zero zero

$$P_{gn} = \frac{1}{M}\sum_{N=1}^{M} p_{Ngn}, \qquad Q_{gn} = \frac{1}{M}\sum_{N=1}^{M} q_{Ngn}$$

nth-order relative moments

Definition of size a_{gn} and argument α_{gn} of nth-order relative moment

Definition $a_{gn} = \sqrt[2n]{P_{gn}^2 + Q_{gn}^2}$ [m], $\alpha_{gn} = \frac{1}{n}\cos^{-1}\frac{P_{gn}}{a_{gn}^n}$ && $\alpha_{gn} = \frac{1}{n}\sin^{-1}\frac{Q_{gn}}{a_{gn}^n}$ $(0 \le \alpha_{gn} < \frac{2\pi}{n})$

$$P_1 = p_{G1}, \quad Q_1 = q_{G1},$$

$$P_2 = p_{G2} + P_{g2}, \quad Q_2 = q_{G2} + Q_{g2},$$

$$P_3 = p_{G3} + 3b_G a_{g2}^2 \cos(\beta_G + 2\alpha_{g2}) + P_{g3}, \quad Q_3 = q_{G3} + 3b_G a_{g2}^2 \sin(\beta_G + 2\alpha_{g2}) + Q_{g3},$$

$$P_4 = p_{G4} + 6b_G^2 a_{g2}^2 \cos(2\beta_G + 2\alpha_{g2}) + 4b_G a_{g3}^3 \cos(\beta_G + 3\alpha_{g3}) + P_{g4},$$

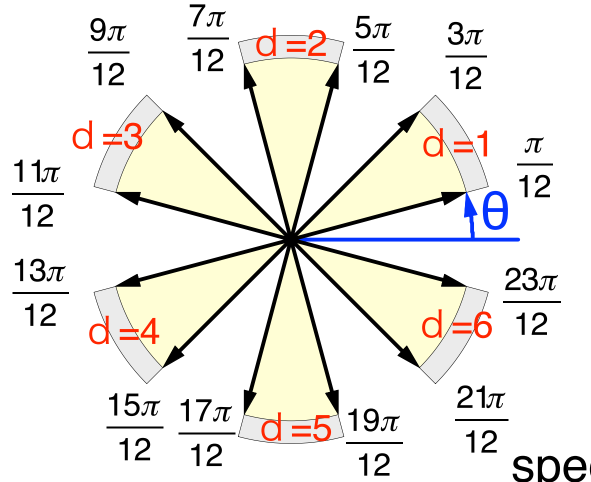
$$Q_4 = q_{G4} + 6b_G^2 a_{g2}^2 \sin(2\beta_G + 2\alpha_{g2}) + 4b_G a_{g3}^3 \sin(\beta_G + 3\alpha_{g3}) + Q_{g4},$$

When a beam is shifted by steering magnet, if we suppose a beam position is changed, but a beam shape (distribution) is not changed, following corollary is obtained.

* Absolute moments, P_n, Q_n, vary.
* Centroid moments, p_{Gn}, q_{Gn}, vary.
* Relative moments, P_{gn}, Q_{gn}, do not vary.
* If relative moments vary, something (formulation, calculation, coefficient, measurement, value of moment, correction or etc.) must be invalid.

Contribution of moment to BPM output and effective aperture radius

Electrode configuration



$$E_r(R,\theta) = \frac{\Lambda}{2\pi R\varepsilon_0}\left(1 + 2\sum_{n=1}^{\infty}\frac{P_n\cos n\theta + Q_n\sin n\theta}{R^n}\right)$$

Output voltage from electrode V_d is proportional to the integral of the surface electric field $E_r(R, \theta)$.

$$V_d \propto \frac{\pi}{12} + \sum_{n=1}^{\infty}\left(P_n\frac{\int_{\frac{(4d-3)\pi}{12}}^{\frac{(4d-1)\pi}{12}}\cos n\theta\, d\theta}{R^n} + Q_n\frac{\int_{\frac{(4d-3)\pi}{12}}^{\frac{(4d-1)\pi}{12}}\sin n\theta\, d\theta}{R^n}\right) = \frac{\pi}{12} + \sum_{n=1}^{\infty}\left(\frac{c_{dn}}{R^n}P_n + \frac{s_{dn}}{R^n}Q_n\right)$$

Geometrical factor

where

$$c_{dn} = \int_{\frac{(4d-3)\pi}{12}}^{\frac{(4d-1)\pi}{12}}\cos n\theta\, d\theta, \quad s_{dn} = \int_{\frac{(4d-3)\pi}{12}}^{\frac{(4d-1)\pi}{12}}\sin n\theta\, d\theta$$

$$c_{11} = c_{61} = f_1,\ c_{31} = c_{41} = -f_1,\ c_{21} = c_{51} = 0$$

$$s_{11} = s_{31} = h_1,\ s_{41} = s_{61} = -h_1,\ s_{21} = 2h_1,\ s_{51} = -2h_1$$

$$c_{12} = c_{32} = c_{42} = c_{62} = f_2,\ c_{22} = c_{52} = -2f_2$$

$$s_{12} = s_{42} = h_2,\ s_{32} = s_{62} = -h_2,\ s_{22} = s_{52} = 0$$

$$c_{13} = c_{23} = c_{33} = c_{43} = c_{53} = c_{63} = 0$$

$$s_{13} = s_{33} = s_{53} = h_3,\ s_{23} = s_{43} = s_{63} = -h_3$$

$$\vdots$$

specific expression

$$V_1 \propto \frac{\pi}{12} + \frac{f_1}{R}P_1 + \frac{h_1}{R}Q_1 + \frac{f_2}{R^2}P_2 + \frac{h_2}{R^2}Q_2 + \frac{0}{R^3}P_3 + \frac{h_3}{R^3}Q_3 - \cdots$$

$$V_2 \propto \frac{\pi}{12} + \frac{0}{R}P_1 + \frac{2h_1}{R}Q_1 - \frac{2f_2}{R^2}P_2 + \frac{0}{R^2}Q_2 + \frac{0}{R^3}P_3 - \frac{h_3}{R^3}Q_3 + \cdots$$

$$V_3 \propto \frac{\pi}{12} - \frac{f_1}{R}P_1 + \frac{h_1}{R}Q_1 + \frac{f_2}{R^2}P_2 - \frac{h_2}{R^2}Q_2 + \frac{0}{R^3}P_3 + \frac{h_3}{R^3}Q_3 - \cdots$$

$$V_4 \propto \frac{\pi}{12} - \frac{f_1}{R}P_1 - \frac{h_1}{R}Q_1 + \frac{f_2}{R^2}P_2 + \frac{h_2}{R^2}Q_2 + \frac{0}{R^3}P_3 - \frac{h_3}{R^3}Q_3 - \cdots$$

$$V_5 \propto \frac{\pi}{12} + \frac{0}{R}P_1 - \frac{2h_1}{R}Q_1 - \frac{2f_2}{R^2}P_2 + \frac{0}{R^2}Q_2 + \frac{0}{R^3}P_3 + \frac{h_3}{R^3}Q_3 + \cdots$$

$$V_6 \propto \frac{\pi}{12} + \frac{f_1}{R}P_1 - \frac{h_1}{R}Q_1 + \frac{f_2}{R^2}P_2 - \frac{h_2}{R^2}Q_2 + \frac{0}{R^3}P_3 - \frac{h_3}{R^3}Q_3 - \cdots$$

Contribution of moment to BPM output and effective aperture radius

We carefully take difference over sum of V_ds.

$$C_1 = \frac{V_1 - V_3 - V_4 + V_6}{V_1 + V_3 + V_4 + V_6} = \frac{\frac{12f_1}{\pi R}P_1 - \frac{12f_5}{\pi R^5}P_5 + \cdots}{1 + \frac{12f_2}{\pi R^2}P_2 - \frac{12f_4}{\pi R^4}P_4 + \cdots} = \frac{\frac{2P_1}{R_{C1P1u}} - \frac{2P_5}{R^5_{C1P5u}} + \cdots}{1 + \frac{2P_2}{R^2_{C1P2d}} - \frac{2P_4}{R^4_{C1P4d}} + \cdots}$$

$$S_2 = \frac{V_1 - V_3 + V_4 - V_6}{V_1 + V_3 + V_4 + V_6} = \frac{\frac{12h_2}{\pi R^2}Q_2 + \frac{12h_4}{\pi R^4}Q_4 + \cdots}{1 + \frac{12f_2}{\pi R^2}P_2 - \frac{12f_4}{\pi R^4}P_4 + \cdots} = \frac{\frac{2Q_2}{R^2_{S2Q2u}} + \frac{2Q_4}{R^4_{S2Q4u}} + \cdots}{1 + \frac{2P_2}{R^2_{S2P2d}} - \frac{2P_4}{R^4_{S2P4d}} + \cdots}$$

$$R_{C1P1u} = \frac{\pi}{6f_1}R,\quad R_{C1P5u} = \sqrt[5]{\frac{\pi}{6f_5}}R,\quad R_{C1P2d} = \sqrt{\frac{\pi}{6f_2}}R,\quad R_{C1P4d} = \sqrt[4]{\frac{\pi}{6f_4}}R,$$

$$R_{S2Q2u} = \sqrt{\frac{\pi}{6h_2}}R,\quad R_{S2Q4u} = \sqrt[4]{\frac{\pi}{6h_4}}R,\quad R_{S2P2d} = \sqrt{\frac{\pi}{6f_2}}R,\quad R_{S2P4d} = \sqrt[4]{\frac{\pi}{6f_4}}R,$$

Above R.....s are named effective aperture radii (R ≦ R.....).
Smaller apature radius corresponds larger contribution.
The smallest order term, ◯ and ◯, is dominant.
Difference over sum, ◯, is a measurand.

Values of effective aperture radii [mm]

R_{C1P1u}	18.69	R_{S2Q2u}	17.59
R_{C1P5u}	17.50	R_{S2Q4u}	17.39
R_{C1P2d}	23.16	R_{S2P2d}	23.16
R_{C1P4d}	19.95	R_{S2P4d}	19.95

Contribution of moment to BPM output and effective aperture radius

Consequently the smallest order moment is expressed as,

$$P_1 = C_1\frac{R_{C1P1u}}{2} + C_1\left(\frac{R_{C1P1u}}{R^2_{C1P2d}}P_2 - \frac{R_{C1P1u}}{R^4_{C1P4d}}P_4 + \cdots\right) + \frac{R_{C1P1u}}{R^5_{C1P5u}}P_5 + \cdots \quad [m], \qquad C_1 = \frac{V_1 - V_3 - V_4 + V_6}{V_1 + V_3 + V_4 + V_6}$$

$$Q_1 = S_1\frac{R_{S1Q1u}}{2} + S_1\left(\frac{R_{S1Q1u}}{R^2_{S1P2d}}P_2 - \frac{R_{S1Q1u}}{R^4_{S1P4d}}P_4 + \cdots\right) - \frac{R_{S1Q1u}}{R^3_{S1Q3u}}Q_3 - \frac{R_{S1Q1u}}{R^5_{S1Q5u}}Q_5 + \cdots \quad [m], \qquad S_1 = \frac{V_1 + V_3 - V_4 - V_6}{V_1 + V_3 + V_4 + V_6}$$

$$P_2 = C_2\frac{R^2_{C2P2u}}{2} + C_2\left(-\frac{R^2_{C2P2u}}{R^2_{C2P2d}}P_2 + \frac{R^2_{C2P2u}}{R^4_{C2P4d}}P_4 + \cdots\right) + \frac{R^2_{C2P2u}}{R^4_{C2P4u}}P_4 + \cdots \quad [m^2], \qquad C_2 = \frac{V_1 + V_3 + V_4 + V_6 - 2(V_2 + V_5)}{V_1 + V_3 + V_4 + V_6 + 2(V_2 + V_5)}$$

$$Q_2 = S_2\frac{R^2_{S2Q2u}}{2} + S_2\left(\frac{R^2_{S2Q2u}}{R^2_{S2P2d}}P_2 - \frac{R^2_{S2Q2u}}{R^4_{S2P4d}}P_4 + \cdots\right) - \frac{R^2_{S2Q2u}}{R^4_{S2Q4u}}Q_4 + \cdots \quad [m^2], \qquad S_2 = \frac{V_1 - V_3 + V_4 - V_6}{V_1 + V_3 + V_4 + V_6}$$

$$Q_3 = S_3\frac{R^3_{S3Q3u}}{2} + \cdots \quad [m^3], \qquad S_3 = \frac{V_1 - V_2 + V_3 - V_4 + V_5 - V_6}{V_1 + V_2 + V_3 + V_4 + V_5 + V_6}$$

: Dominant term

: Correction terms

: Measurand

These expressions using a combination of P_1, Q_1, P_2, Q_2, Q_3 (absolute moments) are simple.
However we actually need relative moments, these formuale are modified so as to be expressed using a combination of P_1, Q_1, P_{g2}, Q_{g2}, Q_{g3} (relative moments).

From now we confine calculations up to 5th-order.

Contribution of moment to BPM output and effective aperture radius

Expressions of **absolute moments** with correction terms which are described by **relative moments**

$$P_1 = C_1\frac{R_{C1P1u}}{2} + C_1\frac{R_{C1P1u}}{R_{C1P2d}^2}\left\{\left(P_1^2 - Q_1^2\right) + P_{g2}\right\} + C_1\frac{R_{C1P1u}}{R_{C1P4d}^4}\left\{\left(P_1^4 - 6P_1^2Q_1^2 + Q_1^4\right) + 6\left(P_1^2 - Q_1^2\right)P_{g2} - 12P_1Q_1Q_{g2} + 4P_1P_{g3} - 4Q_1Q_{g3} + P_{g4}\right\}$$

$$+\frac{R_{C1P1u}}{R_{C1P5u}^5}\left\{\left(P_1^5 - 10P_1^3Q_1^2 + 5P_1Q_1^4\right) + 10\left(P_1^3 - 3P_1Q_1^2\right)P_{g2} - 10\left(3P_1^2Q_1 - Q_1^3\right)Q_{g2} + 10\left(P_1^2 - Q_1^2\right)P_{g3} - 20P_1Q_1Q_{g3} + 5P_1P_{g4} - 5Q_1Q_{g4} + P_{g5}\right\},$$

$$Q_1 = S_1\frac{R_{S1Q1u}}{2} + S_1\frac{R_{S1Q1u}}{R_{S1P2d}^2}\left\{\left(P_1^2 - Q_1^2\right) + P_{g2}\right\} - S_1\frac{R_{S1Q1u}}{R_{S1P4d}^4}\left\{\left(P_1^4 - 6P_1^2Q_1^2 + Q_1^4\right) + 6\left(P_1^2 - Q_1^2\right)P_{g2} - 12P_1Q_1Q_{g2} + 4P_1P_{g3} - 4Q_1Q_{g3} + P_{g4}\right\}$$

$$-\frac{R_{S1Q1u}}{R_{S1Q3u}^3}\left\{\left(3P_1^2Q_1 - Q_1^3\right) + 3Q_1P_{g2} + 3P_1Q_{g2} + Q_{g3}\right\}$$

$$-\frac{R_{S1Q1u}}{R_{S1Q5u}^5}\left\{\left(5P_1^4Q_1 - 10P_1^2Q_1^3 + Q_1^5\right) + 10\left(3P_1^2Q_1 - Q_1^3\right)P_{g2} + 10\left(P_1^3 - 3P_1Q_1^2\right)Q_{g2} + 20P_1Q_1P_{g3} + 10\left(P_1^2 - Q_1^2\right)Q_{g3} + 5Q_1P_{g4} + 5P_1Q_{g4} + Q_{g5}\right\};$$

The terms colored with [pink box] are regarded as ZERO, becouse of unmeasurable relateve moments.

Contribution of moment to BPM output and effective aperture radius

$$P_2 = C_2\frac{R_{C2P2u}^2}{2} - C_2\frac{R_{C2P2u}^2}{R_{C2P2d}^2}\left\{\left(P_1^2 - Q_1^2\right) + P_{g2}\right\}$$

$$+ C_2\frac{R_{C2P2u}^2}{R_{C2P4d}^4}\left\{\left(P_1^4 - 6P_1^2Q_1^2 + Q_1^4\right) + 6\left(P_1^2 - Q_1^2\right)P_{g2} - 12P_1Q_1Q_{g2} + 4P_1P_{g3} - 4Q_1Q_{g3} + P_{g4}\right\}$$

$$+ \frac{R_{C2P2u}^2}{R_{C2P4u}^4}\left\{\left(P_1^4 - 6P_1^2Q_1^2 + Q_1^4\right) + 6\left(P_1^2 - Q_1^2\right)P_{g2} - 12P_1Q_1Q_{g2} + 4P_1P_{g3} - 4Q_1Q_{g3} + P_{g4}\right\}$$

$$Q_2 = S_2\frac{R_{S2Q2u}^2}{2} + S_2\frac{R_{S2Q2u}^2}{R_{S2P2d}^2}\left\{\left(P_1^2 - Q_1^2\right) + P_{g2}\right\}$$

$$- S_2\frac{R_{S2Q2u}^2}{R_{S2P4d}^4}\left\{\left(P_1^4 - 6P_1^2Q_1^2 + Q_1^4\right) + 6\left(P_1^2 - Q_1^2\right)P_{g2} - 12P_1Q_1Q_{g2} + 4P_1P_{g3} - 4Q_1Q_{g3} + P_{g4}\right\}$$

$$- \frac{R_{S2Q2u}^2}{R_{S2Q4u}^4}\left\{\left(4P_1^3Q_1 - 4P_1Q_1^3\right) + 12P_1Q_1P_{g2} + 6\left(P_1^2 - Q_1^2\right)Q_{g2} + 4Q_1P_{g3} + 4P_1Q_{g3} + Q_{g4}\right\}$$

$$Q_3 = S_3\frac{R_{S3Q3u}^3}{2}$$

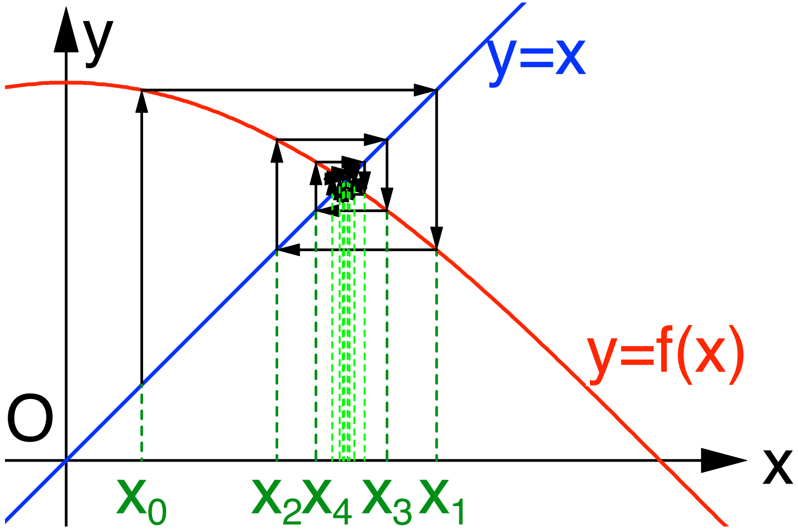
The terms colored with ▭ are regarded as ZERO.

Recursive method for calculations of moments (successive iteration)

How do we calculate the absolute (or relative) moments?

For single variable, e.g. x, suppose an equation can be written as x = F(x).
If F(x) is continuous, smooth, monotone and |dF(x)/dx|<1 in the typical interest region, a successive iteration method can be employed. That is for proper initial x_0 and iteration number i, solution of x can be obtained as the limit at infinity $x = \lim_{i \to \infty} x_i$.

$$x_1 = F(x_0),\ x_2 = F(x_1),\ x_3 = F(x_2),\ x_4 = F(x_3),\ x_5 = F(x_4),\ x_6 = F(x_5),\ \cdots$$



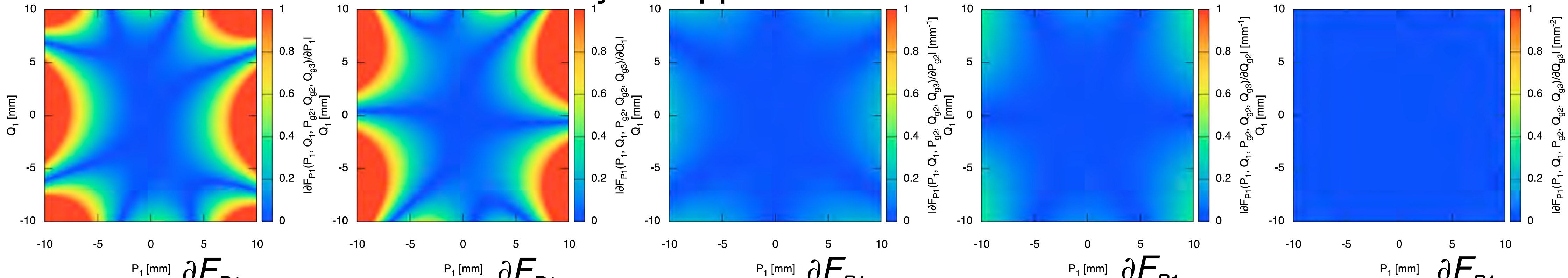
A well-known example of x = F(x) = cos x.

Recursive method for calculations of moments (successive iteration)

In the case of the multiple variable we solve following equations.

$$\begin{cases} P_1 = F_{P1}(P_1, Q_1, P_{g2}, Q_{g2}, Q_{g3}) \\ Q_1 = F_{Q1}(P_1, Q_1, P_{g2}, Q_{g2}, Q_{g3}) \\ P_2 = F_{P2}(P_1, Q_1, P_{g2}, Q_{g2}, Q_{g3}) \\ Q_2 = F_{Q2}(P_1, Q_1, P_{g2}, Q_{g2}, Q_{g3}) \\ Q_3 = F_{Q3}(P_1, Q_1, P_{g2}, Q_{g2}, Q_{g3}) \end{cases}$$

If all function $F_{P1}(P_1, \cdots)$, $F_{Q1}(P_1, \cdots)$, ••• are continuous, smooth, monotone and, all absolute values of partial differential $|\partial F_{P1}/ \partial P_1|$, $|\partial F_{P1}/ \partial Q_1|$, ••• are smaller than 1, the successive iteration method may be applicable.



$\frac{\partial F_{P1}}{\partial P_1}$ $\frac{\partial F_{P1}}{\partial Q_1}$ $\frac{\partial F_{P1}}{\partial P_{g2}}$ $\frac{\partial F_{P1}}{\partial Q_{g2}}$ $\frac{\partial F_{P1}}{\partial Q_{g3}}$

when $P_{g2} = 5\ [mm^2],\ Q_{g2} = 5\ [mm^2],\ Q_{g3} = 10\ [mm^3]$

Recursive method for calculations of moments (successive iteration)

Iteration scheme

• First we calculate P_1, Q_1, P_2, Q_2, Q_3 using no correction term as the initial values of P_{10}, Q_{10}, P_{20} (P_{g20}), Q_{20} (Q_{g20}), Q_{30} (Q_{g30}).

$$P_{10} = C_1 \frac{R_{C1P1u}}{2}$$

$$Q_{10} = S_1 \frac{R_{S1Q1u}}{2}$$

$$P_{20} = C_2 \frac{R^2_{C2P2u}}{2}$$

$$Q_{20} = S_2 \frac{R^2_{S2Q2u}}{2}$$

$$Q_{30} = S_3 \frac{R^3_{S3Q3u}}{2}$$

P_{10}, Q_{10}, P_{g20}, Q_{g20}, Q_{g30}

i = 0

$$P_{11} = F_{P1}\left(P_{10}, Q_{10}, P_{g20}, Q_{g20}, Q_{g30}\right)$$

$$Q_{11} = F_{Q1}\left(P_{10}, Q_{10}, P_{g20}, Q_{g20}, Q_{g30}\right)$$

$$P_{21} = F_{P2}\left(P_{10}, Q_{10}, P_{g20}, Q_{g20}, Q_{g30}\right)$$

$$Q_{21} = F_{Q2}\left(P_{10}, Q_{10}, P_{g20}, Q_{g20}, Q_{g30}\right)$$

$$Q_{31} = F_{Q3}\left(P_{10}, Q_{10}, P_{g20}, Q_{g20}, Q_{g30}\right)$$

P_{11}, Q_{11}, P_{g21}, Q_{g21}, Q_{g31}

i = 1

$$P_{12} = F_{P1}\left(P_{11}, Q_{11}, P_{g21}, Q_{g21}, Q_{g31}\right)$$

$$Q_{12} = F_{Q1}\left(P_{11}, Q_{11}, P_{g21}, Q_{g21}, Q_{g31}\right)$$

$$P_{22} = F_{P2}\left(P_{11}, Q_{11}, P_{g21}, Q_{g21}, Q_{g31}\right)$$

$$Q_{22} = F_{Q2}\left(P_{11}, Q_{11}, P_{g21}, Q_{g21}, Q_{g31}\right)$$

$$Q_{32} = F_{Q3}\left(P_{11}, Q_{11}, P_{g21}, Q_{g21}, Q_{g31}\right)$$

P_{12}, Q_{12}, P_{g22}, Q_{g22}, Q_{g32}

i = 2

$$P_{13} = F_{P1}\left(P_{12}, Q_{12}, P_{g22}, \ldots\right.$$

$$Q_{13} = F_{Q1}\left(P_{12}, Q_{12}, P_{g22}, \ldots\right.$$

$$P_{23} = F_{P2}\left(P_{12}, Q_{12}, P_{g22}, \ldots\right.$$

$$Q_{23} = F_{Q2}\left(P_{12}, Q_{12}, P_{g22} \ldots\right.$$

$$Q_{33} = F_{Q3}\left(P_{12}, Q_{12}, P_{g22} \ldots\right.$$

i = 3

•••

$$P_{g2i} = P_{2i} - (P_{1i}^2 - Q_{1i}^2)$$

$$Q_{g2i} = Q_{2i} - 2P_{1i}Q_{1i}$$

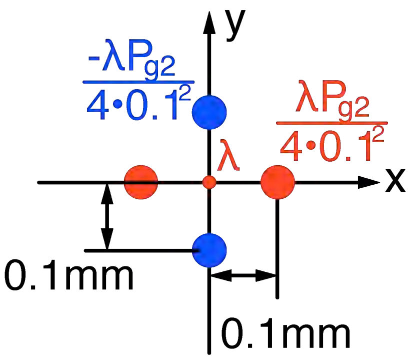
$$Q_{g3i} = Q_{3i} - 3P_{1i}^2 Q_{1i} + Q_{1i}^3 - 3Q_{1i}P_{g2i} - 3P_{1i}Q_{g2i}$$

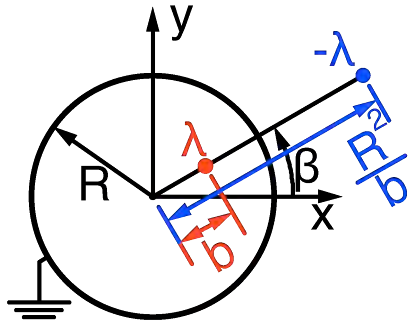
Definition of relative moments

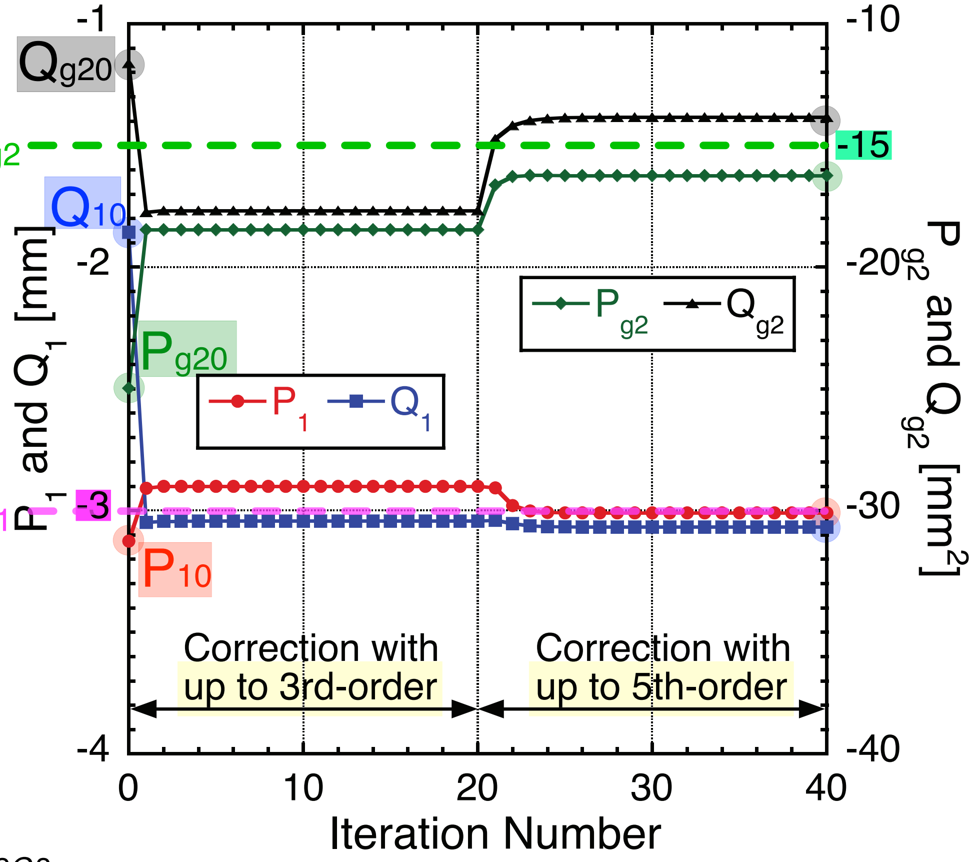
Recursive method for calculations of moments (successive iteration)

Simulation of successive iteration (convergence)

To simulate relative moment we employed electric quadrupoles and sextuples.







To calculate electric field on the electrodes we applied the method of mirror charge.

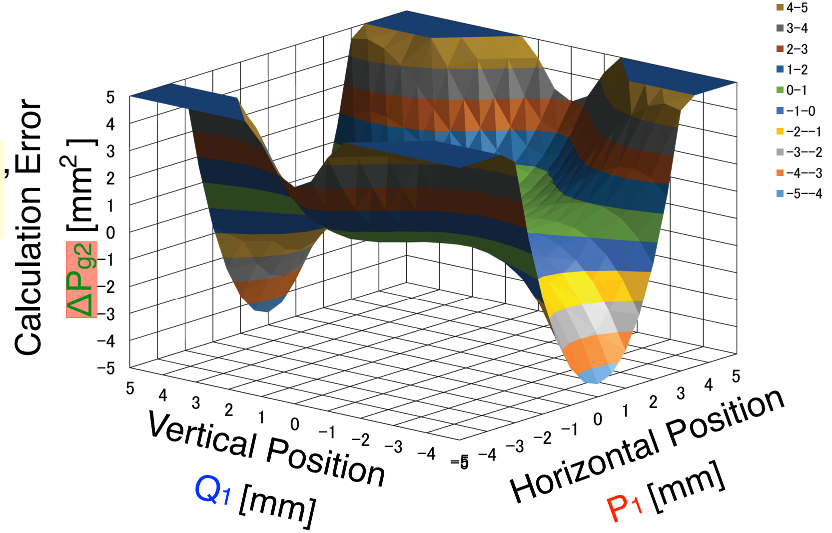
Set moment values : $P_1 = Q_1 =$ **-3** [mm],
$P_{g2} = Q_{g2} =$ **-15** [mm²], $P_{g3} = Q_{g3} = -30$ [mm³]
P_{g3} can not be measured in this system.
Q_{g3} was kept as **zero** because $|Q_{g3}|$ became large.

$$P_{10} = C_1 \frac{R_{C1P1u}}{2}, Q_{10} = S_1 \frac{R_{S1Q1u}}{2}, P_{20} = C_2 \frac{R^2_{C2P2u}}{2}, Q_{20} = S_2 \frac{R^2_{S2Q2u}}{2}$$

Recursive method for calculations of moments (successive iteration)

Comparison of calculation error between the cases of no correction, up to 3rd-order and up to 5th-order

No correction, i.e., P_{g20}

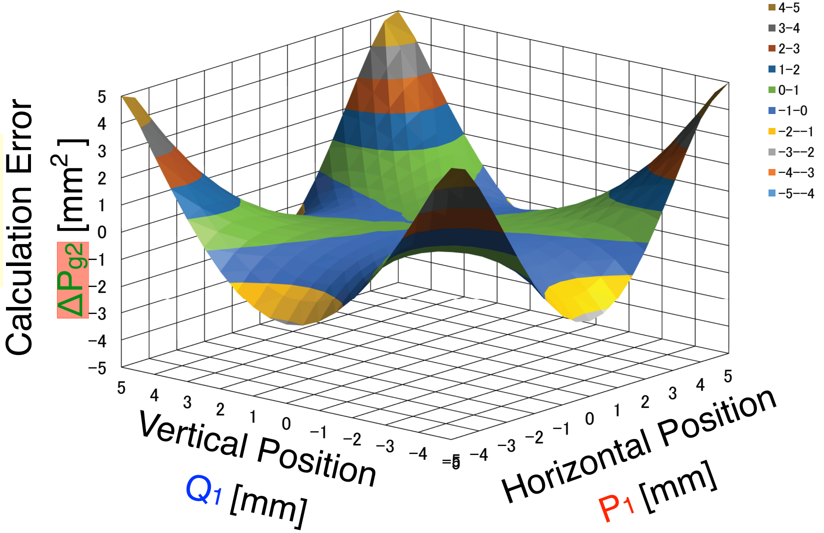


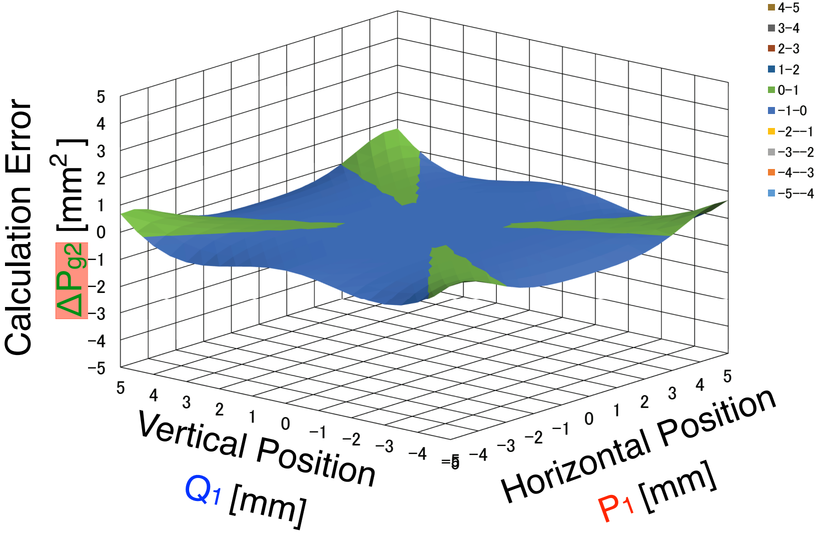
We supposed zero beam size, i.e., all relative momennts are zero, $P_{g2} = Q_{g2} = Q_{g3} = \cdots = 0$.

If attenuation values of electrode (or cable, •••) are not corret, the error distribution is asymmetrical.

-> Entire calibration

Correction up to 3rd-order

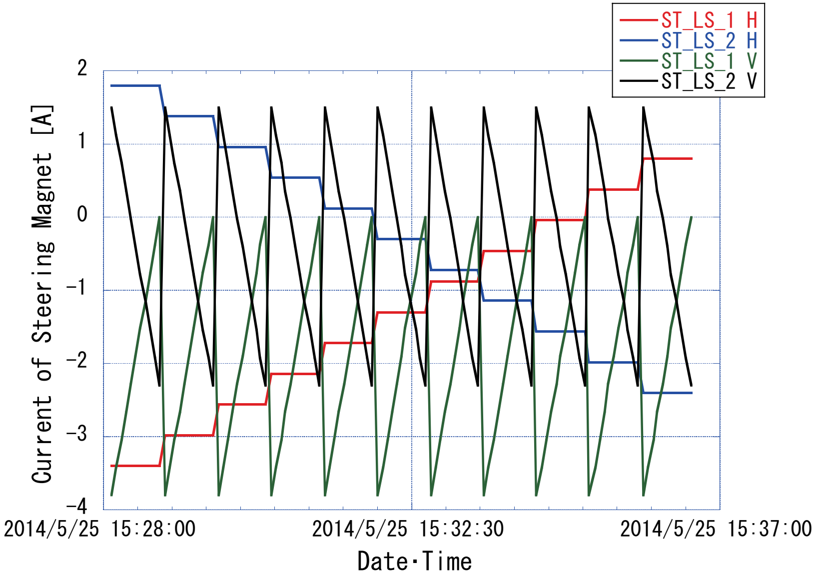


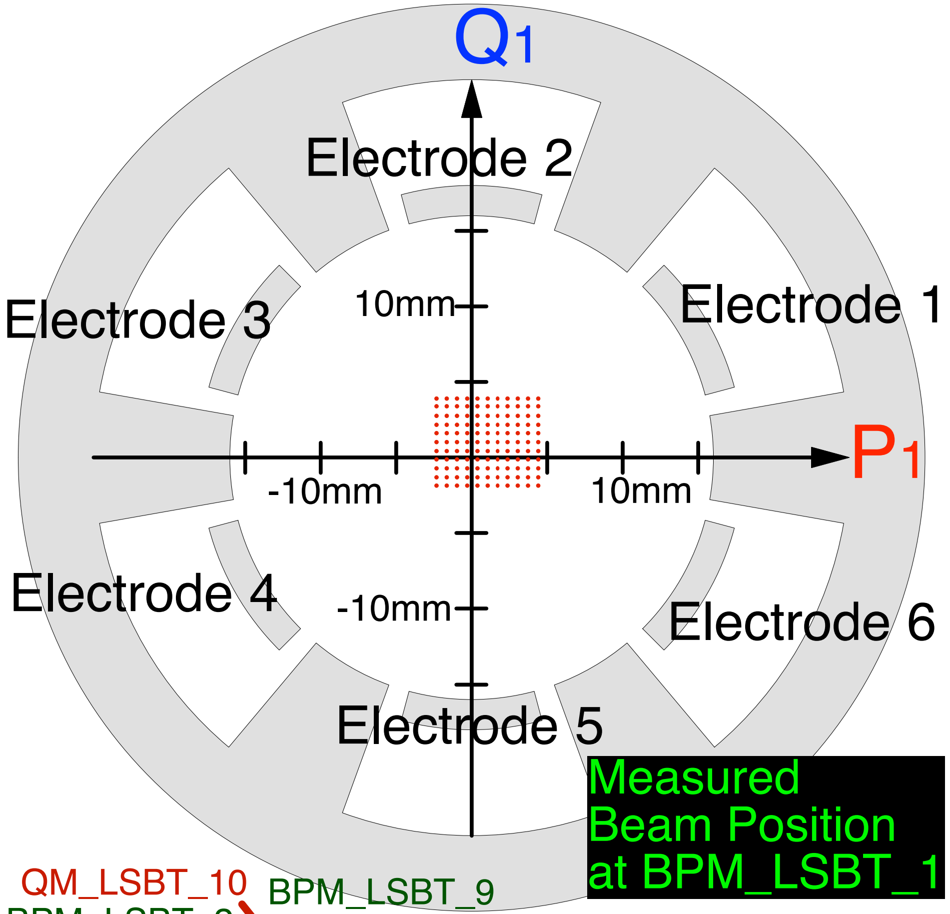


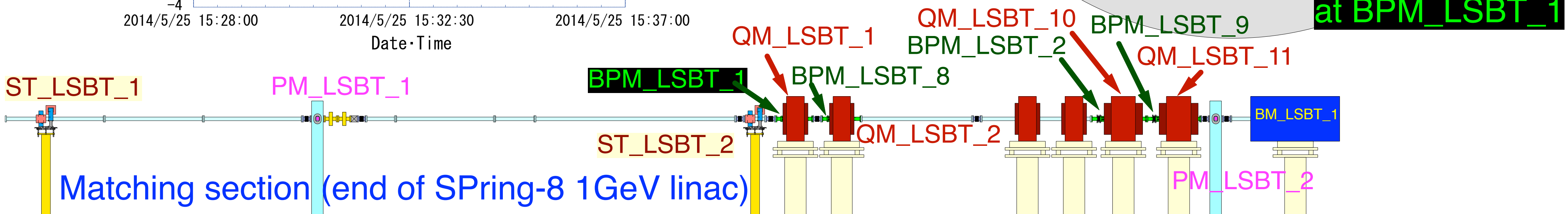
Correction up to 5th-order

Entire calibration (Determination of relative attenuation factor experimentally)

Electron beam is swept horizontally and vertically, then we determine relative attenuation factors of electrode channels so that relative moments are constant. $P_{g2} = \text{constant}, \; Q_{g2} = \text{constant}, \; Q_{g3} \equiv 0$

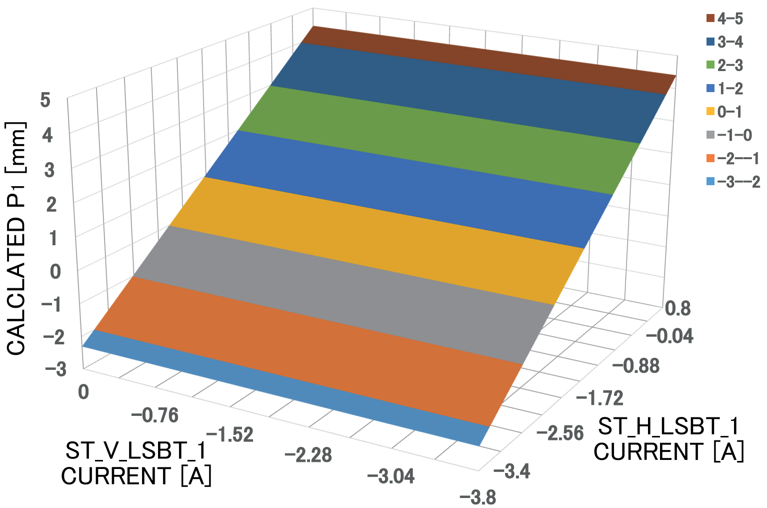


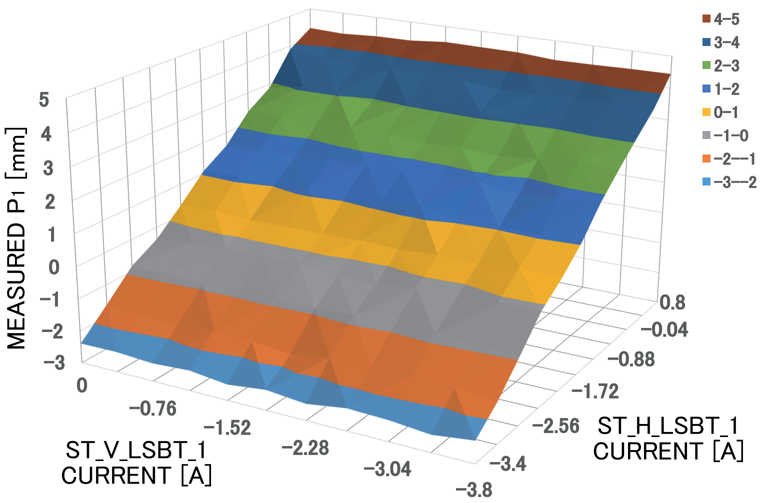




Entire calibration (Determination of relative attenuation factor experimentally)

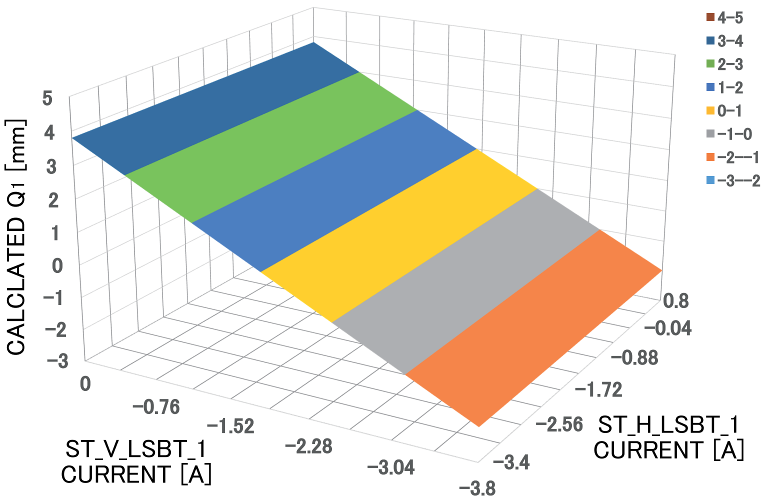
Simulated P_1

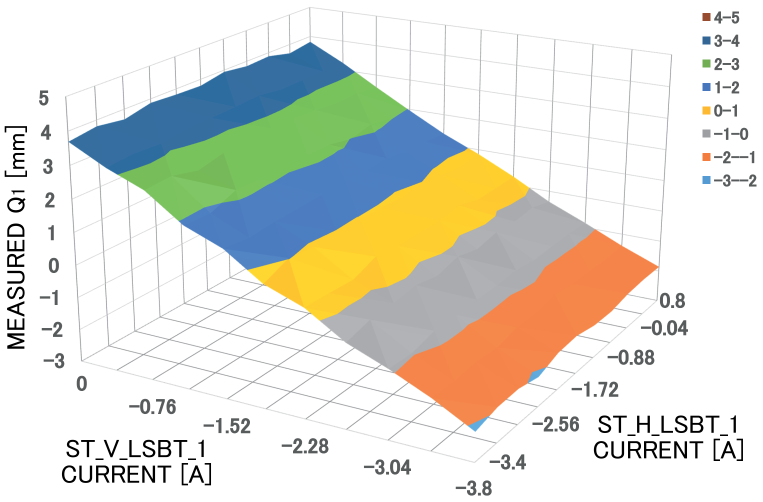




Measured P_1

Simulated Q_1



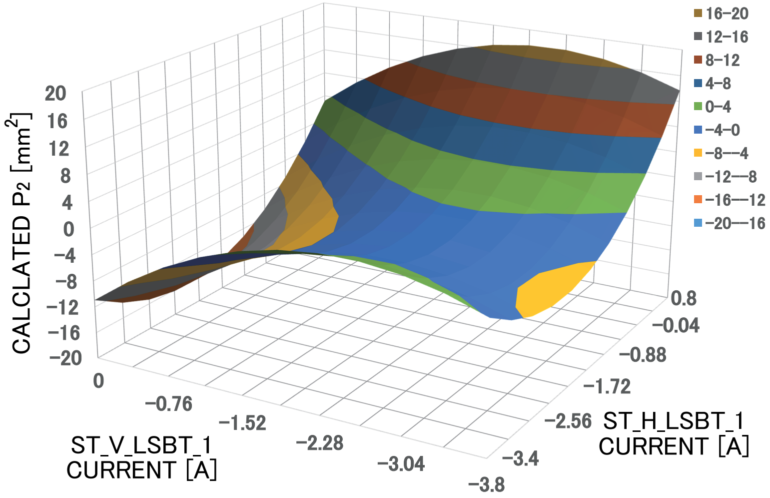


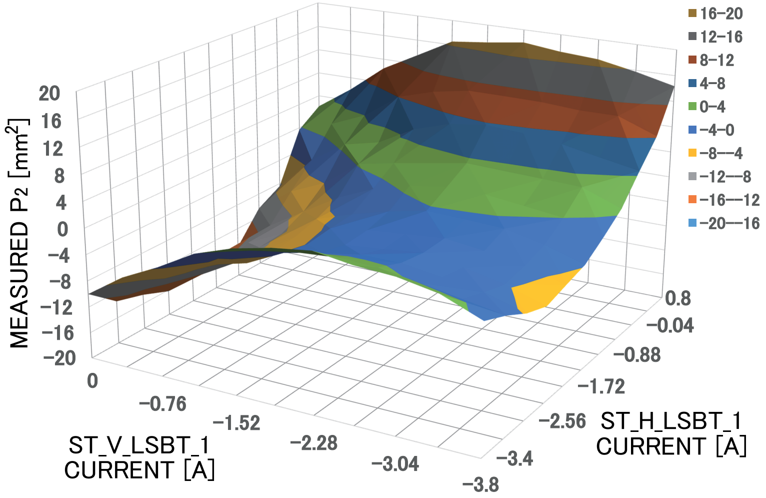
Measured Q_1

Comparison between simulation and experimant

Entire calibration (Determination of relative attenuation factor experimentally)

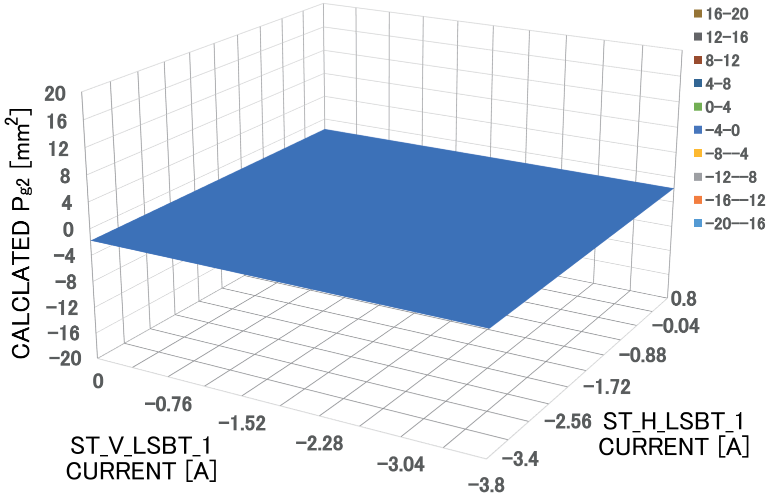
Simulated P_2

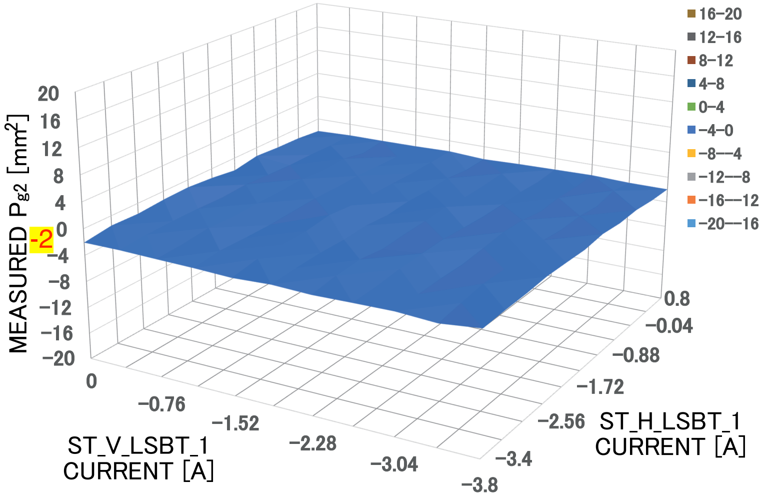




Measured P_2

Simulated P_{g2}



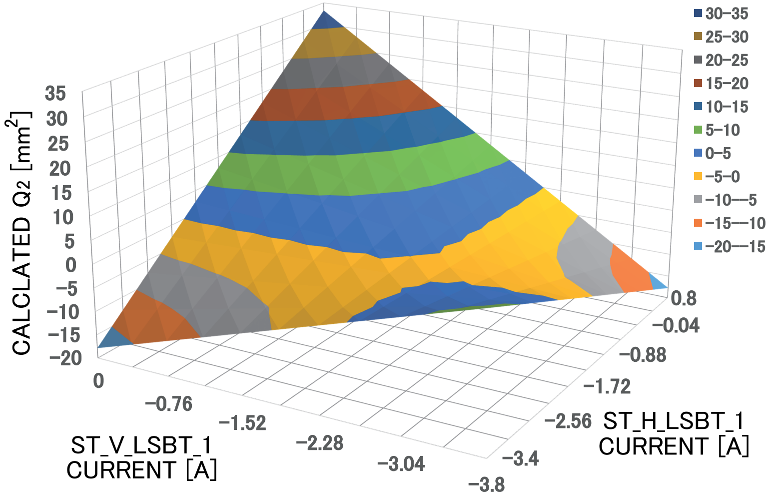


Measured P_{g2}

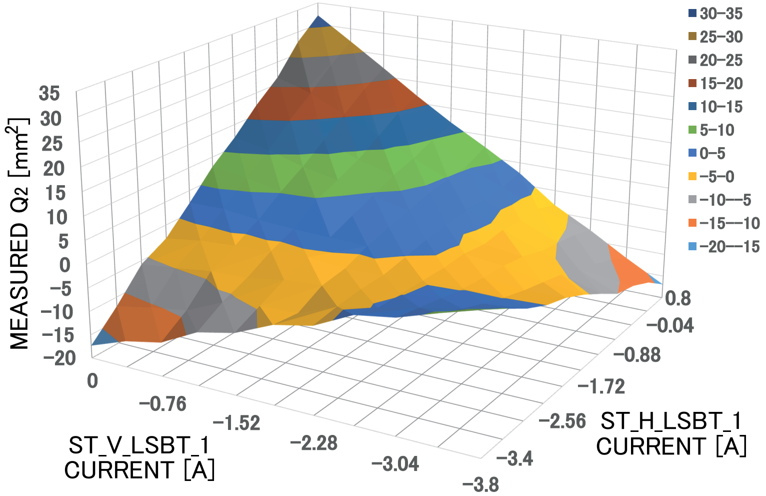
Comparison between simulation and experimant

Entire calibration (Determination of relative attenuation factor experimentally)

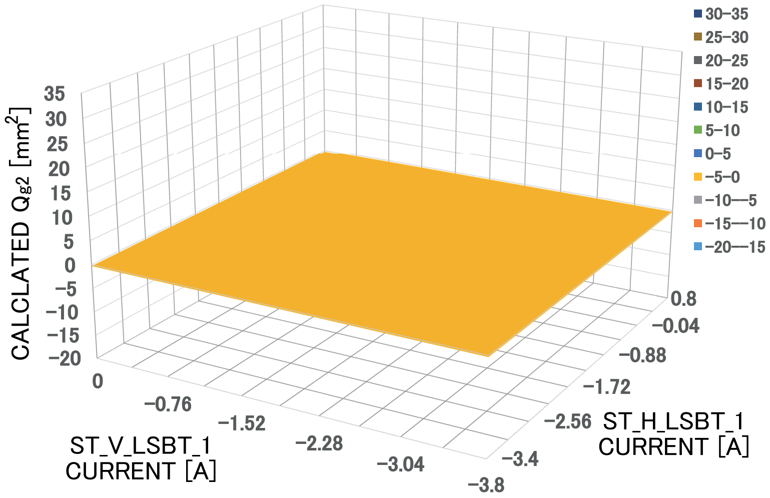
Simulated Q_2



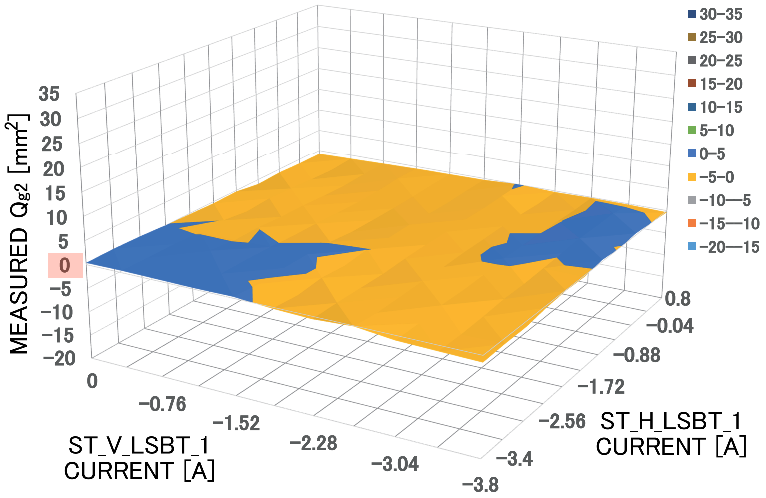
Measured Q_2



Simulated Q_{g2}



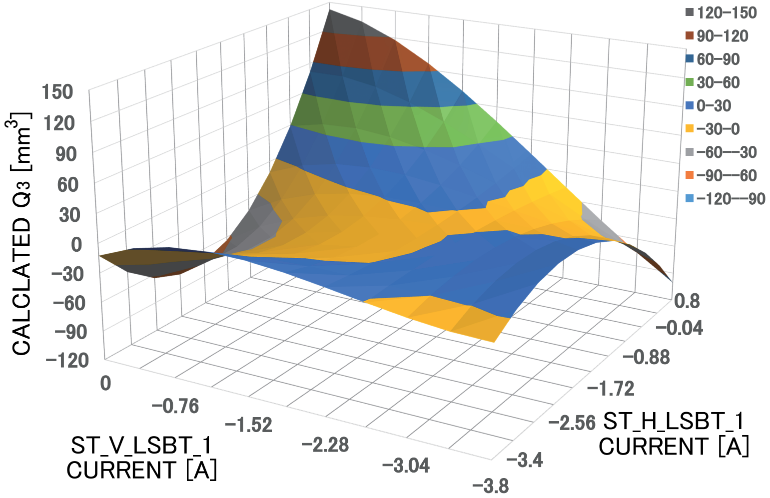
Measured Q_{g2}

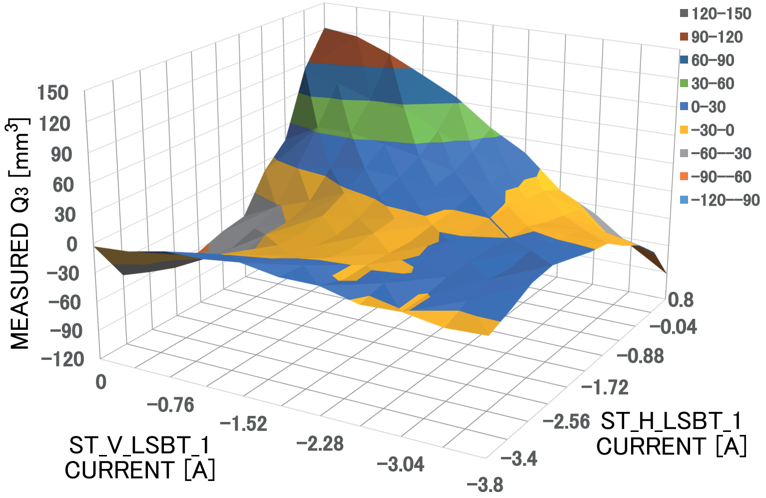


Comparison between simulation and experimant

Entire calibration (Determination of relative attenuation factor experimentally)

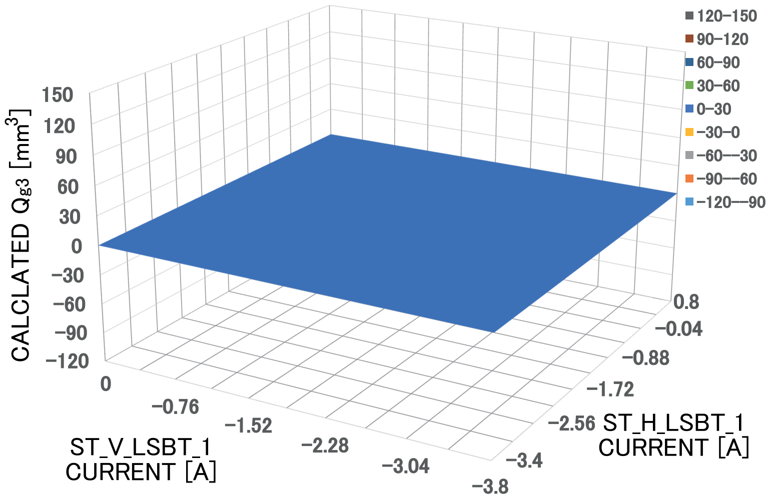
Simulated Q_3

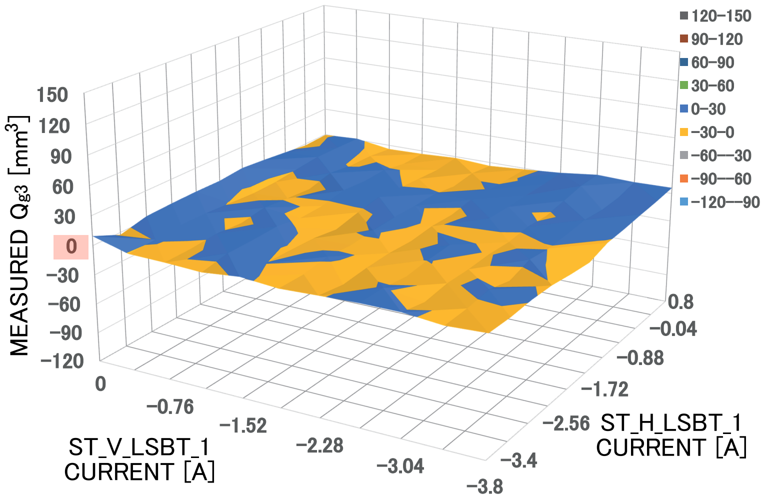




Measured Q_3

Simulated Q_{g3}



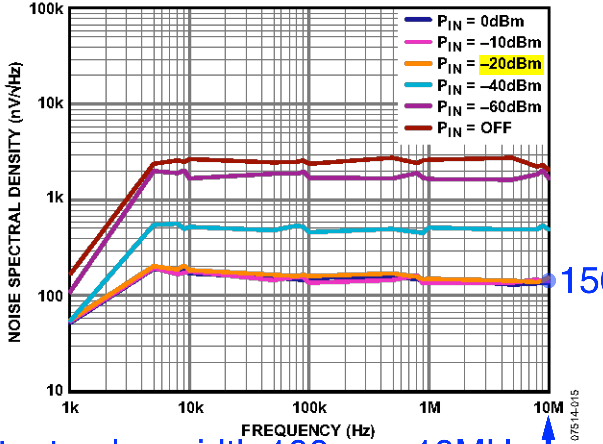


Measured Q_{g3}

Comparison between simulation and experimant

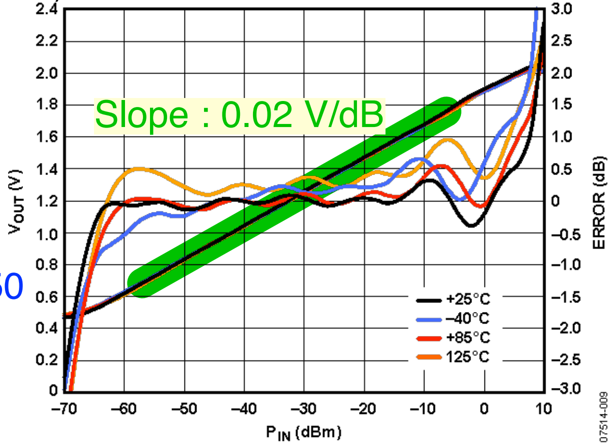
Measurement resolution

From the data sheet of ADL5513(Analog Devices)



150

Output pulse width 100ns ->10MHz



Bandwidth of ADC sampling time : 1MHz

Estimated noise level : $150\times10^{-9}\,[\mathrm{V}/\sqrt{\mathrm{Hz}}]\bullet 1\times10^{3}\,[\sqrt{\mathrm{Hz}}]\,/\,0.02\,[\mathrm{V/dB}]$ = 0.008dB -> 0.1% Error (1 Electrode)

$$P_1 \approx \frac{R_{C1P1u}}{2}C_1,\quad C_1 = \frac{V_1 - V_3 - V_4 + V_6}{V_1 + V_3 + V_4 + V_6}$$

$$Q_2 \approx \frac{R^2_{S2Q2u}}{2}S_2,\quad S_2 = \frac{V_1 - V_3 + V_4 - V_6}{V_1 + V_3 + V_4 + V_6}$$

$$\Delta P_1 \approx \frac{R_{C1P1u}}{2}\Delta C_1 = \frac{R_{C1P1u}}{2}\left(\frac{\partial C_1}{\partial V_1}\Delta V_1 + \frac{\partial C_1}{\partial V_3}\Delta V_3 + \frac{\partial C_1}{\partial V_4}\Delta V_4 + \frac{\partial C_1}{\partial V_6}\Delta V_6\right)$$

$$\Delta Q_2 \approx \frac{R^2_{S2Q2u}}{2}\Delta S_2 = \frac{R^2_{S2Q2u}}{2}\left(\frac{\partial S_2}{\partial V_1}\Delta V_1 + \frac{\partial S_2}{\partial V_3}\Delta V_3 + \frac{\partial S_2}{\partial V_4}\Delta V_4 + \frac{\partial S_2}{\partial V_6}\Delta V_6\right)$$

Measurement resolution

Suppose $V \approx V_1 \approx V_3 \approx V_4 \approx V_6, \Delta V \approx \Delta V_1 \approx \Delta V_3 \approx \Delta V_4 \approx \Delta V_6$.

Take $(\Delta P_1)^2$ or $(\Delta Q_2)^2$, then calculate deviation σ_{P1}^2 or σ_{Q2}^2.

Because $\frac{\partial C_1}{\partial V_i}\left(\text{or } \frac{\partial S_2}{\partial V_i}\right)$ and $\Delta V_j \quad (i \neq j)$ are uncorrelated statistically,

the cross term $\frac{\partial C_1}{\partial V_i}\Delta V_j$ or $\frac{\partial S_2}{\partial V_i}\Delta V_j$ $(i \neq j)$ is vanish.

Finally we obtain following relations;

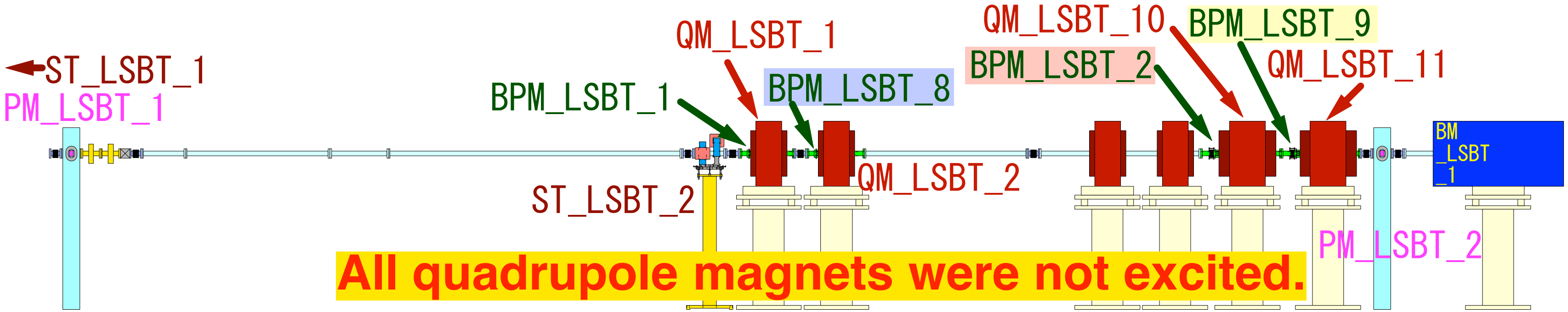
$$\sigma_{P1}^2 \approx \left(\frac{R_{C1P1u}}{4}\right)^2 \frac{\sigma_V^2}{V^2} \Rightarrow \sigma_{P1} \approx \frac{R_{C1P1u}}{4}\frac{\sigma_V}{V}, \quad \sigma_{Q2}^2 \approx \left(\frac{R_{S2Q2u}^2}{4}\right)^2 \frac{\sigma_V^2}{V^2} \Rightarrow \sigma_{Q2} \approx \frac{R_{S2Q2u}^2}{4}\frac{\sigma_V}{V}.$$

Because of $\frac{\sigma_V}{V} \approx 0.001, R_{C1P1u} = 18.69$ [mm], $R_{S2Q2u} = 17.59$ [mm],

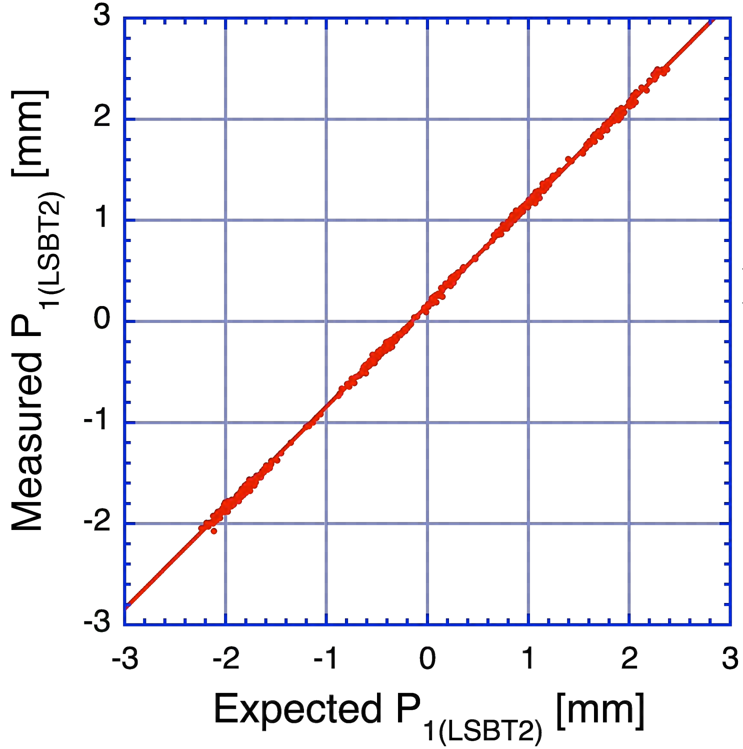
estimated resolutions $\sigma_{P1} \approx 0.005$ [mm] and $\sigma_{Q2} \approx 0.077$ [mm^2] are obtained.

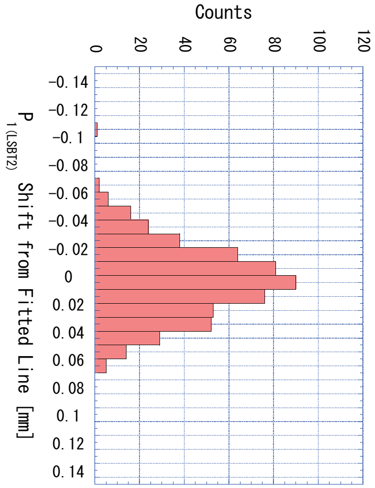
Measurement resolution

Resolution measurement by means of three BPM method in the drift space (on beam)



All quadrupole magnets were not excited.





$$\text{Expected } P_{1(LSBT2)} = \frac{\left(0.620 P_{1(LSBT8)} + 3.064 P_{1(LSBT9)}\right)}{\left(3.064 + 0.620\right)}$$

This standard deviation involves other BPM's.

A standard deviation of single BPM is

$$\sigma_{P1(LSBT2)} = 0.015 \text{ [mm]}.$$

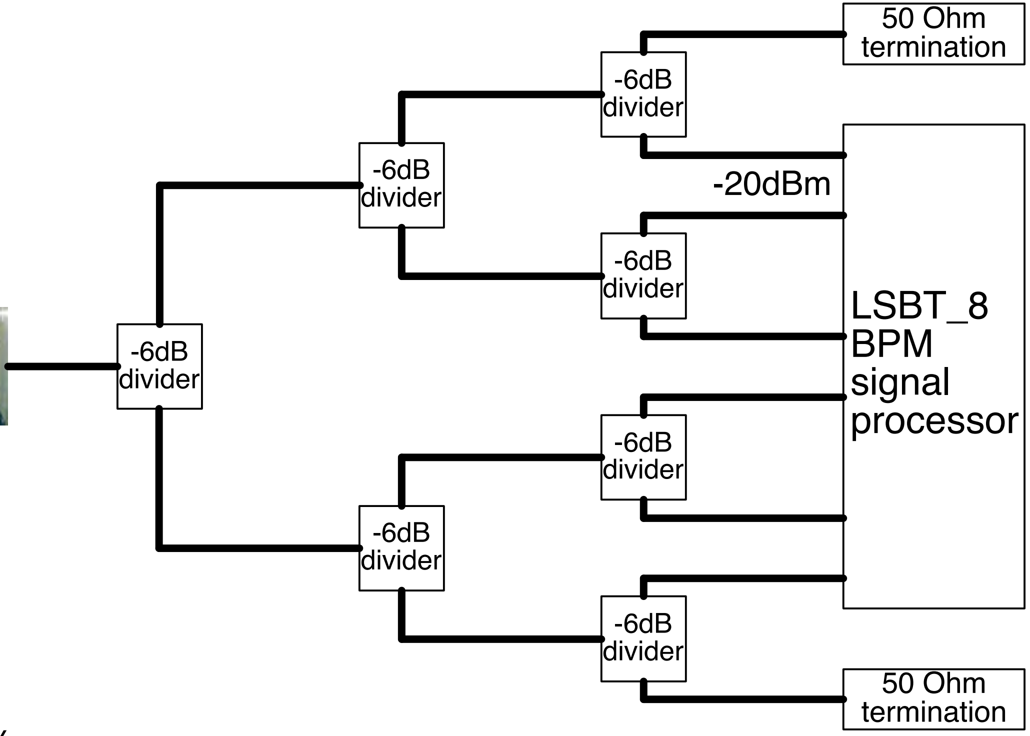
$$\Rightarrow \frac{\sigma_V}{V} = 0.003$$

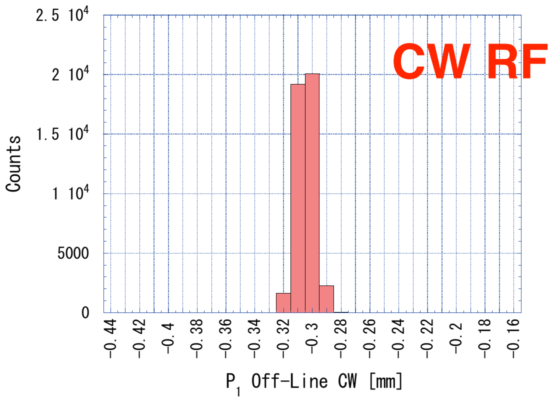
Measurement resolution

Resolution measurement by means of signal generator N5181A (off beam)

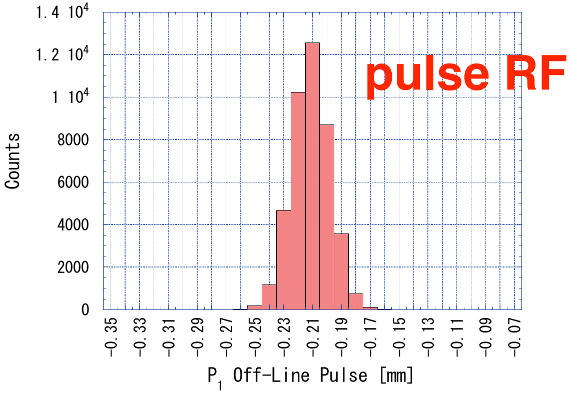
N5181A Agilent Technologies

0dBm CW or Pulse (100ns)





$$\sigma_{P1} = 0.006\ [\mathrm{mm}] \Rightarrow \frac{\sigma_V}{V} = 0.0012$$



$$\sigma_{P1} = 0.013\ [\mathrm{mm}] \Rightarrow \frac{\sigma_V}{V} = 0.0026$$

Comparison of $\frac{\sigma_V}{V}$

	Design	Off beam CW RF	Off beam pulse RF	On beam
$\frac{\sigma_V}{V}$	0.0010	0.0012	0.0026	0.0030

Summary

- SPring-8 BPM detects **wall current** longitudinally and transversely
- Transverse wall current distribution is determined by **beam moments** and **geometrical factor**.
- **Effective aperture radius** is calculated from geometrical factor and relates to an amplitude of BPM output.
- **Absolute moment** is measurable physical quantity. **Relative moment** which depends on the beam shape is exracted from the absolute moment.
- As the moment-correction scheme a **recursive method** was employed.
- To calibrate the attenuation factors of channels the **entire calibration** method was developed.
- Measurement resolutions of horizontal beam positions are 5μm for CW RF measurement, but 15μm for pulse RF measurement.