An Overview of Moonshine

John Duncan

PASCOS 24

Case Western Reserve University,

Cleveland, Ohio, U.S.A.

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1 What is Moonshine?

- Light reflected from the moon.

- An illicit beverage.

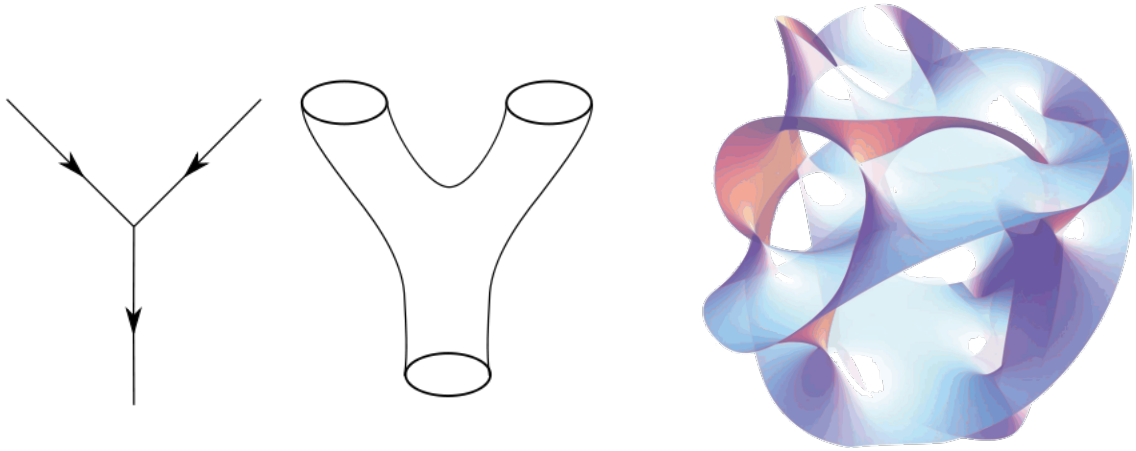
- Lunatic ideas.

- From a summary in *The Times* of a speech given by Rutherford in September 1933:

  "...anyone who looked for a source of power in the transformation of the atoms was talking moonshine."

2 Hidden Symmetry

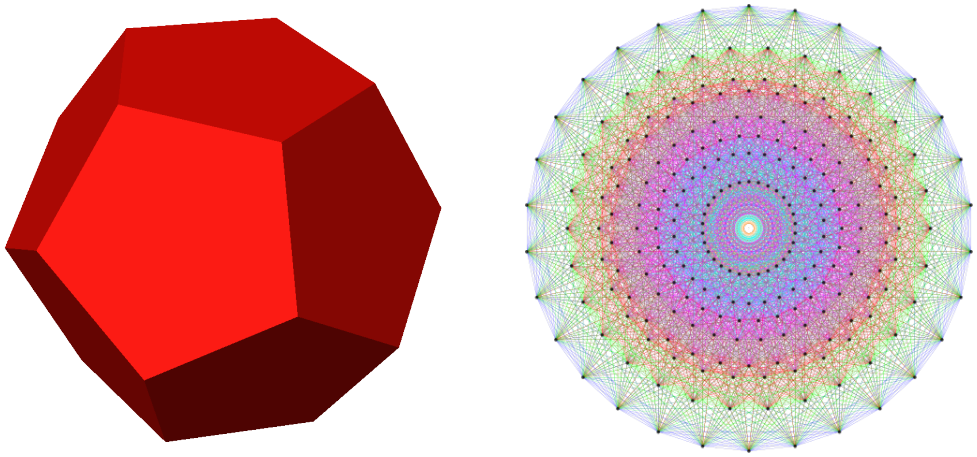
- Really, moonshine is a (growing) family of connections between algebra, number theory and geometry, that promise to be united by theoretical physics.



- The goal of moonshine theory is to uncover and understand the physical origins of hidden finite symmetry in algebraic, analytic and geometric statistics.

3 Finite Simple Groups

- The building blocks of finite symmetry—the *finite simple groups*—were classified over the course of the 20th century.
- There are several infinite families that arise naturally from geometric considerations.



- But in addition there are 26 exceptions, called the *sporadic* simple groups.

4 The Monster

- The largest of the sporadic groups is called the *monster*.

$$\#\mathbb{M} \approx 8 \times 10^{53} \tag{4.1}$$

- Evidence for existence of the monster was found independently by Bernd Fischer and Robert Griess in 1973.

5 Ogg's Observation

- Tits gave his inaugural lecture at the Collége de France on 14 January 1975.

- Tits described some properties of the (then conjectural) monster, including the prime factorization of its order.

$$\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \tag{5.1}$$

- Ogg attended, and noticed that the primes dividing the order of the monster

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 41,\ 47,\ 59,\ 71, \tag{5.2}$$

admit a geometric characterization.

- Ogg (1974): for p a prime, every supersingular elliptic curve over an algebraically closed field of characteristic p has j-invariant in $\mathbb{Z}/p\mathbb{Z}$ if and only if p belongs to the following list.

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 41,\ 47,\ 59,\ 71. \tag{5.3}$$

Une bouteille de Jack Daniels est offerte à celui qui expliquera cette coïncidence.

6 Diophantine Analysis

- Some of the most challenging open problems in mathematics, on elliptic curves in particular, go back to ancient Greece.

- Diophantus (~ 250): given a polynomial in several variables with integer coefficients

$$P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n] \tag{6.1}$$

  find the set of rational solutions

$$V(\mathbb{Q}) := \{(b_1, \ldots, b_n) \in \mathbb{Q}^n \mid P(b_1, \ldots, b_n) = 0\}\,. \tag{6.2}$$

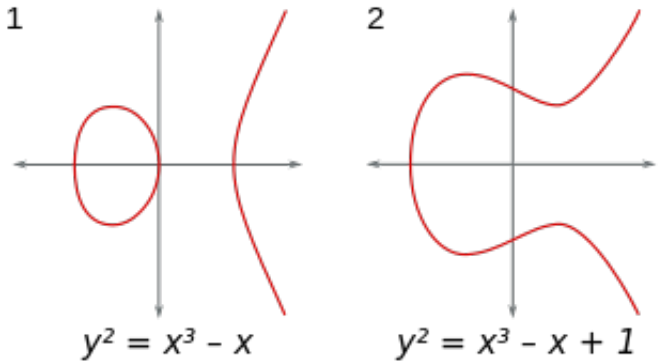
- For $n = 1$ or $d := \deg P = 1$ this is easy.
- For $n = 2$ and $d = 2$ it includes Gauss' theory of quadratic forms.
- For $n = 2$ and $d = 3$ this is the study of elliptic curves.
- For elliptic curves Diophantus' question is a millennium prize problem (Birch–Swinnerton-Dyer conjecture).

7 Elliptic Curves

- An *elliptic curve* over a field $\mathbb{F}$ (e.g. $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$) is an equation of the form

$$y^2 = x^3 + Ax + B \tag{7.1}$$

  for $A, B \in \mathbb{F}$ such that $4A^3 + 27B^2 \neq 0$.



- (The equation (7.1) takes a more general form if char($\mathbb{F}$) is 2 or 3.)

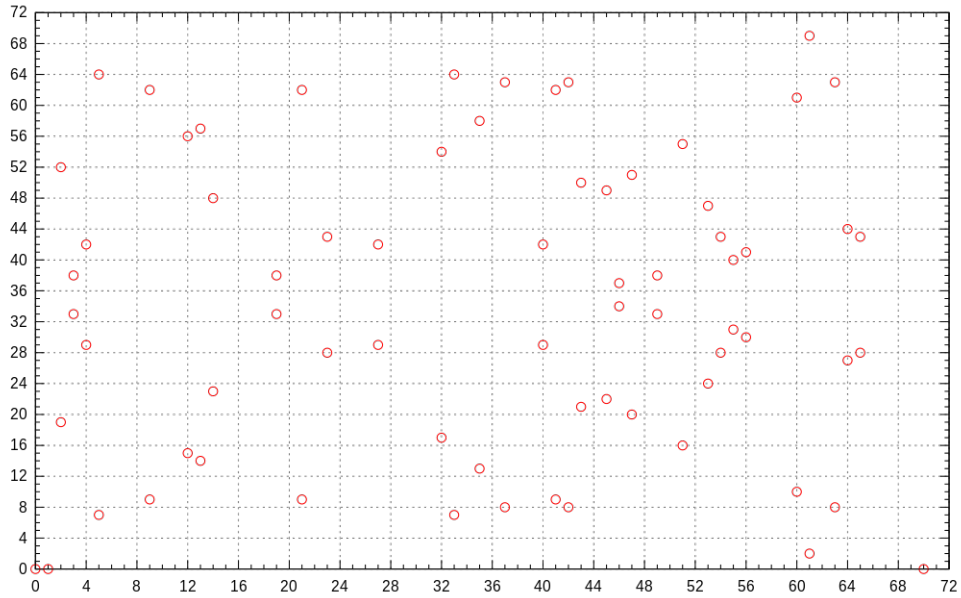
- The *j-invariant* of an elliptic curve $E : y^2 = x^3 + Ax + B$ is

$$j(E) := 1728\frac{4A^3}{4A^3 + 27B^2}. \tag{7.2}$$

- An elliptic curve is called *supersingular* if its endomorphism ring has rank 4.

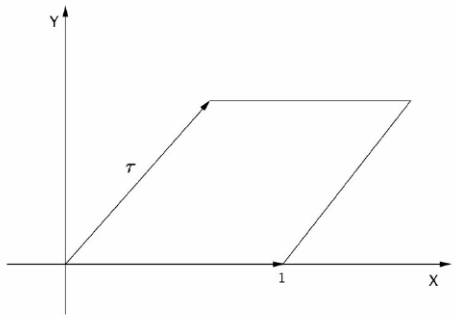
- Supersingular elliptic curves only occur in positive characteristic.

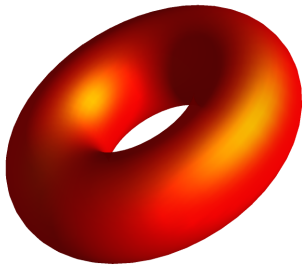
- The curve $E : y^2 = x^3 - x$ is supersingular for $\mathbb{F} = \mathbb{Z}/71\mathbb{Z}$.



8 Complex Elliptic Curves

- Any elliptic curve over $\mathbb{C}$ is isomorphic to $E_\tau := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ for some τ in the complex upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$.





- We have $E_{\tau'} \simeq E_\tau$ if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. So $SL_2(\mathbb{Z})\backslash\mathbb{H}$ is a moduli space for complex elliptic curves.

- The j-invariant defines an injective holomorphic map $SL_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$. So we may regard j as a $SL_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$.

- Setting $q := e^{2\pi i\tau}$ we obtain

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots \tag{8.1}$$

9 McKay's Observation

- Conway–Norton conjectured that there is an embedding of $\mathbb{M}$ in $GL_n(\mathbb{C})$ for $n = 196883$.

- McKay observed that $196884 = 1 + 196883$.

10 Monstrous Moonshine

- Conway–Norton–Thompson conjecture (1978): there exists an $\mathbb{M}$-module $V = \bigoplus V_n$ such that $\sum_n \dim(V_n) q^n = j(\tau) - 744$. More generally, for each $g \in \mathbb{M}$, the graded trace function

$$T_g(\tau) := \sum_n \operatorname{tr}(g|V_n) q^n \tag{10.1}$$

  is a *principal modulus* for its invariance group. That is, T_g induces an injective holomorphic map $\Gamma_g \backslash \mathbb{H} \to \mathbb{C}$, where $\Gamma_g := \{\gamma \in SL_2(\mathbb{R}) \mid T_g(\gamma\tau) = T_g(\tau)\}$.

- The principal modulus property enabled computations in the monster using $SL_2(\mathbb{R})$ (i.e. 2×2 matrices). To paraphrase Conway: it seemed too powerful to be legal!

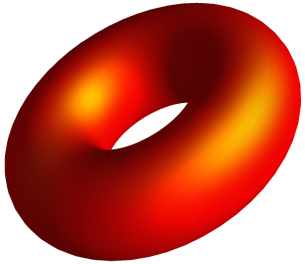
11 Frenkel–Lepowsky–Meurmann and Borcherds

- Frenkel–Lepowsky–Meurman constructed a candidate *moonshine module* $V^\natural = \bigoplus V_n^\natural$ in 1984.

- Borcherds confirmed that the FLM module $V^\natural$ satisfies the Conway–Norton–Thompson conjecture in 1992.

12 Conformal Field Theory

- The Frenkel–Lepowsky–Meurman construction turned out to be the first example of an orbifold (chiral) 2d conformal field theory.



- The target space of the theory is the canonical 2-fold orbifold of the 24-dimensional torus defined by the *Leech lattice*.

13 Ramanujan

- Ramanujan was born in Tamil Nadu, India, on 22 December 1887.
- He studied and worked with Hardy in Cambridge from 1914 to 1919.

- Ramanujan introduced *mock theta functions* in a letter to Hardy dated 12 January 1920.
- He succumbed to disease in his home town Kumbakonam on 26 April 1920.

- From Ramanujan's last letter to Hardy:

  I discovered very interesting functions recently which I call "Mock" ϑ-functions... they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

- It would take mathematicians almost a century to find a theoretical framework for the mock theta functions.

$$f(q) := 1 + \sum_{n>0} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}, \tag{13.1}$$

$$g(q) := 1 + \sum_{n>0} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}. \tag{13.2}$$

- Challenge: one of these is mock and the other is not (i.e. one of these is a modular form). Which is which?

- The function g is a *modular form* (theta-function) in the sense that if $G(\tau) := q^{-\frac{1}{24}} g(q)$ for $\tau \in \mathbb{H}$ where $q = e^{2\pi i \tau}$ then for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have

$$G\left(\frac{a\tau + b}{c\tau + d}\right)\sqrt{c\tau + d} = \epsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) G(\tau) \tag{13.3}$$

  for some function $\epsilon : SL_2(\mathbb{Z}) \to \mathbb{C}^*$.

- Zwegers (2002): the function f is a *mock modular form* in the sense that if $F(\tau) := q^{-\frac{1}{24}} f(q)$ and $\hat{F}(\tau) := F(\tau) + s^*(\tau)$ where $s^*(\tau) = \frac{i}{\sqrt{3}} \int_{-\overline{\tau}}^{i\infty} \frac{s(\tau)}{\sqrt{(-i)(z+\tau)}} \mathrm{d}z$ and $s(\tau) := \sum_{k \equiv 1 \bmod 6} k q^{\frac{k^2}{24}}$ then

$$\hat{F}\left(\frac{a\tau + b}{c\tau + d}\right)\frac{1}{\sqrt{c\tau + d}} = \chi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \hat{F}(\tau) \tag{13.4}$$

  when $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ for some subgroup $\Gamma(2) < SL_2(\mathbb{Z})$ and some function $\chi : \Gamma(2) \to \mathbb{C}^*$.

- The function $s(\tau)$ is called the *shadow* of f.

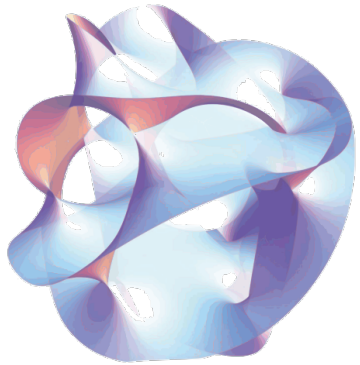
14 Dyson's Dream

- From Dyson's lecture "A Walk Through Ramanujan's Garden" at the Ramanujan Centenary Conference in 1987:

  "The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered... My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions."

15 Superstring Theory

- Calabi–Yau manifolds are candidate consistent geometric backgrounds for superstrings.



- Non-linear sigma models with Calabi–Yau target govern string dynamics in such theories.
- The *elliptic genus* of a non-linear sigma model with Calabi–Yau target X is

$$EG_X(\tau, z) := \operatorname{tr}\left((-1)^{F_L+F_R} y^{J_L(0)} q^{L_L(0)-\frac{c_L}{24}} \bar{q}^{L_R(0)-\frac{c_R}{24}} | \mathcal{H}_{\mathrm{RR}}\right), \tag{15.1}$$

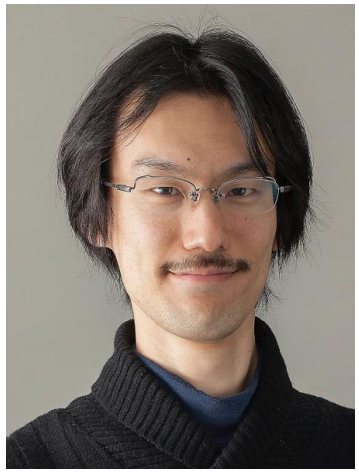
where $\mathcal{H}_{\mathrm{RR}}$ is the Ramond-Ramond sector of an associated $N = (2, 2)$ superconformal field theory, and $q := e^{2\pi i\tau}$ and $y := e^{2\pi iz}$.

16 Mathieu Moonshine

- A superconformal field theory with K3 surface target has $N = (4, 4)$ supersymmetry.
- Eguchi–Ooguri–Tachikawa (2010): if we decompose the K3 elliptic genus into characters $\mathrm{ch}_{h,\ell}$ of the $N = 4$ superconformal algebra

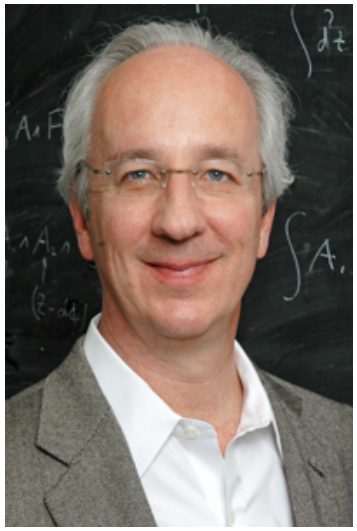
$$EG_{K3} = 20\,\mathrm{ch}_{\frac{1}{4},0} - 2\,\mathrm{ch}_{\frac{1}{4},\frac{1}{2}} + \sum_{n\geq 1} A(n-\tfrac{1}{8})\,\mathrm{ch}_{n+\frac{1}{4},\frac{1}{2}} \tag{16.1}$$

  then $A(n-\frac{1}{8})$ is twice the dimension of an irreducible representation of the sporadic *Mathieu group* M_{24}, for $1 \leq n \leq 5$.

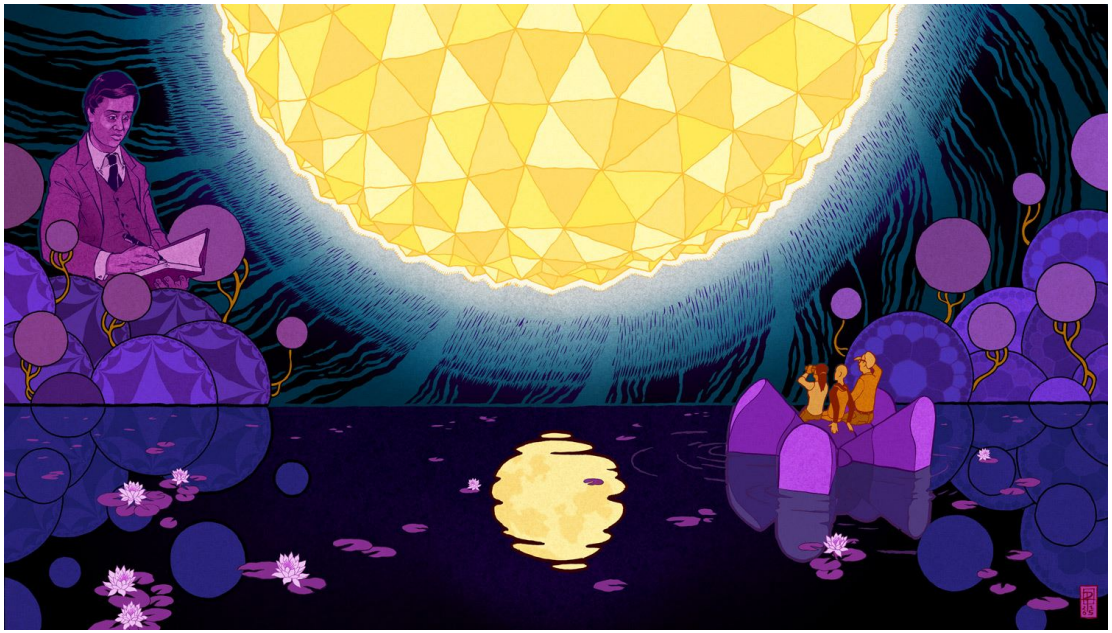


17 Umbral Moonshine

- Cheng–D–Harvey (2012): Mathieu moonshine generalizes to an assignment of mock modular forms to automorphisms of special sphere packings in 24-dimensions.



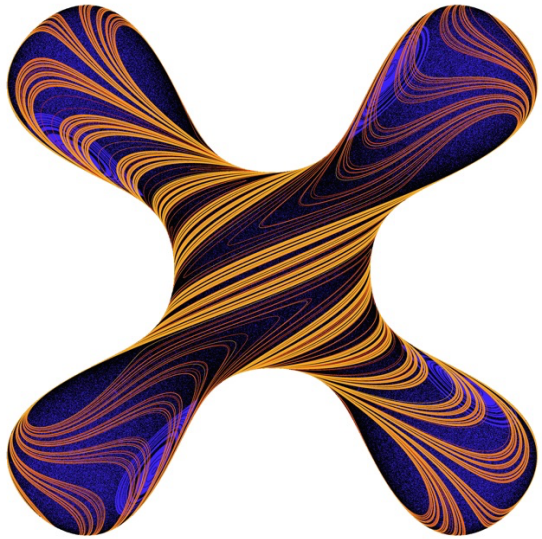
- Cheng–D (2016): All of Ramanujan's mock theta functions appear in umbral moonshine. They can all be defined in terms of principal moduli (like the T_g of monstrous moonshine).



18 K3 Surfaces

- A *complex K3 surface* is a choice of complex structure on the *Fermat quartic*

$$\{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{P}^3. \tag{18.1}$$

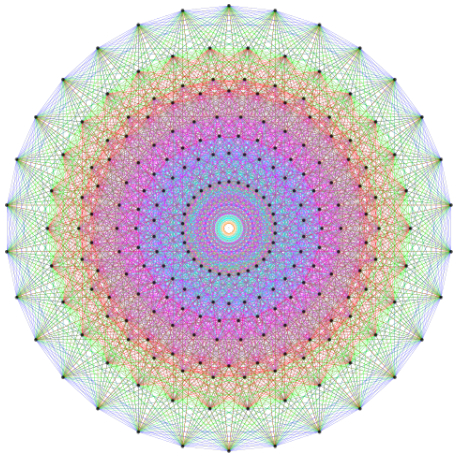


- K3 surfaces are the simplest Calabi–Yau manifolds that are not tori.

- K3 surfaces were so named in 1958 by Weil in honor of Kummer, Kähler, Kodaira and the mountain K2.
- K2 is short for Karakoram 2.
- The mountain Karakoram 3 is also known as Broadpeak.

19 Conway Moonshine

- The monster module corresponds to the compactification of the 26-dimensional bosonic string on the canonical 2-fold orbifold of the (24-dimensional) Leech torus.
- Frenkel–Lepowsky–Meurman (1985): how about the compactification of the 10-dimensional superstring on the canonical 2-fold orbifold of the (8-dimensional) E_8 torus?

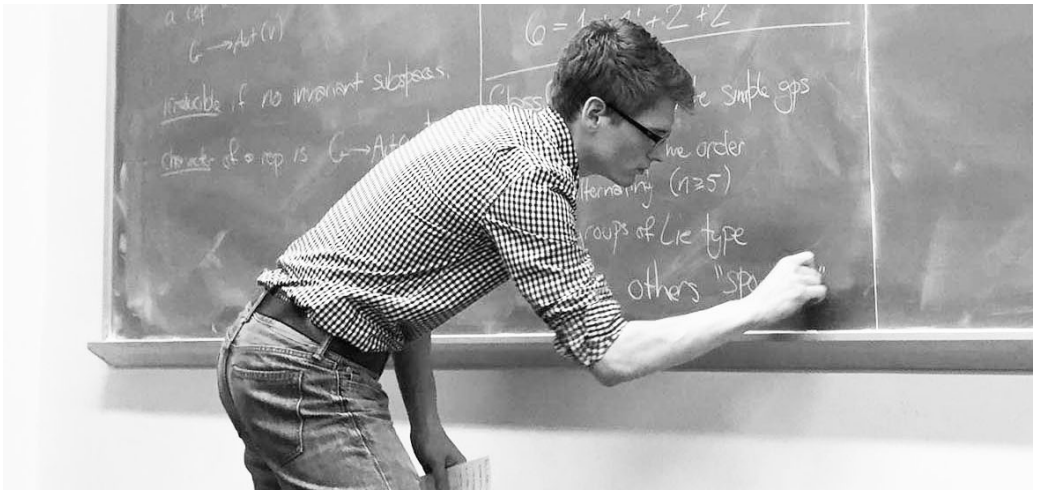


- D (2006): the E_8 suggestion leads to a chiral $N = 1$ superconformal field theory $V^{s\natural} = \bigoplus_n V_n^{s\natural}$ whose symmetry group is the sporadic *Conway group*.

- D–Mack-Crane (2015): for every g in the Conway group the trace function

$$T_g^s(\tau) := \sum_n \operatorname{tr}\left(g(-1)^F | V_n^{s\natural}\right) q^n \tag{19.1}$$

  is a principal modulus for its invariance group.



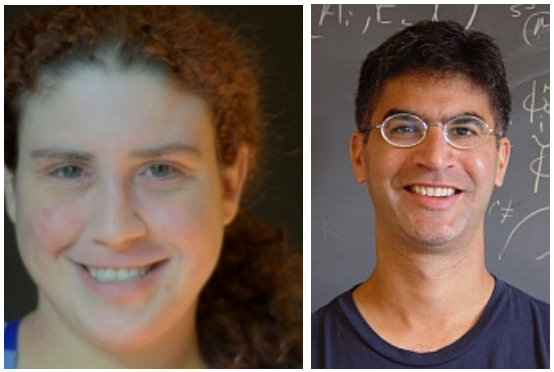
20 Conway and K3

- D–Mack-Crane (2015): a sigma model Π on a K3 surface naturally determines an action of the $N = 2$ superconformal algebra on the Ramond sector of the Conway moonshine module. Moreover, if g is a symmetry of Π then

$$\phi_{\Pi,g}(\tau, z) := \operatorname{tr}\left(g(-1)^F y^{J(0)} q^{L(0)-\frac{c}{24}} | V_{\mathrm{R}}^{s\natural}\right) \tag{20.1}$$

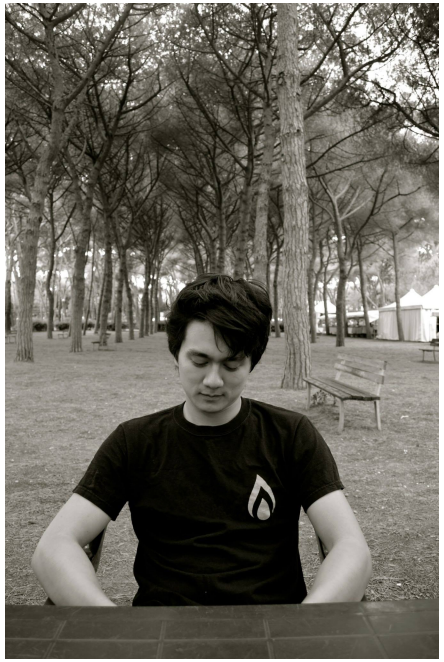
  is a g-equivariant elliptic genus of Π. In particular, for g the identity element $\phi_{\Pi,g}$ is the K3 elliptic genus.

- Cheng–D–Harrison–Kachru conjecture (2015): the functions $\phi_{\Pi,g}$ compute equivariant enumerative invariants of K3 surfaces.



21 Umbral Moonshine Modules

- A uniform construction of the umbral moonshine modules remains open, but special cases have been solved by D–Harvey (2014), D–O'Desky (2017), Cheng–D (2017), and Anagiannis–Cheng–Harrison (2017).
- The first conformal field theoretic constructions were obtained by D–O'Desky (2017).



- "Super blue blood moon" over the Acropolis of Athens on 31 January 2018 (AP)

22 The Birch–Swinnerton-Dyer Conjecture

- Mordell (1922): if $E : y^2 = x^3 + Ax + B$ is an elliptic curve over $\mathbb{Q}$ then

$$E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus E(\mathbb{Q})_{\mathrm{tor}} \tag{22.1}$$

  for some non-negative integer $r = r_E$ called the *rank* of E, where $\#E(\mathbb{Q})_{\mathrm{tor}}$ is finite.

- Set $a_p := p + 1 - \#E(\mathbb{Z}/p\mathbb{Z})$ and let ε_p be 0 or 1 according as p divides $-16(4A^3 + 27B^2)$ or not.

- The modularity theorem (Taniyama, Shimura, Weil, Wiles, Taylor, &c.) states that the *L-function*

$$L_E(s) := \prod_p (1 - a_p p^{-s} + \varepsilon_p p^{1-2s})^{-1} \tag{22.2}$$

  is the Mellin transform of a holomorphic 1-form on $\Gamma_E \backslash \mathbb{H}$ for some $\Gamma_E < SL_2(\mathbb{Z})$.

- The Birch–Swinnerton-Dyer conjecture connects $L_E(s)$ to r_E and $E(\mathbb{Q})_{\text{tor}}$.
- (Strong) Birch–Swinnerton-Dyer conjecture: if $r = r_E$ is the rank of E then the first r derivatives of L_E vanish at $s = 1$, and

$$\frac{1}{\Omega_E}\,\frac{1}{r!}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^r L_E(s)\bigg|_{s=1} = \frac{c_E \#Ш(E)}{(\#E(\mathbb{Q})_{\text{tor}})^2} \tag{22.3}$$

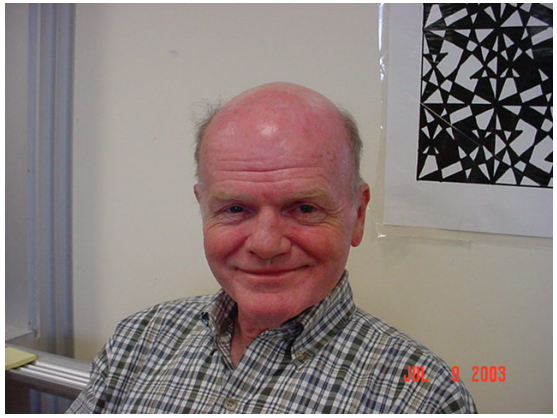
  where Ω_E and c_E are computable.
- The *Tate–Shafarevich group* $Ш(E)$ is hard to compute but for each prime ℓ there is the more accessible ℓ-th *Selmer group* $\mathrm{Sel}_\ell(E)$, which satisfies

$$0 \to E(\mathbb{Q})/\ell E(\mathbb{Q}) \to \mathrm{Sel}_\ell(E) \to Ш(E)[\ell] \to 0, \tag{22.4}$$

  where $Ш(E)[\ell]$ denotes the kernel of multiplication by ℓ on $Ш(E)$.

23 O'Nan Moonshine

- The sporadic *O'Nan group* was discovered by O'Nan in 1976.



- In contrast to most (i.e. 20 out of 26) of the sporadic groups it is not visible inside the monster.

- D–Mertens–Ono (2017): there is a virtual graded ON-module $W^{ON} = \bigoplus W_D^{ON}$ whose graded dimension is a distinguished modular form of weight $\frac{3}{2}$.

24 Hidden Symmetry in Elliptic Curves

- For $D \in \mathbb{Z}$ define

$$E_{14} \otimes D : \qquad y^2 = x^3 + 5805D^2x - 285714D^3 \tag{24.1}$$

$$E_{15} \otimes D : \qquad y^2 = x^3 - 12987D^2x - 263466D^3 \tag{24.2}$$

- Duncan–Mertens–Ono (2017): for $D < 0$ the O'Nan moonshine module can be used to obstruct the vanishing of $\mathrm{Sel}_7(E_{14} \otimes D)$ and $\mathrm{Sel}_5(E_{15} \otimes D)$.

ON