RENORMALONS IN A GENERAL QUANTUM FIELD THEORY

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OUTLINE

1. Renormalons. Introduction
2. Connection with the renormalization group.
3. Generalization to an arbitrary QFT
4. Iterative solution of the renormalization group equations
5. Toy model example
6. Contact with realistic models
7. Conclusions

DIVERGENCES IN QFT. (DYSON 1951)

INSTANTONS (Lipatov 1976)

- In 1976 Lipatov realized that divergences in the perturbative series are related to instantons. In fact

$$F(e^2) = \sum_{k=0}^{\infty} = a_k (e^2)^k$$

Where

$$a_k \sim n!$$

Consider the following example in one dimension: (t'Hooft 1979)

$$G(\lambda) = \int dx e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4}$$

$$G(\lambda) = \int_0^{\infty} dz F(z) e^{-z/\lambda}$$

$$F(z) = \int dx \delta(z - S(x)/\lambda) \qquad S(x) = -\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4$$

F(z) is called the Borel transform of G

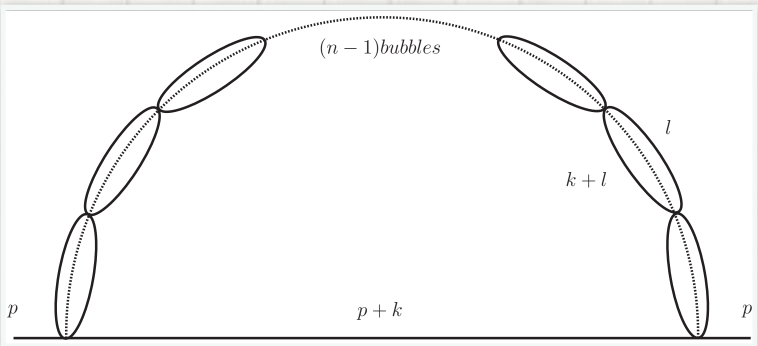
F(z) diverges when

$$\sim dS/dx|_{x=\bar{x}} = 0 \qquad S(\bar{x}) = \lambda z$$

This is the classical equation of motion and the solution is the instanton. They are related with barrier penetration (Coleman 1977) and can be treated semiclassically

RENORMALONS ('T HOOFT 1979)

One type of Feynman diagram that goes as n! (after renormalization)



$$R_n = \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p+k)^2 - m^2 + i\epsilon} \frac{1}{(-i\lambda)^{n-1}} [B(k)]^n \,,$$

$$B(k) = \frac{(-i\lambda)^2}{2} \int \frac{d^4l}{(2\pi)^4} \frac{i}{(k+l)^2 - m^2 + i\epsilon} \frac{i}{l^2 - m^2 + i\epsilon} \,.$$

For large Euclidean momentum k:

$$R_n \sim i\lambda^{n+1} \left(\frac{\beta_0}{2}\right)^n \frac{1}{16} \frac{m^4}{\mu^2} n! \qquad \lambda' = \beta_0 \lambda^2 \qquad \lambda' \equiv \mu \frac{d\lambda}{d\mu}$$

The Borel transform is $\mathcal{B}^{(1)}(\lambda^{n+1}) \equiv \frac{z^n}{n!}$

$$\mathcal{B}^{(1)}(\sum_n R_n) \propto \sum_n (\frac{\beta_0}{2} z)^n \equiv \sum_n \mathcal{B}^{(1)}_n(z) = \frac{1}{1 - \frac{\beta_0}{2} z}$$

$$z_{pole} = \frac{2}{\beta_0}$$

CONNECTION WITH THE RENORMALIZATION GROUP (PARISI 1978-1979)

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\mu} - n\gamma\right)\Gamma_R^{(n)}(p_1, ..., p_n; \lambda(\mu); \mu) = 0$$

$$\beta \simeq \beta_0\lambda^2 + ...$$

$$\gamma \simeq 1 + ...$$

It can be written as

$$\mu\frac{d}{d\mu}\Gamma_R^{(n)}(\lambda(\mu)) = n\gamma\Gamma_R^{(n)}(\lambda(\mu))$$

$$\frac{d}{d\lambda}\ln\Gamma_R^{(n)}(\lambda(\mu)) = n\gamma/\beta(g)$$

$$\gamma/\beta \simeq \left(\frac{1}{\beta_0\lambda^2} + z_1\frac{1}{\lambda} + z_2\right)$$

The general solution is of the form

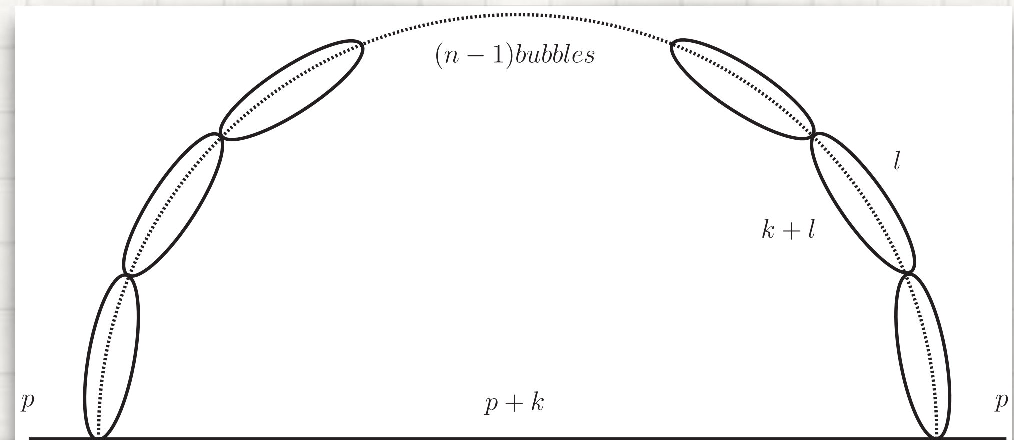
$$z_{pole} = 2/\beta_0$$

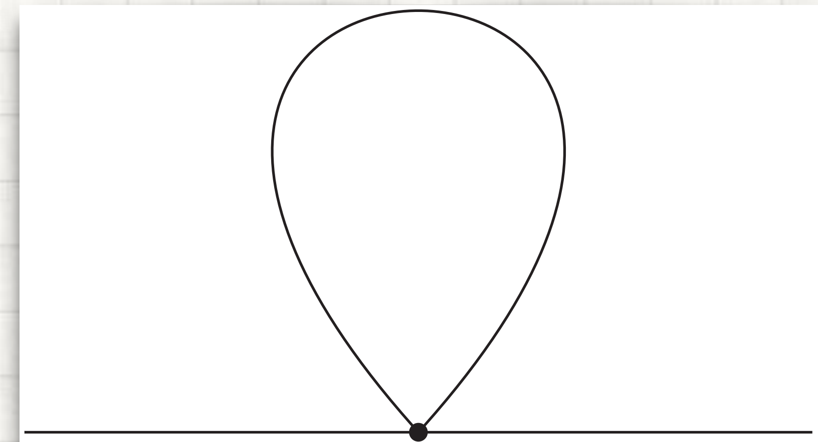
$$\Gamma_R^{(n)}(\lambda(\mu)) \propto e^{\frac{n}{2}\frac{z_{pole}}{\lambda}}\left(\frac{\lambda(\mu)}{\lambda(\mu_0)}\right)^{\frac{n}{2}z_1}\Gamma_R^{(n)}(\lambda(\mu_0))$$

Borel transform

$$G(g) = \int F(z)e^{-\frac{z}{\lambda}}dz$$

CONNECTION WITH THE RENORMALIZATION GROUP (PARISI 1978-1979)





For large Euclidean momentum there exist the equivalence

$$R_n \sim i\lambda^{n+1}\left(\frac{\beta_0}{2}\right)^n \frac{1}{16}\frac{m^4}{\mu^2} n! \qquad \lambda' = \beta_0\lambda^2$$

$$\begin{aligned}\sum_n R_n &= \sum_n i\lambda^{n+1}\left(\frac{\beta_0}{2}\right)^n \frac{1}{16}\frac{m^4}{\mu_0^2} n! \\ &= \frac{im^4}{8\pi^2}\int \frac{dl_E^2}{2(l_E^2)^2}\frac{\lambda(\mu_0)}{1-\frac{\beta_0}{2}\lambda(\mu_0)^2\ln(l_E^2/\mu_0^2)} \\ &\simeq \frac{im^4}{8\pi^2}\int \frac{dl_E^2}{2(l_E^2)^2}\lambda(l_E)\end{aligned}$$

That in the multi-coupling case. (This requires an analytical solution of the RGE's)

$$\mathcal{B}\left(\int \frac{dl_E^2}{(l_E^2)^2}\lambda(l_E)\right) \longrightarrow \mathcal{B}\left(\int \frac{dl_E^2}{(l_E^2)^2}\lambda_i(l_E)\right)$$

ITERATIVE SOLUTION OF THE RGE'S

. (J.-H. He, *Variational iteration method for autonomous ordinary differential systems*, *Applied Mathematics and Computation* 114 (2000) 115–123)

It turns that the more general problem of the solution of ordinary non-linear differential equations has been studied by He. In particular for the RGE's

$$\lambda_i' = \sum_{n,m=1}^{N} \beta_i^{nm}\lambda_n\lambda_m \equiv \beta_i^{nm}\lambda_n\lambda_m \,,$$

$$\lambda_i^{(n)}(t) = \lambda_i^{(n-1)}(t) - \int_0^t \left(\frac{d\lambda_i^{(n-1)}}{ds} - \sum_{j,k}^{N} \beta_i^{jk}\lambda_j^{(n-1)}(s)\lambda_k^{(n-1)}(s) \right) ds \,.$$

Integrating one finds

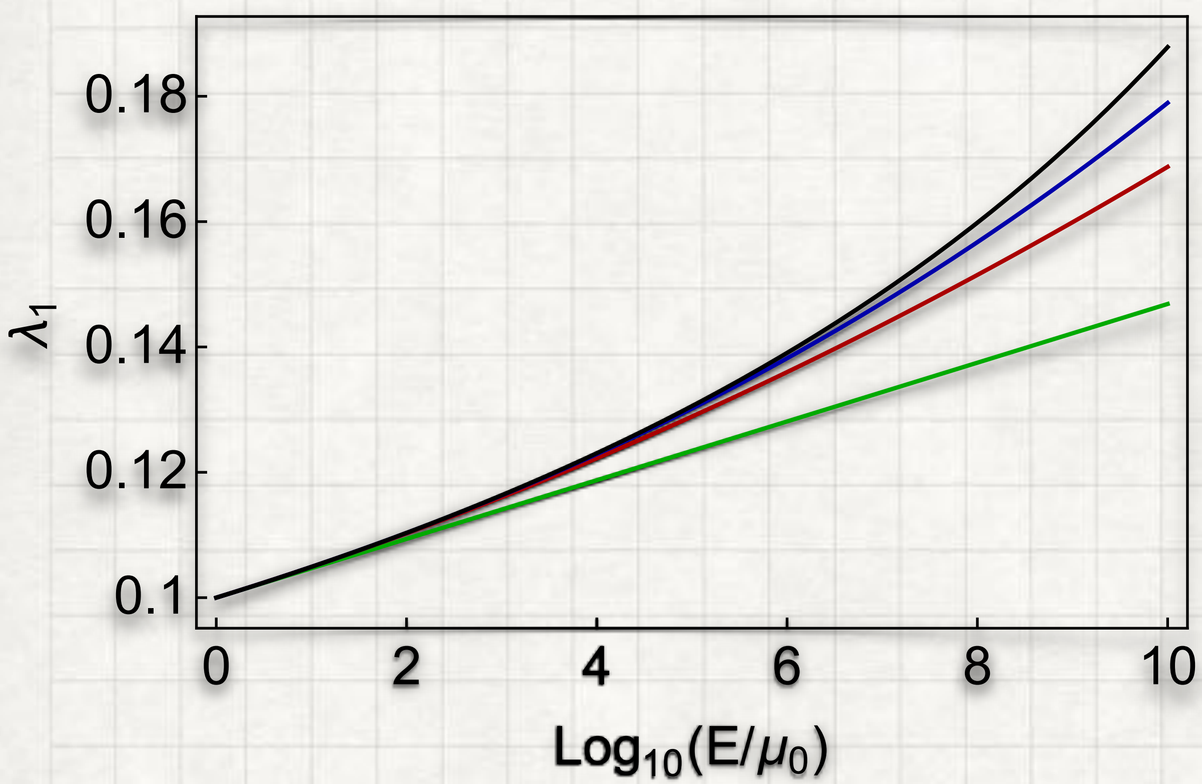
$$\lambda_i(t) = \lambda_i(0) + t v_i + M_i^k s(1)_k t^2 + \sum_{n=3}^{\infty} s(n)_i \frac{t^n}{n} \qquad t \equiv \ln(\mu/\mu_0)$$

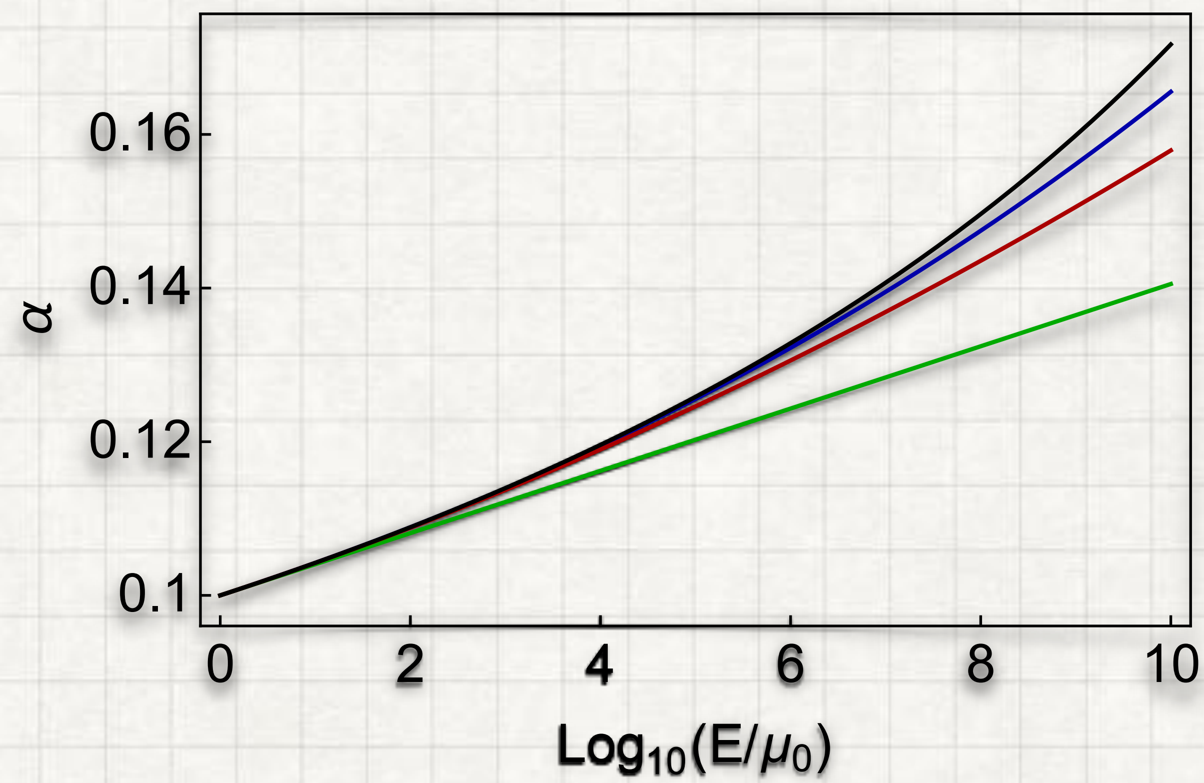
$$s(n)_i = M_i^k s(n-1)_k + \beta_i^{kl} \sum_{m=1}^{n-2} s(m)_k s(n-1-m)_l \,,$$

$$s(1)_i \equiv v_i \qquad v_i \equiv \beta_i^{mn}\lambda_m(0)\lambda_n(0), \quad M_i^k = \frac{\partial s(1)_i}{\partial \lambda_k} \equiv (\beta_i^{mk} + \beta_i^{km})\lambda_m(0) \,.$$

ITERATIVE SOLUTION OF THE RGE'S

$$V(\phi_1, \phi_2) \supset \lambda_1 \phi_1^4 + \lambda_2 \phi_2^4 + \alpha \phi_1^2 \phi_2^2 \,,$$





These are the comparison of the first three orders in t with the usual numerical solution of the RGE's

Algorithm for finding the divergences in the Borel series

$$\mathcal{B}\left(\int \frac{dl_E^2}{(l_E^2)^2}\lambda_i(l_E)\right)$$

$$\mathcal{F}_{n,i} \equiv \frac{n!}{2^n}\text{Coefficient}(\lambda_i(t), t^n)$$
$$\mathcal{F}_{n,i} \mapsto \mathcal{B}_{n,i}(z_1, z_2, ..., z_N)$$
$$|\mathcal{B}_{n,i}(z_1, z_2, ..., z_N)| \leq \mathcal{B}_n^{(1)}(z)|_{z_{pole}=2/\beta}\,,$$

Using the comparison test we can locate the divergence of the multivariate Borel series, eventually on the positive axis and therefore the Renormalons

The generalized Borel transform

$$\lambda_j^k \to z_j^{k-1}/(k-1)!$$

$$\lambda_j^0 \to \delta(z_j)$$

The integral form of the generalized Borel transform

$$\int dz_1 dz_2 ... dz_N e^{-(\frac{z_1}{\lambda_1}+\frac{z_2}{\lambda_2}+...+\frac{z_N}{\lambda_N})}\mathcal{B}_{n,i}(z_1, z_2, ..., z_N)\,.$$

Leading Renormalons. One coupling much larger with respect To the others

$$\lambda_i(t) = \delta_{ji}\lambda_j(0) + \sum_{n=1}^{\infty} t^n (a_n \lambda_j(0)^2 + A_n)$$

Applying the Borel transform

$$\prod_{k\neq j} \delta(z_k) \left(\delta_{ji} + \sum_{n=1}^{\infty} a_n (\frac{z_i}{2})^n \right)$$

Using the comparison test

$$\frac{1}{2^n} |a_n| (z_j)^n_{pole} = \left(\frac{\beta}{2} z_{pole} \right)^n = 1$$

$$(z_j)_{pole} \equiv \lim_{n\to\infty} 2/\sqrt[n]{|a_n|}$$

This definition is the generalization of the t' Hooft result when one coupling is large with respect to the others

Another important limit: all couplings are of the same size. A power counting argument

In this case one finds

$$\beta_i \to \beta$$

$$z_i \to z$$

$$z_{pole} \sim \frac{2}{N\beta}$$

N is the number of couplings in the theory under consideration

This result shows how the position of the renormalon's singularity changes with the number of couplings. Large N push the renormalon singularity to smaller values, therefore worsening the issue of perturbative renormalizability

TOY MODEL EXAMPLE (N = 3)

The potential

$$V(\phi_1, \phi_2) \supset \lambda_1 \phi_1^4 + \lambda_2 \phi_2^4 + \alpha \phi_1^2 \phi_2^2 \,,$$

The RGE's for this model are

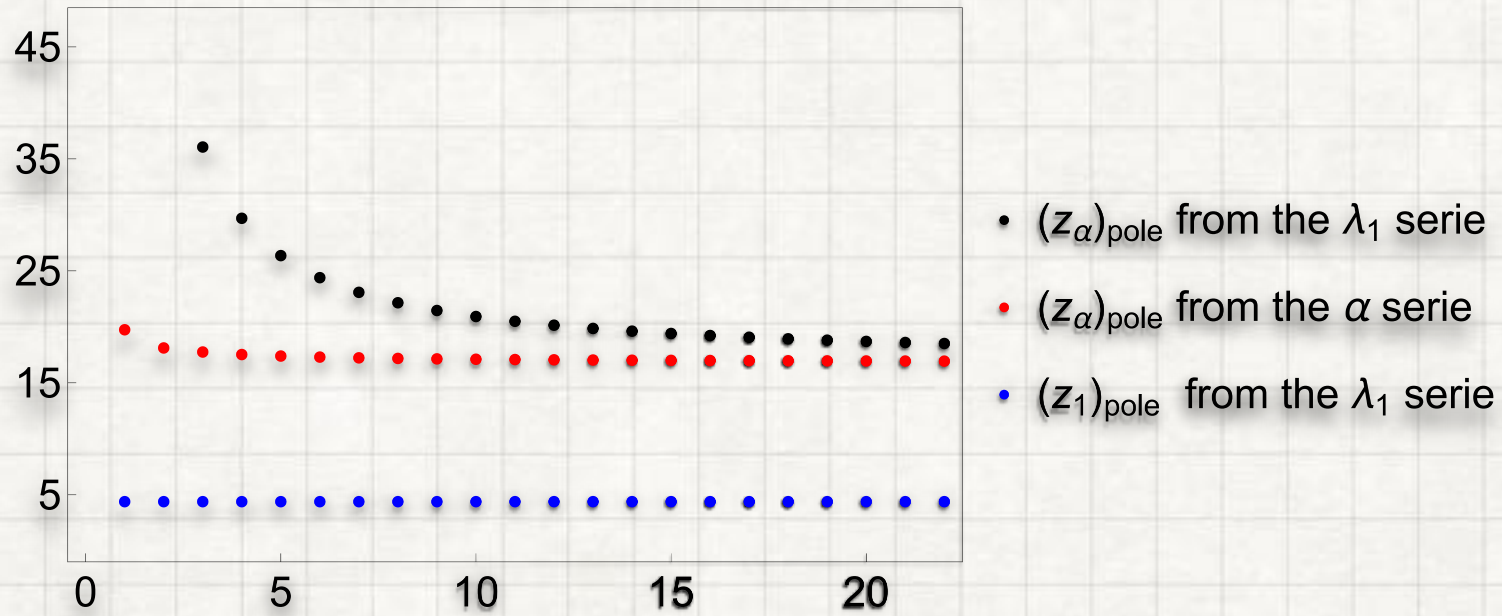
$$\lambda_1' = \frac{1}{16\pi^2}(72\lambda_1^2 + 2\alpha^2) = \beta_{11}\lambda_1^2 + \beta_{13}\alpha^2 \,,$$
$$\lambda_2' = \frac{1}{16\pi^2}(72\lambda_2^2 + 2\alpha^2) = \beta_{21}\lambda_2^2 + \beta_{23}\alpha^2 \,,$$
$$\alpha' = \frac{1}{16\pi^2}[16\alpha^2 + 24\alpha(\lambda_1 + \lambda_2)] = \beta_{33}\alpha^2 + \beta_{31}\alpha\lambda_1 + \beta_{32}\alpha\lambda_2 \,,$$

We parametrize so that usual criteria for convergence may applied to the variable R

$$z_1 = a \times R,\ z_2 = b \times R,\ z_\alpha = c \times R \,,$$

$$R \in \mathbb{R}^+ \qquad a, b, c \in \mathbb{C}$$

TOY MODEL. LEADING RENORMALON CASE



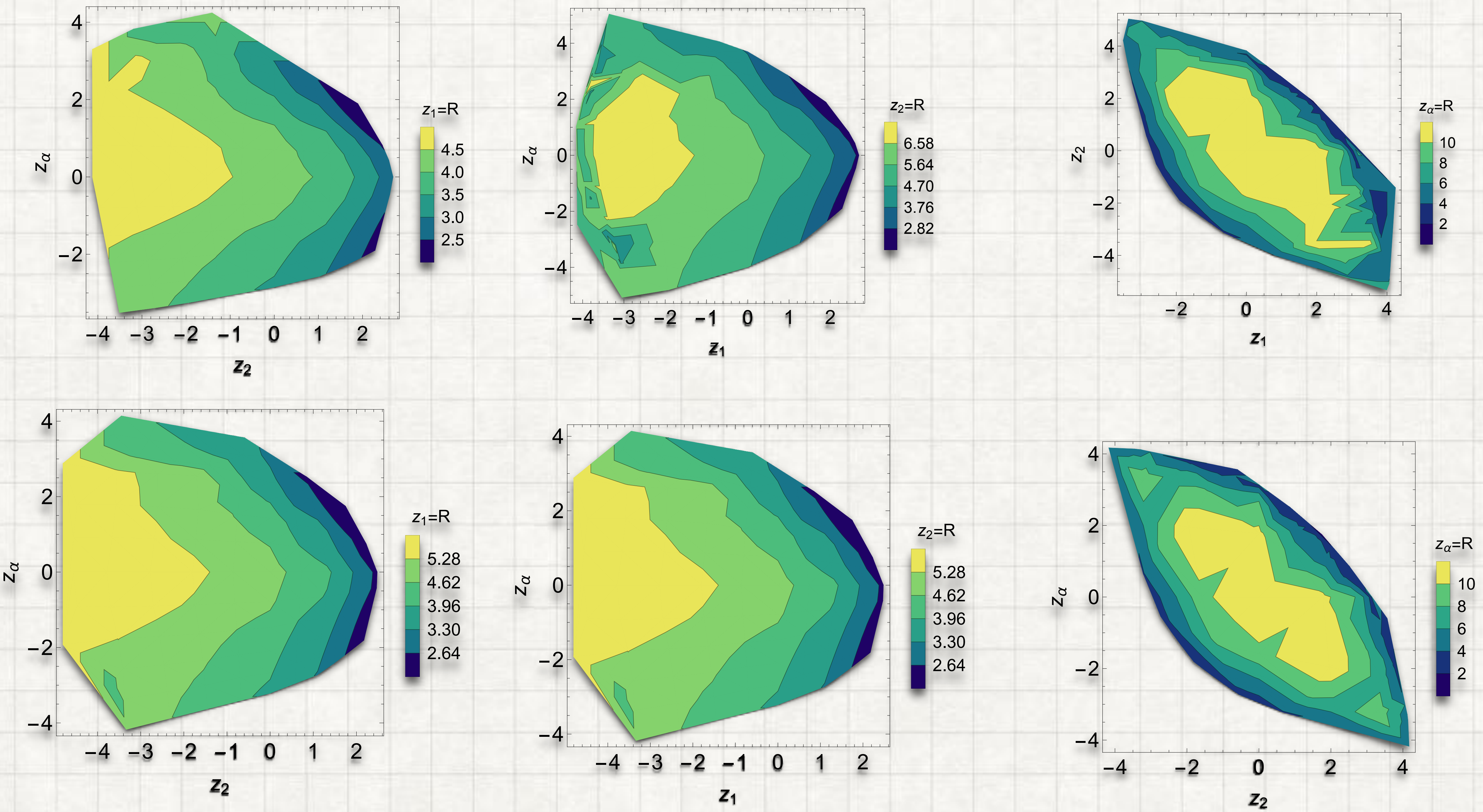
Asymptotic values for the poles in the Borel variable in the leading renormalon case

$$\delta(z_1)\delta(z_2)\times(z_\alpha)^{(1)}_{pole} = \frac{2}{\beta_{33}}\delta(z_1)\delta(z_2)$$

$$\delta(z_1)\delta(z_2)\times(z_\alpha)^{(2)}_{pole} = \pm\frac{2\sqrt{2}}{\sqrt{\beta_{13}\beta_{31}+\beta_{23}\beta_{32}+2\beta_{33}^2}}\delta(z_1)\delta(z_2)$$

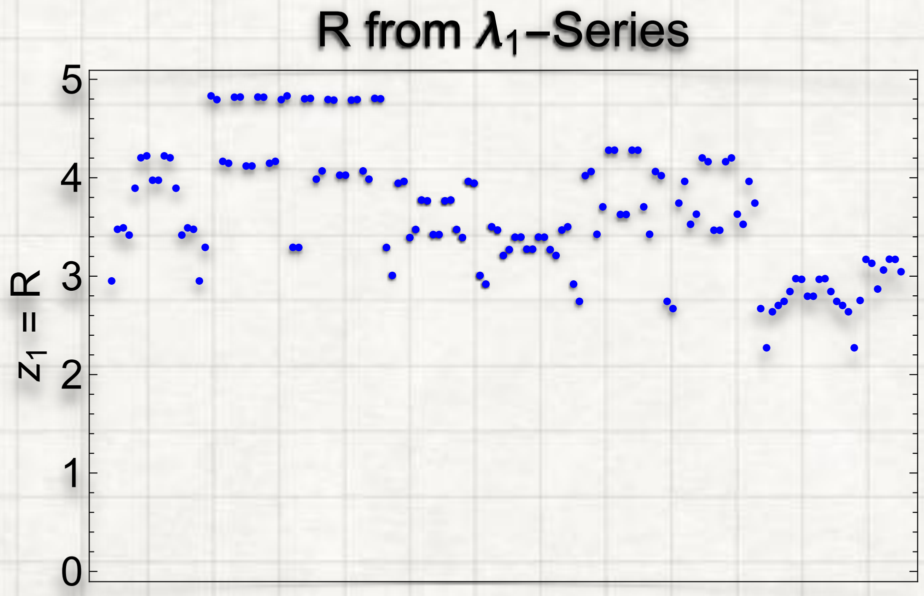
$$\delta(z_1)\delta(z_2)\times(z_\alpha)^{(3)}_{pole} = \frac{2}{\sqrt[3]{\beta_{33}^3+\beta_{13}\beta_{31}\beta_{33}+\beta_{23}\beta_{32}\beta_{33}}}\delta(z_1)\delta(z_2)\,,$$

NON-LEADING CASE. REAL Z'S



$$z_1 = aR, z_2 = bR, z_\alpha = cR. \qquad a, b, c \in \mathbb{R}$$

NON-LEADING CASE. COMPLEX Z'S



$z_1 \in \mathbb{R}$

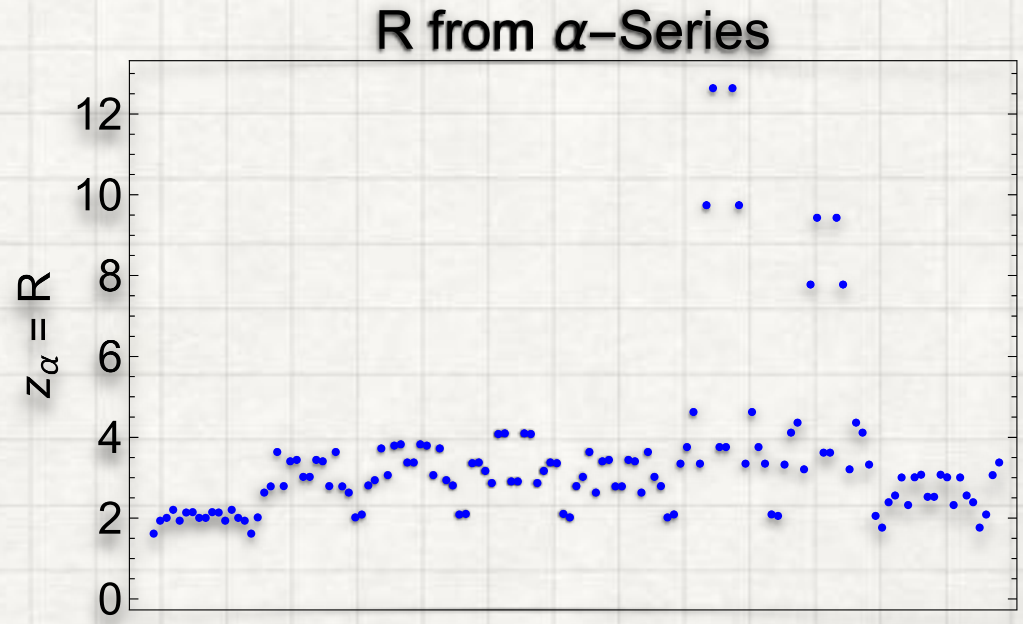
$z_2 = b + \tilde{b}i$

$z_\alpha = c + \tilde{c}i$

$z_\alpha \in \mathbb{R}$

$z_1 = b + \tilde{b}i$

$z_2 = c + \tilde{c}i$



CONTACT WITH REALISTIC THEORIES

- There are BSM scenarios with a non trivial scalar potential (such as MSSM, 2HDM, mLRSM and GUTs). Renormalons may be important for all these examples

- Renormalon's singularities might be important in these examples. The perturbativity issue has already been studied in mLRSM in "A. Maiezza, M. Nemevsˇek and F. Nesti, Perturbativity and mass scales in the minimal left-right symmetric model, Phys. Rev. D94 (2016) 035008, [1603.00360]" and "A. Maiezza, G. Senjanovic´ and J. C. Vasquez, Higgs sector of the minimal left-right symmetric theory, Phys. Rev. D95 (2017) 095004, [1612.09146]"

- The position of the Landau poles at 1-loop can be calculated with the iterative solution of the RGEs

$$\Lambda_{i,Landau} = \mu_0 \exp[\lim_{n\to\infty} |s(n)_i|^{-1/n}],$$

CONCLUSIONS

- We have extended the concept of renormalons to a QFT with an arbitrary number of fields and couplings
- Our aim of to find regions in the parameter space of any model where the perturbative renormalizability is guaranteed
- Analogous to the seminal work of t' Hooft, the renormalons can be identified in terms of the one loop beta functions
- In the generic case regions in the parameter's space of the model emerges where the perturbative renormalization fails
- We provide a method to find such regions both analytically (with some assumptions) and numerically, and then to infer bounds on the couplings of a given model
- We have used a simple toy model with two coupled scalar fields in order to illustrate the emergence of the renormalons
- In summary, any theory might be perturbatively ill-defined even below the Landau poles. To our knowledge, there is no a quantitatively precise measure of the non-perturbative regime of a generic QFT. We have tried to fill up this gap by requiring the Borel resumability in a framework with an arbitrary number of fields and couplings. This enables one to determine safe regions in the parameter space of a given model, where perturbation theory can be consistently used and resummed.

Thank you