QCD (for Colliders)
Lecture 2

Gavin Salam, CERN
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Yesterday:
➤ QCD Lagrangian
➤ Running coupling
➤ Soft gluon emission & its divergences

Today
➤ Real–virtual cancellation
➤ Factorisation
➤ Parton Distribution Functions (PDFs)
➤ Total cross sections & their perturbative series
GLUON EMISSION FROM A QUARK

Consider an emission with
- energy $E \ll \sqrt{s}$ ("soft")
- angle $\theta \ll 1$
  ("collinear" wrt quark)

Examine correction to some hard process with cross section $\sigma_0$

$$d\sigma \simeq \sigma_0 \times \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\theta}$$

This has a divergence when $E \to 0$ or $\theta \to 0$
[in some sense because of quark propagator going on-shell]
How come we get finite cross sections?

Divergences are present in both real and virtual diagrams.

If you are “inclusive”, i.e. your measurement doesn’t care whether a soft/collinear gluon has been emitted then the real and virtual divergences cancel.
Beyond inclusive cross sections: **infrared and collinear (IRC) safety**

**For an observable’s distribution to be calculable in [fixed-order] perturbation theory, the observable should be infra-red safe, i.e. insensitive to the emission of soft or collinear gluons. In particular if $\vec{p}_i$ is any momentum occurring in its definition, it must be invariant under the branching**

$$\vec{p}_i \rightarrow \vec{p}_j + \vec{p}_k$$

whenever $\vec{p}_j$ and $\vec{p}_k$ are parallel [collinear] or one of them is small [infrared].

[QCD and Collider Physics (Ellis, Stirling & Webber)]

**Examples**

Multiplicity of gluons is not IRC safe

[modified by soft/collinear splitting]

Energy of hardest particle is not IRC safe

[modified by collinear splitting]

Energy flow into a cone is IRC safe

[soft emissions don’t change energy flow, collinear emissions don’t change its direction]
A proton–proton collision: INITIAL STATE

proton

proton
A proton–proton collision: FINAL STATE

(actual final-state multiplicity ~ several hundred hadrons)
A proton–proton collision: FILLING IN THE PICTURE
A proton–proton collision: SIMPLIFYING IN THE PICTURE
\[ \sigma (h_1 h_2 \rightarrow ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left( \mu_R^2 \right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} (x_1, \mu_F^2) f_{j/h_2} (x_2, \mu_F^2) \]

\[ \times \hat{\sigma}_{ij \rightarrow ZH+X} (x_1 x_2 s, \mu_R^2, \mu_F^2) + O \left( \frac{\Lambda^2}{M_W^4} \right), \]
The master equation — **Factorisation**

\[
\sigma (h_1 h_2 \rightarrow ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left( \mu_R^2 \right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} \left( x_1, \mu_F^2 \right) f_{j/h_2} \left( x_2, \mu_F^2 \right) \times \hat{\sigma}_{ij \rightarrow ZH+X} \left( x_1 x_2 s, \mu_R^2, \mu_F^2 \right) + \mathcal{O} \left( \frac{\Lambda^2}{M_W^4} \right),
\]

Parton distribution function (PDF): e.g. number of up anti-quarks carrying fraction \( x_2 \) of proton’s momentum
\[ \sigma(h_1h_2 \rightarrow ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left( \mu_R^2 \right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} \left( x_1, \mu_F^2 \right) f_{j/h_2} \left( x_2, \mu_F^2 \right) \]

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Parton distribution function (PDF): e.g. number of up quarks carrying fraction \( x_1 \) of proton’s momentum
\[
\sigma(h_1 h_2 \rightarrow ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left( \mu^2_R \right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} \left( x_1, \mu^2_F \right) f_{j/h_2} \left( x_2, \mu^2_F \right) \\
\times \hat{\sigma}^{(n)}_{i,j \rightarrow ZH + X} \left( x_1 x_2 s, \mu^2_R, \mu^2_F \right) + O \left( \frac{\Lambda^2}{M^4_W} \right),
\]

**Perturbative sum over powers of the strong coupling:** typically we know first 2-4 orders.

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**THE MASTER EQUATION — FACTORIZATION**

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**Diagram:**

- Z and H are bosons, b and \( \bar{b} \) are quarks.
- \( \mu^+ \) and \( \mu^- \) are muons.
- \( \hat{\sigma} \) is the inclusive cross section.

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**Explanation:**

The master equation describes the inclusive cross section for the process \( h_1 h_2 \rightarrow ZH + X \) as a perturbative sum over powers of the strong coupling constant, \( \alpha_s \), with the factorization scale, \( \mu_R \), and the factorization scale, \( \mu_F \). The cross section is given by an infinite series of terms, each contributing to the total cross section at a given order in perturbation theory. The terms involve the parton densities, \( f_{i/h} \), and the hadronic branching ratio, \( \hat{\sigma}^{(n)}_{i,j \rightarrow ZH + X} \), which is the hadronic cross section corrected for perturbative effects. The \( O \left( \frac{\Lambda^2}{M^4_W} \right) \) term represents higher-order corrections, typically first order in the strong coupling and the ratio of scales, \( M_W \) being the top quark mass.
\[
\sigma (h_1 h_2 \rightarrow ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n \left( \frac{\mu_R^2}{\mu_F^2} \right) \sum_{i,j} \int dx_1 dx_2 f_i / h_1 \left( x_1, \frac{\mu^2_F}{\mu^2_R} \right) f_j / h_2 \left( x_2, \frac{\mu^2_F}{\mu^2_R} \right) \\
\times \hat{\sigma}^{(n)}_{ij \rightarrow ZH+X} \left( x_1 x_2 s, \frac{\mu^2_F}{\mu^2_R}, \frac{\mu^2_F}{\mu^2_R} \right) + \mathcal{O} \left( \frac{\Lambda^2}{M_W^4} \right),
\]

At each perturbative order \( n \) we have a specific “hard matrix element” (sometimes several for different subprocesses)
\[ \sigma (h_1h_2 \rightarrow ZH + X) = \sum_{n=0}^{\infty} \alpha_s^n (\mu_R^2) \sum_{i,j} \int dx_1dx_2 f_{i/h_1} (x_1, \mu_F^2) f_{j/h_2} (x_2, \mu_F^2) \times \hat{\sigma}^{(n)}_{ij \rightarrow ZH+X} (x_1x_2s, \mu_R^2, \mu_F^2) + O \left( \frac{\Lambda^2}{M_W^4} \right), \]

Additional corrections from non-perturbative effects (higher “twist”, suppressed by powers of QCD scale (\( \Lambda \)) / hard scale)
PARTON DISTRIBUTION FUNCTIONS (PDFs)
Hadron-hadron is complex because of two incoming partons — so start with simpler Deep Inelastic Scattering (DIS).

Kinematic relations:

\[ x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xys \]

\[ \sqrt{s} = \text{c.o.m. energy} \]

- \( Q^2 \) = photon virtuality \( \leftrightarrow \) transverse resolution at which it probes proton structure
- \( x \) = longitudinal momentum fraction of struck parton in proton
- \( y \) = momentum fraction lost by electron (in proton rest frame)
Deep Inelastic scattering (DIS): example

\[ Q^2 = 25030 \text{ GeV}^2; \quad y = 0.56; \quad x = 0.50 \]
Write DIS X-section to zeroth order in $\alpha_s$ (‘quark parton model’):

$$\frac{d^2\sigma^{em}}{dx dQ^2} \sim \frac{4\pi\alpha^2}{xQ^4} \left( \frac{1 + (1 - y)^2}{2} F^{em}_{2} + \mathcal{O}(\alpha_s) \right)$$

$$\propto F^{em}_{2} \quad \text{[structure function]}$$

$$F_2 = x(e^2_u u(x) + e^2_d d(x)) = x \left( \frac{4}{9} u(x) + \frac{1}{9} d(x) \right)$$

$[u(x), d(x): \text{parton distribution functions (PDF)}]$

**NB:**

- use perturbative language for interactions of up and down quarks
- but distributions themselves have a *non-perturbative* origin.
Higher order corrections from initial state splittings?

For initial state splitting, hard process occurs \textit{after splitting}, and momentum entering hard process is modified: \( p \rightarrow zp \).

\[
\sigma_{g+h}(p) \simeq \sigma_h(zp) \frac{\alpha_s C_F}{\pi} \frac{dz}{1 - z} \frac{dk_t^2}{k_t^2}
\]

For virtual terms, momentum entering hard process is unchanged

\[
\sigma_{V+h}(p) \simeq -\sigma_h(p) \frac{\alpha_s C_F}{\pi} \frac{dz}{1 - z} \frac{dk_t^2}{k_t^2}
\]

Total cross section gets contribution with \textit{two different hard X-sections}

\[
\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \int \frac{dk_t^2}{k_t^2} \frac{dz}{1 - z} \left[ \sigma_h(zp) - \sigma_h(p) \right]
\]

NB: We assume \( \sigma_h \) involves momentum transfers \( \sim Q \gg k_t \), so ignore extra transverse momentum in \( \sigma_h \)
Higher order corrections from initial state splittings?

\[ \sigma_{g+h} + \sigma_{V+h} \approx \frac{\alpha_s C_F}{\pi} \int_0^{Q^2} \frac{dk_t^2}{k_t^2} \int \frac{dz}{1-z} [\sigma_h(zp) - \sigma_h(p)] \]

- In soft limit (\(z \to 1\)), \(\sigma_h(zp) - \sigma_h(p) \to 0\): *soft divergence cancels.*
- For \(1 - z \neq 0\), \(\sigma_h(zp) - \sigma_h(p) \neq 0\), so \(z\) integral is non-zero but finite.

**BUT:** \(k_t\) integral is just a factor, and is *infinite*

This is a collinear \((k_t \to 0)\) divergence.

Cross section with incoming parton is not collinear safe!

This always happens with coloured initial-state particles

So how do we do QCD calculations in such cases?
Parton distributions and DGLAP

➢ Write up-quark distribution in proton as

\[ u(x, \mu_F^2) \]

➢ Perturbative collinear (IR) divergence absorbed into the parton distribution (NB divergence not physical: non-perturbative physics provides a physical cutoff)

➢ \( \mu_F \) is the factorisation scale — a bit like the renormalisation scale (\( \mu_R \)) for the running coupling.

➢ As you vary the factorisation scale, the parton distributions evolve with a renormalisation-group type equation

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations
**DGLAP EQUATION**

*take derivative* wrt factorization scale $\mu^2$

$$
\frac{dq(x, \mu^2)}{d \ln \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 dz \ p_{qq}(z) \frac{q(x/z, \mu^2)}{z} - \frac{\alpha_s}{2\pi} \int_0^1 dz \ p_{qq}(z) \ q(x, \mu^2)
$$

$p_{qq}$ is real $q \leftrightarrow q$ splitting kernel: $p_{qq}(z) = C_F \frac{1 + z^2}{1 - z}$
DGLAP EQUATION

Awkward to write real and virtual parts separately. Use more compact notation:

\[
\frac{dq(x, \mu^2)}{d \ln \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 dz \left( P_{qq}(z) \frac{q(x/z, \mu^2)}{z} \right),
\]

\[P_{qq} = C_F \left( \frac{1 + z^2}{1 - z} \right)_+\]

This involves the *plus prescription*:

\[
\int_0^1 dz \ [g(z)]_+ f(z) = \int_0^1 dz \ g(z) f(z) - \int_0^1 dz \ g(z) f(1)
\]

\(z = 1\) divergences of \(g(z)\) cancelled if \(f(z)\) sufficiently smooth at \(z = 1\)
Proton contains both quarks and gluons — so DGLAP is a matrix in flavour space:

$$\frac{d}{d \ln Q^2} \begin{pmatrix} q \\ g \end{pmatrix} = \begin{pmatrix} P_{q\leftarrow q} & P_{q\leftarrow g} \\ P_{g\leftarrow q} & P_{g\leftarrow g} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}$$

(In general, matrix spanning all flavors, anti-flavors, $P_{qq'} = 0$ (LO), $P_{\bar{q}g} = P_{qg}$)

Splitting functions are:

$$P_{qg}(z) = T_R \left[ z^2 + (1-z)^2 \right], \quad P_{gq}(z) = C_F \left[ \frac{1 + (1-z)^2}{z} \right],$$

$$P_{gg}(z) = 2C_A \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \delta(1-z) \frac{(11C_A - 4n_f T_R)}{6}.$$

Have various symmetries / significant properties, e.g.

- $P_{qg}, P_{gg}$: symmetric $z \leftrightarrow 1 - z$ (except virtuals)
- $P_{qq}, P_{gg}$: diverge for $z \rightarrow 1$ soft gluon emission
- $P_{gg}, P_{gq}$: diverge for $z \rightarrow 0$ Implies PDFs grow for $x \rightarrow 0$

2015 EPS HEP prize to Bjorken, Altarelli, Dokshitzer, Lipatov & Parisi
\[ P^{(1)}_{ps}(x) = 4 \, C_F n_f \left( \frac{20}{9} \, x - 2 + 6x - 4H_0 + x^2 \left[ \frac{8}{3} H_0 - \frac{56}{9} \right] + (1 + x) \left[ 5H_0 - 2H_{0,0} \right] \right) \]

\[ P^{(1)}_{qg}(x) = 4 \, C_A n_f \left( \frac{20}{9} \, x - 2 + 25x - 2\rho_{qg}(-x)H_{-1,0} - 2\rho_{qg}(x)H_{1,1} + x^2 \left[ \frac{44}{3} H_0 - \frac{218}{9} \right] \right) + 4 \, (1 - x) \left[ H_{0,0} - 2H_0 + xH_1 \right] - 4\zeta_2x - 6H_{0,0} + 9H_0 \right) + 4 \, C_F n_f \left( 2\rho_{qg}(x) \left[ H_{1,0} + H_{1,1} + H_2 - \zeta_2 \right] + 4x^2 \left[ H_0 + H_{0,0} + \frac{5}{2} \right] + 2(1 - x) \left[ H_0 + H_{0,0} - 2xH_1 + \frac{29}{4} \right] - \frac{15}{2} - H_{0,0} - \frac{1}{2}H_0 \right) \]

\[ P^{(1)}_{gq}(x) = 4 \, C_A C_F \left( \frac{1}{x} + 2\rho_{gq}(x) \left[ H_{1,0} + H_{1,1} + H_2 - \frac{11}{6} H_1 \right] - x^2 \left[ \frac{8}{3} H_0 - \frac{44}{9} \right] + 4\zeta_2 - 2 - 7H_0 + 2H_{0,0} - 2H_1x + (1 + x) \left[ 2H_{0,0} - 5H_0 + \frac{37}{9} \right] - 2\rho_{gq}(-x)H_{-1,0} \right) - 4 \, C_F n_f \left( \frac{2}{3} x \right) - p_{gq}(x) \left[ \frac{2}{3} H_1 - \frac{10}{9} \right] \right) + 4 \, C_F^2 \left( \rho_{gq}(x) \left[ 3H_1 - 2H_{1,1} \right] + (1 + x) \left[ H_{0,0} - \frac{7}{2} + \frac{7}{2} H_0 \right] - 3H_{0,0} + 1 - \frac{3}{2}H_0 + 2H_1x \right) \]

\[ P^{(1)}_{gg}(x) = 4 \, C_A n_f \left( 1 - x - \frac{10}{9} \rho_{gg}(x) - \frac{13}{9} \left( \frac{1}{x} - x^2 \right) - \frac{2}{3} (1 + x)H_0 - \frac{2}{3} \delta(1 - x) \right) + 4 \, C_A^2 \left( 27 + (1 + x) \left[ \frac{11}{3} H_0 + 8H_{0,0} - \frac{27}{2} \right] + 2\rho_{gg}(-x) \left[ H_{0,0} - 2H_{-1,0} - \zeta_2 \right] - \frac{67}{9} \left( \frac{1}{x} - x^2 \right) - 12H_0 \right) - \frac{44}{3} x^2 H_0 + 2\rho_{gg}(x) \left[ \frac{67}{18} - \zeta_2 + H_{0,0} + 2H_{1,0} + 2H_2 \right] + \delta(1 - x) \left[ \frac{8}{3} + 3\zeta_3 \right] \right) + 4 \, C_F n_f \left( 2H_0 + \frac{2}{3} x + \frac{10}{3} x^2 - 12 + (1 + x) \left[ 4 - 5H_0 - 2H_{0,0} \right] - \frac{1}{2} \delta(1 - x) \right) .

\[ NLO: \]

\[ P_{ab} = \frac{\alpha_s}{2\pi} P^{(0)} + \frac{\alpha_s^2}{16\pi^2} P^{(1)} \]

Curci, Furmanski & Petronzio ‘80
NNLO, $P_{ab}^{(2)}$: Moch, Vermaseren & Vogt '04
Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond

S. Moch\textsuperscript{a}, B. Ruijl\textsuperscript{b,c}, T. Ueda\textsuperscript{b}, J.A.M. Vermaseren\textsuperscript{b} and A. Vogt\textsuperscript{d}

\begin{align*}
P_{ab} &= \frac{\alpha_s}{2\pi} P_{ab}^{(0)} \\
&\quad + \left( \frac{\alpha_s}{2\pi} \right)^2 P_{ab}^{(1)} \\
&\quad + \left( \frac{\alpha_s}{2\pi} \right)^3 P_{ab}^{(2)} \\
&\quad + \left( \frac{\alpha_s}{2\pi} \right)^4 P_{ab}^{(3)}
\end{align*}

DGLAP evolution (initial quarks only)

Take example evolution starting with just quarks:

\[
\partial_{\ln Q^2} q = P_{q\leftrightarrow q} \otimes q \\
\partial_{\ln Q^2} g = P_{g\leftrightarrow q} \otimes q
\]

- quark is depleted at large \( x \)
- gluon grows at small \( x \)
DGLAP evolution (initial quarks only)

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\]

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- gluon grows at small \( x \)
DGLAP evolution (initial gluons only)

2nd example: start with just gluons.

\[
\begin{align*}
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\partial_{\ln Q^2} g &= P_{g \leftarrow g} \otimes g
\end{align*}
\]

- gluon is depleted at large \(x\).
- high-\(x\) gluon feeds growth of small \(x\) gluon & quark.
DGLAP evolution (initial gluons only)

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\[
\frac{\partial}{\partial \ln Q^2} q = P_{q \leftrightarrow g} \otimes g
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\]

- gluon is depleted at large \( x \).
- high-\( x \) gluon feeds growth of small \( x \) gluon & quark.
DGLAP evolution (initial gluons only)

\[ xq(x, Q^2), \ xg(x, Q^2) \]

\[ xg(x, Q^2) \]
\[ xq + xqbar \]

\[ Q^2 = 27.0 \text{ GeV}^2 \]

2nd example: start with just gluons.

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DGLAP evolution (initial gluons only)

\[ xq(x, Q^2), \ xg(x, Q^2) \]

\[ Q^2 = 35.0 \text{ GeV}^2 \]

2nd example: start with just gluons.

\[
\begin{align*}
\frac{\partial}{\partial \ln Q^2} q &= P_{q\leftrightarrow g} \otimes g \\
\frac{\partial}{\partial \ln Q^2} g &= P_{g\leftrightarrow g} \otimes g
\end{align*}
\]

- gluon is depleted at large \( x \).
- high-\( x \) gluon feeds growth of small \( x \) gluon & quark.
DGLAP evolution (initial gluons only)

\[ xq(x, Q^2), xg(x, Q^2) \]

\[ Q^2 = 60.0 \text{ GeV}^2 \]

2nd example: start with just gluons.

\[
\frac{\partial}{\partial \ln Q^2} q = P_{q\leftarrow g} \otimes g \\
\frac{\partial}{\partial \ln Q^2} g = P_{g\leftarrow g} \otimes g
\]

- gluon is depleted at large \( x \).
- high-\( x \) gluon feeds growth of small \( x \) gluon & quark.
DGLAP evolution (initial gluons only)

\[ x_q(x,Q^2), x_g(x,Q^2) \]

\[ Q^2 = 90.0 \text{ GeV}^2 \]

2nd example: start with just gluons.

\[ \frac{\partial}{\partial \ln Q^2} q = P_{q \leftrightarrow g} \otimes g \]
\[ \frac{\partial}{\partial \ln Q^2} g = P_{g \leftrightarrow g} \otimes g \]

- gluon is depleted at large \( x \).
- high-\( x \) gluon feeds growth of small \( x \) gluon & quark.
DGLAP evolution (initial gluons only)

- $xg(x, Q^2)$
- $xq(x, Q^2)$

$Q^2 = 150.0 \text{ GeV}^2$

2nd example: start with just gluons.

\[
\begin{align*}
\partial_{\ln Q^2} q &= P_{q\leftrightarrow g} \otimes g \\
\partial_{\ln Q^2} g &= P_{g\leftrightarrow g} \otimes g
\end{align*}
\]

- Gluon is depleted at large $x$.
- High-$x$ gluon feeds growth of small $x$ gluon & quark.

DGLAP evolution:
- Partons lose momentum and shift towards smaller $x$.
- High-$x$ partons drive growth of low-$x$ gluon.
determining the gluon

which is critical at hadron colliders (e.g. ttbar, Higgs dominantly produced by gluon-gluon fusion), but not directly probed in Deep-Inelastic-Scattering
Consider DIS data — $F_2(x,Q^2)$ — in a world where the proton just had quarks

Fit quark distributions to $F_2(x, Q_0^2)$, at *initial scale* $Q_0^2 = 12 \text{ GeV}^2$.

NB: $Q_0$ often chosen lower.

Assume there is no gluon at $Q_0^2$:

$$g(x, Q_0^2) = 0$$

Use DGLAP equations to evolve to higher $Q^2$; compare with data.
Consider DIS data — $F_2(x,Q^2)$ — in a world where the proton just had quarks.

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Use DGLAP equations to evolve to higher $Q^2$; compare with data.
Consider DIS data — $F_2(x,Q^2)$ — in a world where the proton just had quarks

$F_2^p(x,Q^2)$

DGLAP: $g(x,Q_0^2) = 0$

**ZEUS**

**NMC**

$Q^2 = 35.0 \text{ GeV}^2$

Fit quark distributions to $F_2(x, Q_0^2)$, at *initial scale* $Q_0^2 = 12 \text{ GeV}^2$.

NB: $Q_0$ often chosen lower

Assume there is no gluon at $Q_0^2$:

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Use DGLAP equations to evolve to higher $Q^2$; compare with data.
Consider DIS data — $F_2(x,Q^2)$ — in a world where the proton just had quarks

Fit quark distributions to $F_2(x, Q_0^2)$, at initial scale $Q_0^2 = 12 \text{ GeV}^2$. NB: $Q_0$ often chosen lower

Assume there is no gluon at $Q_0^2$:

$$g(x, Q_0^2) = 0$$

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$$g(x, Q_0^2) = 0$$

Use DGLAP equations to evolve to higher $Q^2$; compare with data.

COMPLETE FAILURE to reproduce data evolution
Consider DIS data — $F_2(x,Q^2)$ — with specially tuned gluon

If gluon $\neq 0$, splitting

\[ g \rightarrow q \bar{q} \]

generates extra quarks at large $Q^2$ $\Rightarrow$ faster rise of $F_2$

Global PDF fits ($CT$, $MMHT$, $NNPDF$, etc.) choose gluon distribution that leads to the correct $Q^2$ evolution.
Consider DIS data — $F_2(x,Q^2)$ — with specially tuned gluon

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Global PDF fits (CT, MMHT, NNPDF, etc.) choose gluon distribution that leads to the correct $Q^2$ evolution.
Resulting gluon distribution is **HUGE**!

Carries **47% of proton’s momentum**
(at scale of 100 GeV)

Crucial in order to satisfy momentum sum rule.

Large value of gluon has big impact on phenomenology
Figure 2.1: The kinematic coverage of the NNPDF3.1 dataset in the \( (x, Q^2) \) plane.
Three Global PDF Fits: CT14, MMHT2014, NNPDF30/31

Precision of today's PDFs (from PDF4LHC)

Determining full PDFs

Back to factorization

\[
\frac{d}{d \log Q^2} \left( \frac{u(x, Q^2)}{u(x, Q_0^2)} \right) \text{ [ref]}
\]

\[
\frac{s(x, Q^2)}{s(x, Q_0^2)} \text{ [ref]}
\]
Lepton charge asym. v. CT14 @ D0 & CMS

- **Electron charge asymmetry**
  - D0, L=9.7 fb⁻¹
  - CMS, √s=7 TeV, L=4.7 [fb]⁻¹

- **Muon charge asymmetry**
  - P_T > 35 GeV
  - CT14, 68% C.L.

**MMHT v. Z rapidity @ CMS**

- CMS, 60 < M < 120 GeV
- MMHT vs. CMS
- NLO, ATLAS jets (7 TeV), 0.0 < |y| < 0.3

**ATLAS inclusive jets ratio to MMHT**

- Data/Theory vs. p_T [GeV]
- 100 to 1000 GeV
Let us now turn to the strangeness PDF. Constraints on strangeness PDF from CCFR, NuTeV, and LHC experiments

NLO, ATLAS jets (7 TeV), 0.0 < |y| < 0.3

ATLAS inclusive jets ratio to MMHT

CMS Z, 60 < M < 120 GeV

MMHT v. Z rapidity @ CMS

D0, L=9.7 fb^{-1}
NB: top-quark mass choice affects this plot

cross-section ratios \((W^+/W^-, \text{ttbar}/Z)\) show tensions with some PDFs

\[ \alpha_s(M_Z) = 0.113 \]
FINAL REMARKS ON PDFS

➤ In range \(10^{-3} < x < 0.1\), core PDFs (up, down, gluon) known to ~ few % accuracy

➤ For many LHC applications, you can use PDF4LHC15 set, which merges CT14, MMHT2014, NNPDF30

➤ Situation is not full consensus: e.g. ABMP group claims substantially different gluon distribution

For visualisations of PDFs and related quantities, a good place to start is

http://apfel.mi.infn.it/ (ApfelWeb)
We discussed the “Master” formula

$$\sigma (h_1 h_2 \rightarrow W + X) = \sum_{n=0}^{\infty} \alpha_s^n \left( \mu_R^2 \right) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} (x_1, \mu_F^2) f_{j/h_2} (x_2, \mu_F^2)$$

$$\times \hat{\sigma}^{(n)}_{i,j \rightarrow W + X} (x_1 x_2 s, \mu_R^2, \mu_F^2) + O \left( \frac{\Lambda^2}{M_W^4} \right),$$

and its main inputs

- the strong coupling \(\alpha_s\)
- Parton Distribution Functions (PDFs)

Next: we discuss the actual scattering cross section
We discussed the “Master” formula

\[
\sigma (h_1 h_2 \rightarrow W + X) = \sum_{n=0}^{\infty} \alpha_s^n (\mu_R^2) \sum_{i,j} \int dx_1 dx_2 f_{i/h_1} (x_1, \mu_F^2) f_{j/h_2} (x_2, \mu_F^2)
\]

\[
\times \hat{\sigma}_{i,j \rightarrow W+X}^{(n)} (x_1 x_2 s, \mu_R^2, \mu_F^2) + \mathcal{O} \left( \frac{\Lambda^2}{M_W^4} \right),
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\]

and its main inputs

- the strong coupling \( \alpha_s \)
- Parton Distribution Functions (PDFs)

Next: we discuss the actual scattering cross section
the hard cross section

\[ \sigma \sim \sigma_2 \alpha_s^2 + \sigma_3 \alpha_s^3 + \sigma_4 \alpha_s^4 + \sigma_5 \alpha_s^5 + \cdots \]

LO NLO NNLO N3LO
INGREDIENTS FOR A CALCULATION (generic 2→2 process)

<table>
<thead>
<tr>
<th>LO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tree</td>
</tr>
<tr>
<td>2→2</td>
</tr>
</tbody>
</table>

2

To illustrate the concepts, we don’t care what the particles are — just draw lines.
INGREDIENTS FOR A CALCULATION (generic $2 \rightarrow 2$ process)

**LO**

- **Tree** $2 \rightarrow 2$

**NLO**

- **Tree** $2 \rightarrow 3$
- **1-loop** $2 \rightarrow 2$

$\times$ + complex conj.

To illustrate the concepts, we don’t care what the particles are — just draw lines.
INGREDIENTS FOR A CALCULATION (generic 2\rightarrow2 process)

Tree
2\rightarrow4

1-loop
2\rightarrow3

2-loop
2\rightarrow2

1-loop
2\rightarrow2

NNLO

\[ + \text{complex conj.} \]
\[
\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = R_0 \left( 1 + 0.32\alpha_s + 0.14\alpha_s^2 - 0.47\alpha_s^3 - 0.59316\alpha_s^4 + \cdots \right)
\]

[\alpha_s \equiv \alpha_s(\sqrt{s_{e^+e^-}})]

This is one of the few quantities calculated to N4LO

Good convergence of the series at every order
(at least for \(\alpha_s(M_Z) = 0.118\))

Baikov et al., 1206.1288
(numbers for \(\gamma\)-exchange only)
$\sigma(pp \to H) = (961 \text{ pb}) \times (\alpha_s^2 + 10.4\alpha_s^3 + 38\alpha_s^4 + 48\alpha_s^5 + \cdots)$

$\alpha_s \equiv \alpha_s(M_H/2)$

$\sqrt{s_{pp}} = 13 \text{ TeV}$

*Anastasiou et al., 1602.00695 (ggF, hEFT)*

pp→H (via gluon fusion) is one of only two hadron-collider processes known at N3LO (the other is pp→H via weak-boson fusion)

The series does not converge well (explanations for why are only moderately convincing)
On previous page, we wrote the series in terms of powers of $\alpha_s(M_H/2)$

But we are free to rewrite it in terms of $\alpha_s(\mu)$ for any choice of “renormalisation scale” $\mu$.

$$\sigma(pp \rightarrow H) = \sigma_0 \times \alpha_s^2(\mu)$$
On previous page, we wrote the series in terms of powers of $\alpha_s(M_H/2)$

But we are free to rewrite it in terms of $\alpha_s(\mu)$ for any choice of “renormalisation scale” $\mu$.

\[
\sigma(pp \to H) = \sigma_0 \times \left( \alpha_s^2(\mu) + (10.4 + 2b_0 \ln \frac{\mu^2}{\mu_0^2}) \alpha_s^3(\mu) \right)
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\]
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\]
 SCALE DEPENDENCE

➢ On previous page, we wrote the series in terms of powers of $\alpha_s(M_H/2)$

➢ But we are free to rewrite it in terms of $\alpha_s(\mu)$ for any choice of “renormalisation scale” $\mu$.

$$\sigma(pp \rightarrow H) = \sigma_0 \times \left( \alpha_s^2(\mu) \right) + (10.4 + 2b_0 \ln \frac{\mu^2}{\mu_0^2})\alpha_s^3(\mu)$$

scale dependence (an intrinsic uncertainty) gets reduced as you go to higher order
Convention: “theory uncertainty” (i.e. from missing higher orders) is estimated by change of cross section when varying $\mu$ in range $1/2 \rightarrow 2$ around central value

Here, only the renorm. scale $\mu$ has been varied. In real life you need to change renorm. and factorisation scales.
Convention: “theory uncertainty” (i.e. from missing higher orders) is estimated by change of cross section when varying $\mu$ in range $1/2 \rightarrow 2$ around central value.

Here, only the renorm. scale $\mu(=\mu_R)$ has been varied. In real life you need to change renorm. and factorisation ($\mu_F$) scales.
WHAT DO WE KNOW?

➤ LO: almost any process (with MadGraph, ALPGEN, etc.)

➤ NLO: most processes (with MCFM, NLOJet++, MG5_aMC@NLO, POWHEG, OpenLoops/Blackhat/NJet/Gosam/etc.+Sherpa)

➤ NNLO: all 2→1 and most 2→
  (top++, DY/HNNLO, FEWZ, MATRIX, MCFM, NNLOJet, etc.)

➤ N3LO: pp → Higgs via gluon fusion and weak-boson fusion both in approximations (EFT, QCD₁ × QCD₂)

➤ NLO EW corrections, i.e. relative αEW rather than αs:
  most 2→1, 2→2 and 2→3
EXTRA SLIDES
Higgs cross sections

Figure 178: The SM Higgs boson production cross sections as a function of the LHC centre of mass energy.
how close are scale variations to being 1σ uncertainty?
Bagnaschi, Cacciari, Guffanti, Jenniches (1409.5036)

<table>
<thead>
<tr>
<th>Observable</th>
<th>Leading order in $\alpha_s$</th>
<th>Highest known order in $\alpha_s$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pp \rightarrow H$</td>
<td>2</td>
<td>4</td>
<td>HIGLU [27, 28]</td>
</tr>
<tr>
<td>$pp \rightarrow b\bar{b} \rightarrow H$</td>
<td>0</td>
<td>2</td>
<td>bbh@nnlo [29]</td>
</tr>
<tr>
<td>$pp \rightarrow t\bar{t}$</td>
<td>2</td>
<td>4</td>
<td>top++ [30]</td>
</tr>
<tr>
<td>$pp \rightarrow Z \rightarrow e^+e^-$</td>
<td>0</td>
<td>2</td>
<td>DYNNLO [31]</td>
</tr>
<tr>
<td>$pp \rightarrow W^+ \rightarrow e^+\nu_e$</td>
<td>2</td>
<td>2</td>
<td>DYNNLO</td>
</tr>
<tr>
<td>$pp \rightarrow W^- \rightarrow e^-\bar{\nu}_e$</td>
<td>0</td>
<td>2</td>
<td>DYNNLO</td>
</tr>
<tr>
<td>$pp \rightarrow Z^* \rightarrow ZH$</td>
<td>0</td>
<td>2</td>
<td>vh@nnlo [32]</td>
</tr>
<tr>
<td>$pp \rightarrow W^{\pm*} \rightarrow W^{\pm}H$</td>
<td>0</td>
<td>2</td>
<td>vh@nnlo</td>
</tr>
<tr>
<td>$pp \rightarrow b\bar{b}$</td>
<td>2</td>
<td>3</td>
<td>MCFM [33, 34]</td>
</tr>
<tr>
<td>$pp \rightarrow Z + j$</td>
<td>1</td>
<td>2</td>
<td>MCFM</td>
</tr>
<tr>
<td>$pp \rightarrow Z + 2j$</td>
<td>2</td>
<td>3</td>
<td>MCFM</td>
</tr>
<tr>
<td>$pp \rightarrow W^{\pm} + j$</td>
<td>1</td>
<td>2</td>
<td>MCFM</td>
</tr>
<tr>
<td>$pp \rightarrow W^{\pm} + 2j$</td>
<td>2</td>
<td>3</td>
<td>MCFM</td>
</tr>
<tr>
<td>$pp \rightarrow ZZ$</td>
<td>0</td>
<td>1</td>
<td>MCFM</td>
</tr>
<tr>
<td>$pp \rightarrow WW$</td>
<td>0</td>
<td>1</td>
<td>MCFM</td>
</tr>
</tbody>
</table>

Table 2: List of hadronic observables used in the global survey.

Figure 3.2: Fraction of observables whose known higher order is found to be contained within the uncertainty interval given by renormalisation and factorisation scale variation between $\mu_{r,f} = Q/r$ and $\mu_{r,f} = rQ$ with the constraint $1/r \leq \mu_r/\mu_f \leq r$. Only the seven points at the extremes and at the centre of the scale-variation interval are used. NNLO-evolved PDFs are used with all perturbative orders.
For many processes NNLO scale band is ~±2%
But only in 3/17 cases is NNLO (central) within NLO scale band…