Heavy Flavor Wilson Coefficients for Deeply Inelastic Scattering

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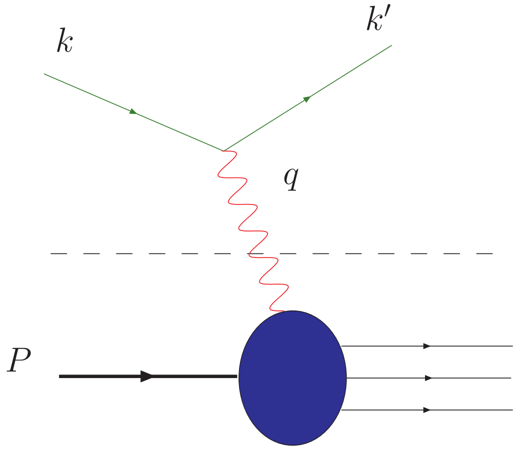
- Introduction and Theory Status
- The Method
- Renormalization
- Asymptotic 3–Loop Results (Fixed Moments) & Anomalous Dimensions
- Conclusions

1. Introduction

Deep–Inelastic Scattering:



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjorken–x}$$

$$\nu := \frac{Pq}{M} \ ,$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle$$

$$\text{unpol.} \left\{ \ = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \right.$$

$$\text{pol.} \left\{ \quad -\frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq} p^\beta\right) g_2(x,Q^2)\right] \ . \right.$$

Structure Functions: $F_{2,L}$, $\quad g_{1,2}$

contain light and heavy quark contributions.

Theory Status of Heavy Quark Corrections

Leading Order : $F_{2,L}(x,Q^2)$ [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Leading Order : $g_1(x,Q^2)$ [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

Leading Order : $g_2(x,Q^2)$ [Blümlein, Ravindran, van Neerven, 2003]

Soft Resummation: $F_{2,L}(x,Q^2)$ [Laenen & Moch, 1998; Alekhin & Moch, 2008]

Next-to-Leading Order : $F_{2,L}(x,Q^2)$ [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, Blümlein, S.K., 2007]

Mellin–space expressions: [Alekhin, Blümlein, 2003.].

Next-to-Leading Order : $g_1(x,Q^2)$ asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1997; Bierenbaum, Blümlein, S.K., 2009]

Next-to-Next-to-Leading Order : $F_L(x,Q^2)$ asymptotic:

[Blümlein, De Freitas, S.K., van Neerven, 2006.]

$O(\alpha_s^3)$: Light flavor Wilson coefficients: [Moch, Vermaseren, Vogt, 2005.]

$\Longrightarrow$ 3-Loop corrections needed to obtain the same accuracy as reached for the light flavor contributions.

Need for the Calculation:

- Heavy flavor (charm) contributions to DIS structure functions are rather large
  $\Longrightarrow$ Precision understanding of structure functions is required

- Precision determination of $\alpha_s(M_Z^2)$:

$\alpha_s(M_Z^2) = 1141^{+0.0020}_{-0.0022}$	N^3LO	NS-analysis [1]
$\alpha_s(M_Z^2) = 1135^{+0.0014}_{-0.0014}$	N^2LO	S+NS-analysis [2]

$$\textbf{Aim}: \quad \delta\alpha_s/\alpha_s \ < 1\ \%.$$

[1] Blümlein, Böttcher, Guffanti, 2007; [2] Alekhin, Blümlein, S.K., Moch, 2009.

- Precise determination of the gluon and sea quark distributions.

Goal:

- Calculation of the heavy flavor Wilson coefficients to higher orders for $Q^2 \geq 25\,\mathrm{GeV}^2$ [sufficient in many applications].
- Application to the polarized (NLO) and transversity (NLO,NNLO) case.

2. The Method

- In Bjorken limit, $\{Q^2,\ \nu\} \rightarrow \infty$, x fixed, at twist $\tau = 2$–level: mass factorization of the structure functions into Wilson coefficients and parton densities:

$$\underbrace{F_i(x,Q^2)}_{\text{structure functions}} = \sum_j \underbrace{\mathcal{C}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{parton densities, non-perturbative}},$$

- Process dependent Wilson coefficients contain both light and heavy flavor contributions:

$$\mathcal{C}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right) = C_{i,j}^{\text{light}}\left(x, \frac{Q^2}{\mu^2}\right) + \mathsf{H}_{i,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m_k^2}{\mu^2}\right), k = c, b\ .$$

- Heavy quark contributions given by heavy quark Wilson coefficients

$$\mathsf{H}_{(2,L),i}^{\text{S}}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{Q^2}\right) = \underbrace{H_{(2,L),i}^{\text{S}}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{Q^2}\right)}_{\gamma + q_{heavy} \rightarrow X} + \underbrace{L_{(2,L),i}^{\text{S,NS}}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{Q^2}\right)}_{\gamma + q_{light} \rightarrow X}$$

- Consider only one species of heavy quarks

- Factorization for $F_2^{Q\overline{Q}}(x, Q^2)$ at the level of twist $\tau = 2$:

$$
\begin{aligned}
F_2^{Q\overline{Q}}(n_f, x, Q^2, m^2) = \sum_{k=1}^{n_f} e_k^2 \Biggl\{ & \quad L_{2,q}^{\mathsf{NS}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(n_f, x, \mu^2) + f_{\overline{k}}(n_f, x, \mu^2)\right] \\
& + \tilde{L}_{2,q}^{\mathsf{PS}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(n_f, x, \mu^2) \\
& + \tilde{L}_{2,g}^{\mathsf{S}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(n_f, x, \mu^2) \Biggr\} \\
+ e_Q^2 \Biggl\{ & \quad H_{2,q}^{\mathsf{PS}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(n_f, x, \mu^2) \\
& + H_{2,g}^{\mathsf{S}}\left(n_f, x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(n_f, x, \mu^2) \Biggr\} .
\end{aligned}
$$

- In the limit $Q^2 \gg m^2$ [$Q^2 \approx 10\ m^2$ for F_2, g_1]: massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H^{\rm S}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\rm S}_{k,i}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\rm S}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}} .$$

- Formula holds for completely inclusive quantities $\Longrightarrow$ Virtual heavy quarks are included.
- Similar formula for $L^{S,NS}_{(2,L),i}$. Holds for polarized and unpolarized case.
- OMEs obey expansion

$$A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\rm S,NS}_k|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A^{\rm S,NS,(l)}_{k,i}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- OMEs occur as well in the definition of a variable flavor number scheme starting from a fixed flavor number scheme.

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- Expansion up to $O(a_s^3)$ of L_2 and H_2 reads

$$
\begin{aligned}
L_{2,q}^{\mathsf{NS}}(n_f) &= a_s^2\left[A_{qq,Q}^{\mathsf{NS},(2)}(n_f)+\hat{C}_{2,q}^{\mathsf{NS},(2)}(n_f)\right]+a_s^3\left[A_{qq,Q}^{\mathsf{NS},(3)}(n_f)+A_{qq,Q}^{\mathsf{NS},(2)}(n_f)C_{2,q}^{\mathsf{NS},(1)}(n_f)+\hat{C}_{2,q}^{\mathsf{NS},(3)}(n_f)\right]\\
\tilde{L}_{2,q}^{\mathsf{PS}}(n_f) &= a_s^3\left[\tilde{A}_{qq,Q}^{\mathsf{PS},(3)}(n_f)+A_{gq,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)+\hat{\tilde{C}}_{2,q}^{\mathsf{PS},(3)}(n_f)\right]\\
\tilde{L}_{2,g}^{\mathsf{S}}(n_f) &= a_s^2 A_{gg,Q}^{(1)}(n_f)\tilde{C}_{2,g}^{(1)}(n_f+1)+a_s^3\Big[\tilde{A}_{qg,Q}^{(3)}(n_f)+A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(2)}(n_f+1)\\
&\quad +A_{gg,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{Qg}^{(1)}(n_f)\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)+\hat{\tilde{C}}_{2,g}^{(3)}(n_f)\Big]\\
H_{2,q}^{\mathsf{PS}}(n_f) &= a_s^2\left[A_{Qq}^{\mathsf{PS},(2)}+\tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right]+a_s^3\left[A_{Qq}^{\mathsf{PS},(3)}+\tilde{C}_{2,q}^{\mathsf{PS},(3)}(n_f+1)+A_{gq,Q}^{(2)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{Qq}^{\mathsf{PS},(2)}\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)\right]\\
H_{2,g}^{\mathsf{S}}(n_f) &= a_s\left[A_{Qg}^{(1)}+\tilde{C}_{2,g}^{(1)}(n_f+1)\right]+a_s^2\left[A_{Qg}^{(2)}+A_{Qg}^{(1)}\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)+A_{gg,Q}^{(1)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+\tilde{C}_{2,g}^{(2)}(n_f+1)\right]\\
&+ a_s^3\Big[A_{Qg}^{(3)}+A_{Qg}^{(2)}\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)+A_{gg,Q}^{(2)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{gg,Q}^{(1)}\ \tilde{C}_{2,g}^{(2)}(n_f+1)\\
&\quad +A_{Qg}^{(1)}\left[C_{2,q}^{\mathsf{NS},(2)}(n_f+1)+\tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right]+\tilde{C}_{2,g}^{(3)}(n_f+1)\Big]\ .
\end{aligned}
$$

- n_f–dependence non–trivial: $\hat{f}(n_f)\equiv f(n_f+1)-f(n_f)$, $\tilde{f}(n_f)\equiv f(n_f)/n_f$.

- Highlighted terms are (partially) due to virtual heavy quark loops and have to be included for an inclusive description in the $\overline{\text{MS}}$–scheme.

- Comparison to exact order $O(a_s^2)$ result: asymptotic formulae valid for $Q^2 \gtrsim 20\ (\text{GeV}/c)^2$ in case of $F_2^{c\bar{c}}(x,Q^2)$ and $Q^2 \gtrsim 800\ (\text{GeV}/c)^2$ for $F_L^{c\bar{c}}(x,Q^2)$

Renormalization

- Mass renormalization (on–mass shell scheme)
- Charge renormalization: MOM scheme for the gluon propagator.
  MOM scheme → $\overline{\text{MS}}$ scheme:

$$a_s^{\text{MOM}} = a_s^{\overline{\text{MS}}} - \beta_{0,Q}\ln\left(\frac{m^2}{\mu^2}\right){a_s^{\overline{\text{MS}}}}^2 + \left[\beta_{0,Q}^2\ln^2\left(\frac{m^2}{\mu^2}\right) - \beta_{1,Q}\ln\left(\frac{m^2}{\mu^2}\right) - \beta_{1,Q}^{(1)}\right]{a_s^{\overline{\text{MS}}}}^3 .$$

  $\Longrightarrow$ Accounts at NLO for virtual heavy quark loops.
- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities into Γ–factors Γ_{NS}, $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1}\hat{A}_{lm}\Gamma_{mj}^{-1}$$

$\Longrightarrow$ $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3 loops.
$\Longrightarrow$ All 2–loop $O(\varepsilon)$–terms in the unpolarized case are known.
[Bierenbaum, Blümlein, S.K., Schneider, 2008; Bierenbaum, Blümlein, S.K., 2009.]

4. Fixed moments at 3–Loop

Contributing OMEs:

Singlet	A_{Qg}	A_{Qg}	$A_{gg,Q}$	$A_{gq,Q}$	} mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		
Non–Singlet	$A_{qq,Q}^{\rm NS,+}$	$A_{qq,Q}^{\rm NS,-}$	$A_{qq,Q}^{\rm NS,v}$		

- Unpolarized anomalous dimensions are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]
  $\Longrightarrow$ All terms needed for the renormalization of unpolarized 3–Loop heavy OMEs are present.
  $\Longrightarrow$ The calculation provides first independent checks on $\gamma_{qg}^{(2)}$, $\gamma_{qq}^{(2),\rm PS}$ and on respective color projections of $\gamma_{qq}^{(2),\rm NS\pm}$, $\gamma_{gg}^{(2)}$ and $\gamma_{gq}^{(2)}$.

- The calculation proceeds in the same way in the polarized case.

- Calculation in Mellin space.
  For fixed N: three–loop "self-energy" type diagrams with an operator insertion
  $\Longrightarrow$ Calculation using MATAD [Steinhauser, 2001] and FORM [Vermaseren, 2000].

General structure of the result: the PS –case

$$A_{Qq}^{(3),\mathsf{PS},\overline{\mathrm{MS}}} = \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{48}\Big\{\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 16\beta_{0,Q}\Big\}\ln^3\left(\frac{m^2}{\mu^2}\right)$$
$$+\frac{1}{8}\Big\{-4\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\left(\beta_0+\beta_{0,Q}\right) + \hat{\gamma}_{qg}^{(0)}\left(\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)}\right) - \gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}\Big\}\ln^2\left(\frac{m^2}{\mu^2}\right)$$
$$+\frac{1}{16}\Big\{8\,\hat{\gamma}_{qq}^{(2),\mathsf{PS}} - 8n_f\hat{\tilde{\gamma}}_{qq}^{(2),\mathsf{PS}} - 32a_{Qq}^{(2),\mathsf{PS}}(\beta_0+\beta_{0,Q}) + 8\hat{\gamma}_{qg}^{(0)}a_{gq,Q}^{(2)} - 8\gamma_{gq}^{(0)}a_{Qg}^{(2)}$$
$$-\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}\left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 8\beta_{0,Q}\right)\Big\}\ln\left(\frac{m^2}{\mu^2}\right)$$
$$+4(\beta_0+\beta_{0,Q})\overline{a}_{Qq}^{(2),\mathsf{PS}} + \gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)} - \hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} + \zeta_3\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{48}\left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0\right)$$
$$+\frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(1)}\zeta_2}{16} + C_F\left(-(4+\frac{3}{4}\zeta_2)\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\mathsf{PS}} + 12a_{Qq}^{(2),\mathsf{PS}}\right) + a_{Qq}^{(3),\mathsf{PS}}\,.$$

All terms but $a_{Qq}^{(3),\mathsf{PS}}$ known for all N.

- There are similar formulas for the remaining OMEs.

5. Results

Using MATAD, we calculated the terms ($\approx$ 250 days of computer time)

$$A_{Qq}^{(3),\mathrm{PS}} : \quad (2,4,..,12); \quad A_{qq,Q}^{(3),\mathrm{PS}}, A_{gq,Q}^{(3)} : \quad (2,4,..,14);$$
$$A_{qq,Q}^{(3),\mathrm{NS}\pm} : \quad (2,3,..,14); \quad A_{Q(q)g}^{(3)}, A_{gg,Q}^{(3)} : \quad (2,4,..,10);$$

and find agreement with the predictions obtained from renormalization.

Example: non–logarithmic term of $A_{Qg}^{(3)}$ for $N = 2$

$$\begin{aligned}A_{Qg}^{(3),\overline{\mathrm{MS}}}(\mu^2=m^2, N=2) = {} & T_F {C_A}^2 \left(\frac{174055}{4374} - \frac{88}{9}\mathsf{B}_4 + 72\zeta_4 - \frac{29431}{324}\zeta_3\right) \\ & + T_F C_F C_A \left(-\frac{18002}{729} + \frac{208}{9}\mathsf{B}_4 - 104\zeta_4 + \frac{2186}{9}\zeta_3 - \frac{64}{3}\zeta_2 + 64\zeta_2\ln(2)\right) \\ & + T_F {C_F}^2 \left(-\frac{8879}{729} - \frac{64}{9}\mathsf{B}_4 + 32\zeta_4 - \frac{701}{81}\zeta_3 + 80\zeta_2 - 128\zeta_2\ln(2)\right) + {T_F}^2 C_A \left(-\frac{21586}{2187} + \frac{3605}{162}\zeta_3\right) \\ & + {T_F}^2 C_F \left(-\frac{55672}{729} + \frac{889}{81}\zeta_3 + \frac{128}{3}\zeta_2\right) + n_f {T_F}^2 C_A \left(-\frac{7054}{2187} - \frac{704}{81}\zeta_3\right) + n_f {T_F}^2 C_F \left(-\frac{22526}{729} + \frac{1024}{81}\zeta_3 - \frac{64}{3}\zeta_2\right) .\end{aligned}$$

The constant terms: N = 10 $a_{Qg}^{(3)}$ & $a_{qg,Q}^{(3)}$:

$$
\begin{aligned}
a_{Qg}^{(3)}\Big|_{N=10} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Bigg\{ n_f T_F \Bigg(C_A \left[-\frac{1505896}{245025}\zeta_3 + \frac{189965849}{188669250}\zeta_2 + \frac{297277185134077151}{15532837481700000} \right] \\
& + C_F \left[\frac{62292104}{13476375}\zeta_3 - \frac{49652772817}{93391278750}\zeta_2 - \frac{1178560772273339822317}{107642563748181000000} \right] \Bigg) + C_A^2 \Bigg[-\frac{563692}{81675} B_4 \\
& + \frac{483988}{9075}\zeta_4 - \frac{103652031822049723}{415451499724800}\zeta_3 - \frac{20114890664357}{581101290000}\zeta_2 \\
& + \frac{6830363463566924692253659}{685850575063965696000000} \Bigg] + C_A C_F \Bigg[\frac{1286792}{81675} B_4 - \frac{643396}{9075}\zeta_4 \\
& - \frac{7618971674774379 07}{33236119977984000}\zeta_3 + \frac{15455008277}{660342375}\zeta_2 + \frac{87220147948647179788995 7487}{29928025093700321280000 00} \Bigg] \\
& + C_F^2 \Bigg[-\frac{11808}{3025} B_4 + \frac{53136}{3025}\zeta_4 + \frac{9636017147214304991}{7122025709568000}\zeta_3 + \frac{14699237127551}{15689734830000}\zeta_2 \\
& - \frac{2479301473496359601488696 54541}{1481437242138165903360000 00} \Bigg] + T_F C_A \Bigg[\frac{4206955789}{377338500}\zeta_2 + \frac{123553074914173}{5755172290560}\zeta_3 \\
& + \frac{2323118975810619964 5229}{633397356480430080000} \Bigg] + T_F C_F \Bigg[-\frac{502987059528463}{113048027136000}\zeta_3 + \frac{24683221051}{46695639375}\zeta_2 \\
& - \frac{1831993118263044461191 2149}{1410892611560158003200000} \Bigg] - \frac{896}{1485} T_F^2 \zeta_3 \Bigg\} .
\end{aligned}
$$

$$
\begin{aligned}
a_{qg,Q}^{(3)}\Big|_{N=10} = \; & n_f T_F^2 \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Bigg\{ C_A \left[-\frac{1505896}{245025}\zeta_3 + \frac{1109186999}{377338500}\zeta_2 + \frac{6542127929072987}{191763425700000} \right] \\
& + C_F \left[\frac{62292104}{13476375}\zeta_3 - \frac{83961181063}{93391278750}\zeta_2 - \frac{353813854966442889041}{21528512749636200000} \right] \Bigg) \Bigg\}
\end{aligned}
$$

- We obtain e.g. for the moments of the $\hat{\gamma}_{qg}^{(2)}$ anomalous dimension

N	$\hat{\gamma}_{qg}^{(2)}/T_F$
2	$(1+2n_f)T_F\left(\frac{8464}{243}C_A-\frac{1384}{243}C_F\right)+\frac{\zeta_3}{3}\left(-416C_AC_F+288C_A^2+128C_F^2\right)-\frac{7178}{81}C_A^2+\frac{556}{9}C_AC_F-\frac{8620}{243}C_F^2$
4	$(1+2n_f)T_F\left(\frac{4481539}{303750}C_A+\frac{9613841}{3037500}C_F\right)+\frac{\zeta_3}{25}\left(2832C_A^2-3876C_AC_F+1044C_F^2\right)$ $-\frac{295110931}{3037500}C_A^2+\frac{278546497}{2025000}C_AC_F-\frac{757117001}{12150000}C_F^2$
6	$(1+2n_f)T_F\left(\frac{86617163}{11668860}C_A+\frac{1539874183}{340341750}C_F\right)+\frac{\zeta_3}{735}\left(69864C_A^2-94664C_AC_F+24800C_F^2\right)$ $-\frac{58595443051}{653456160}C_A^2+\frac{1199181909343}{8168202000}C_AC_F-\frac{2933980223981}{40841010000}C_F^2$
8	$(1+2n_f)T_F\left(\frac{10379424541}{2755620000}C_A+\frac{7903297846481}{1620304560000}C_F\right)+\zeta_3\left(\frac{128042}{1575}C_A^2-\frac{515201}{4725}C_AC_F+\frac{749}{27}C_F^2\right)$ $-\frac{24648658224523}{289340100000}C_A^2+\frac{4896295442015177}{32406091200000}C_AC_F-\frac{4374484944665803}{56710659600000}C_F^2$
10	$(1+2n_f)T_F\left(\frac{1669885489}{988267500}C_A+\frac{1584713325754369}{323600780868750}C_F\right)+\zeta_3\left(\frac{1935952}{27225}C_A^2-\frac{2573584}{27225}C_AC_F+\frac{70848}{3025}C_F^2\right)$ $-\frac{21025430857658971}{255684567600000}C_A^2+\frac{926990216580622991}{6040547909550000}C_AC_F-\frac{1091980048536213833}{13591232796487500}C_F^2$

- Agreement for the terms $\propto T_F$ of the anomalous dimensions $\gamma_{ij}^{(2),\mathsf{NS}^\pm,\ \mathsf{S},\ \mathsf{PS}}$ with [Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]
- How far can we go ? $N = 14$ in some cases; generally: $N = 10 \Longrightarrow$ Phenomenology
- Unfortunately not enough to perform the automatic fixed moments $\to$ all moments turn. [Blümlein, Kauers, S.K., Schneider, 0902.4091 [hep-ph]].
- With B. Tödtli: Calculation of moments $N = 1, ..., 13$ of the transversity heavy OMEs $A_{qq}^{h,(2,3)}$ [Blümlein, S.K., Tödtli, 0909.1547 [hep-ph]].
  $\Longrightarrow$ Agreement with anomalous dimensions $\gamma_{qq}^{h,(1,2)}$ from
  [Kumano, 1997; 2–Loop: Hayashigaki, Kanazawa, Koike, 1997; Vogelsang, 1998; 3 Loop, N$\leq$ 8: Gracey, 2006]

6. Conclusions

- The heavy flavor contributions to F_2 are rather large in the region of lower values of x.
- QCD precision analyses require the description of the heavy quark contributions to 3–loops.
- Complete analytic results are known in the region $Q^2 \gg m^2$ at NLO for $F_{2,L}^{Q\overline{Q}}(x, Q^2), g_{1,2}^{Q\overline{Q}}(x, Q^2)$. They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients.
- $F_L^{Q\overline{Q}}(x, Q^2)$ is known to NNLO for $Q^2 \gg m^2$.
- The calculation of fixed moments of the massive operator matrix elements at $O(a_s^3)$ has been finished for $N = 10, 12, 14$
  $\Longrightarrow F_2^{Q\overline{Q}}(x, Q^2)$ to NNLO for $Q^2 \gg m^2$.
  $\Longrightarrow$ Logarithmic terms are known for all N.
- We also calculate the matrix elements necessary to transform from the FFNS to the VFNS.
- First phenomenological parametrization to come up soon.
- Moments of the fermionic contributions to the 3-loop anomalous dimensions have been confirmed for the first time by an independent calculation.