On Approximation Methods in the Study of Boson Stars

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Motivation

- Dark matter (DM) theorized - what is it?
  - Axions – possible candidate
    - Light ($m_{QCD} = 10^{-5} eV$), electrically neutral, spinless
    - Can form gravitationally bound Bose-Einstein condensates (BEC's), called boson stars
- Need to properly formulate behavior

[2] PhysRev.187.1767
[3] PhysRevLett.38.1440

Boson stars

- Described by Gross-Pitaevskii + Poisson (GPP) eq.

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + V_g\,\psi + \frac{\lambda}{8\,m^2}\,|\psi|^2\,\psi \qquad \nabla^2 V_g = 4\pi\,G\,m^2\,|\psi|^2$$

  - Attractive (–λ) or repulsive (+λ) self-interactions
  - Difficult to solve exactly – need approximation methods
    - Numerical
    - Variational – ansatz dependent

[4] J.Stat.Phys.87.1353
[5] PhysRevA.59.1461

Approximation methods

- Numerical method
  - Solves time-independent GPP to arbitrary precision
  - Time-dependent solutions cumbersome

- Variational method
  - Can easily be used for static or dynamic problems
  - Find minimum energy solutions for given ansatz

- Find static state parameters using numerical and variational – make comparisons across ansatze

Variational method

$$\psi(r) = \sqrt{\frac{N}{\sigma^3 C_2}} F\left(\frac{r}{\sigma}\right) = \frac{m^{5/2}}{M_P\,|\lambda|} \sqrt{\frac{n}{\rho^3 C_2}} F\left(\frac{r}{\rho}\right) \qquad \sigma = \sqrt{|\lambda|}\frac{M_P}{m^2}\,\rho \qquad N = \frac{M_P}{m\,\sqrt{|\lambda|}}\,n$$

$$\frac{E(\rho)}{m\,N} = \frac{m^2}{{M_P}^2\,|\lambda|}\left(\frac{D_2}{2\,C_2}\frac{1}{\rho^2} - \frac{B_4}{2\,{C_2}^2}\,\frac{n}{\rho} + \mathrm{sgn}(\lambda)\frac{C_4}{16\,{C_2}^2}\frac{n}{\rho^3}\right)$$

$$\frac{\mu(\rho)}{m} = \frac{m^2}{{M_P}^2\,|\lambda|}\left(\frac{D_2}{2\,C_2}\frac{1}{\rho^2} - \frac{B_4}{{C_2}^2}\,\frac{n}{\rho} + \mathrm{sgn}(\lambda)\frac{C_4}{8\,{C_2}^2}\frac{n}{\rho^3}\right)$$

$$C_k = 4\,\pi\int d\xi\,\xi^2\,F(\xi)^k \qquad \xi = \frac{r}{\rho}$$

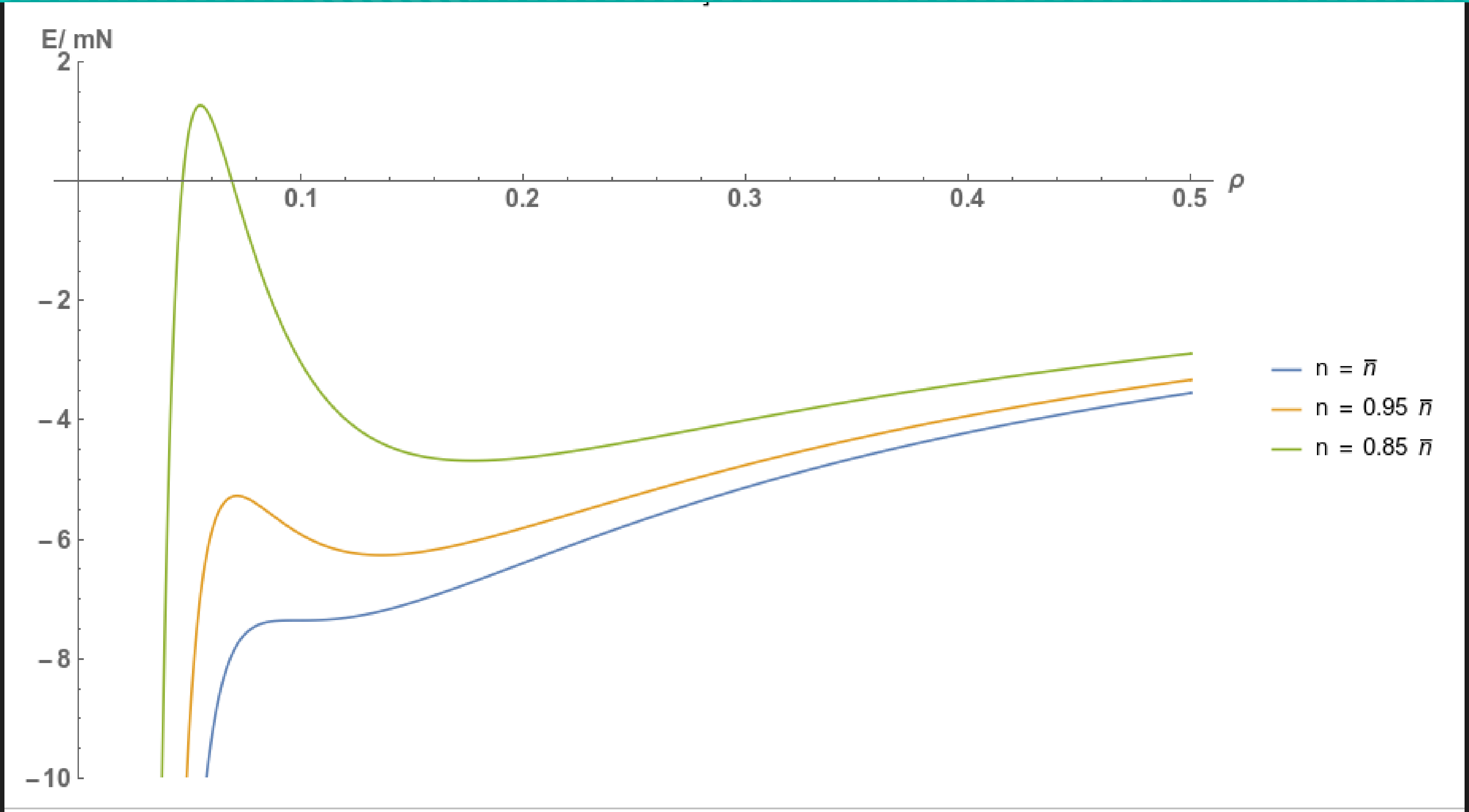
$$D_2 = 4\,\pi\int d\xi\,\xi^2\,F'(\xi)^2$$

$$B_4 = 32\,\pi^2\int d\xi\,\xi\,F(\xi)^2\int_0^\xi d\eta\,\eta^2\,F(\eta)^2$$

[1] arXiv:1809.08598

[6] Cambridge/books - Pethick & Smith

Energy functional – attractive



$$\rho_d = \frac{C_2\, D_2}{B_4\, n}\left[1 + \sqrt{1 - \frac{n^2}{\bar{n}^2}}\right]$$

$$\bar{n} = \sqrt{\frac{8}{3}}\,\frac{C_2\, D_2}{\sqrt{B_4\, C_4}}$$

[1] arXiv:1809.08598

Ansatze

$$\psi_A(r) = \begin{cases} \sqrt{\dfrac{N}{\pi^{3/2}\,\sigma^3}}\, e^{-r^2/2\,\sigma^2} & \text{(Gaussian (G)} \\ \sqrt{\dfrac{N}{\pi\,\sigma^3}}\, e^{-r/\sigma} & \text{(Exponential (E)} \\ \sqrt{\dfrac{N}{7\,\pi\,\sigma^3}} \left(1+\dfrac{r}{\sigma}\right) e^{-r/\sigma} & \text{(Linear + Exponential (LE)} \\ \sqrt{\dfrac{3\,N}{\pi^3\,\sigma^3}}\, \mathrm{sech}\left(\dfrac{r}{\sigma}\right) & \text{(Sech (S)} \\ \sqrt{\dfrac{N}{\pi\,\sigma^3}} \left(1+\dfrac{1}{a^5}-\dfrac{16}{a(1+a)^3}\right)^{-1/2} \left(e^{-r/\sigma}-\dfrac{1}{a}e^{-a\,r/\sigma}\right) & \text{(Double Exponential (DE}_a\text{))} \end{cases}$$

- Efficacy of ansatz varies
  - Need to make quantitative comparison
- Increase efficiency by addition of free parameters
  - DE can fit numerical solution or other ansatze

[7] arXiv:1103.2050
[8] arXiv:1412.3430
[9] arXiv:1710.04729
[1] arxiv:1809.08598

Numerical method

$$\tilde{\nabla}^2 \tilde{\psi} = 2\,\tilde{V}\,\tilde{\psi} + \frac{\mathrm{sgn}(\lambda)}{4}\tilde{\psi}^3 \qquad \tilde{\nabla}^2 \tilde{V} = 4\pi\,\tilde{\psi}^2$$

$$\tilde{\psi}(0) \to 0 \quad \text{and} \quad \tilde{V} \to \text{constant} \quad \text{as} \quad \tilde{r} \to \infty$$

$$E(\tilde{\psi}) = \frac{m^2}{M_P\,|\lambda|^{3/2}} \int d^3\tilde{r} \left[\frac{1}{2}|\tilde{\nabla}\tilde{\psi}|^2 + \frac{1}{2}\left(\tilde{V} + \tilde{\mu}\right)\,\tilde{\psi}^2 + \frac{\mathrm{sgn}(\lambda)}{16}\tilde{\psi}^4\right] \qquad \tilde{\mu} = -\lim_{\tilde{r}\to\tilde{r}_0} \tilde{V}$$

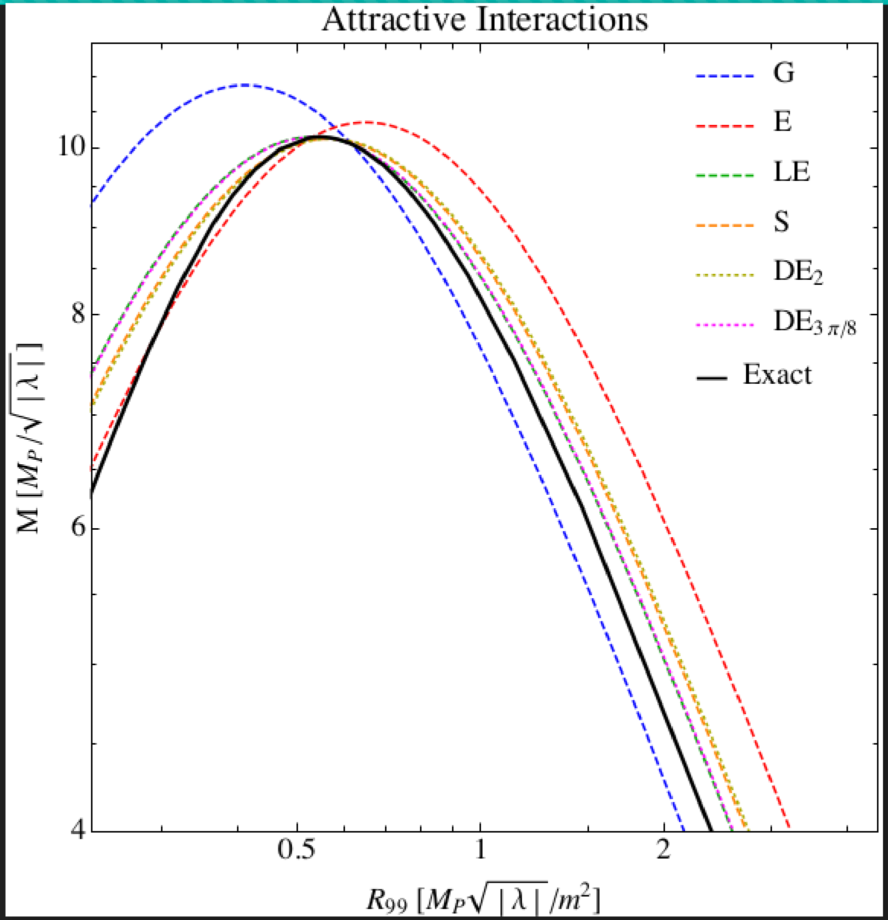
- Parameters to compare
  - Energy and chemical potential
  - Mass vs. R99 curves
  - Deviations

$$\Delta_r(n) \equiv \sqrt{\left[\frac{\langle r^2\rangle - \langle r^2\rangle_A}{\langle r^2\rangle}\right]^2 + \left[\frac{\langle r\rangle - \langle r\rangle_A}{\langle r\rangle}\right]^2 + \left[\frac{R_{99} - R_{99}^A}{R_{99}}\right]^2}$$

$$\Delta_\psi(n) \equiv \frac{\int d^3\tilde{r}\left[\tilde{\psi}(\tilde{r}) - \tilde{\psi}_A(\tilde{r})\right]^2}{\int d^3\tilde{r}\,\tilde{\psi}(\tilde{r})^2}$$

[1] arXiv:1809.08598

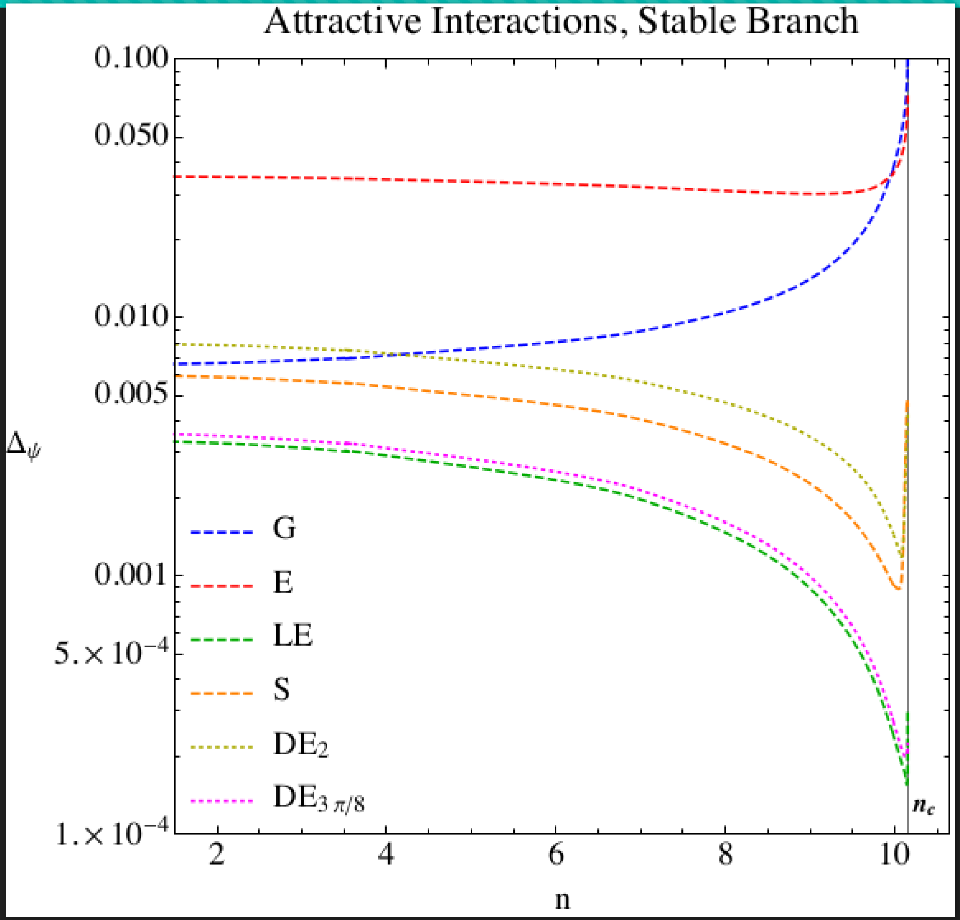
M vs. R99 – attractive



- Notable results
  - Lin+exp best fit
  - Exp good fit for unstable branch (small R99), poor fit for stable branch (large R99)
  - Gauss used often in literature – poor fit along unstable branch, poor estimate of critical mass
  - Double exp with $a = 3\pi/8$ to reproduce critical mass and lin+exp
  - Double exp with $a = 2$ to reproduce sech

[1] arXiv:1809.08598

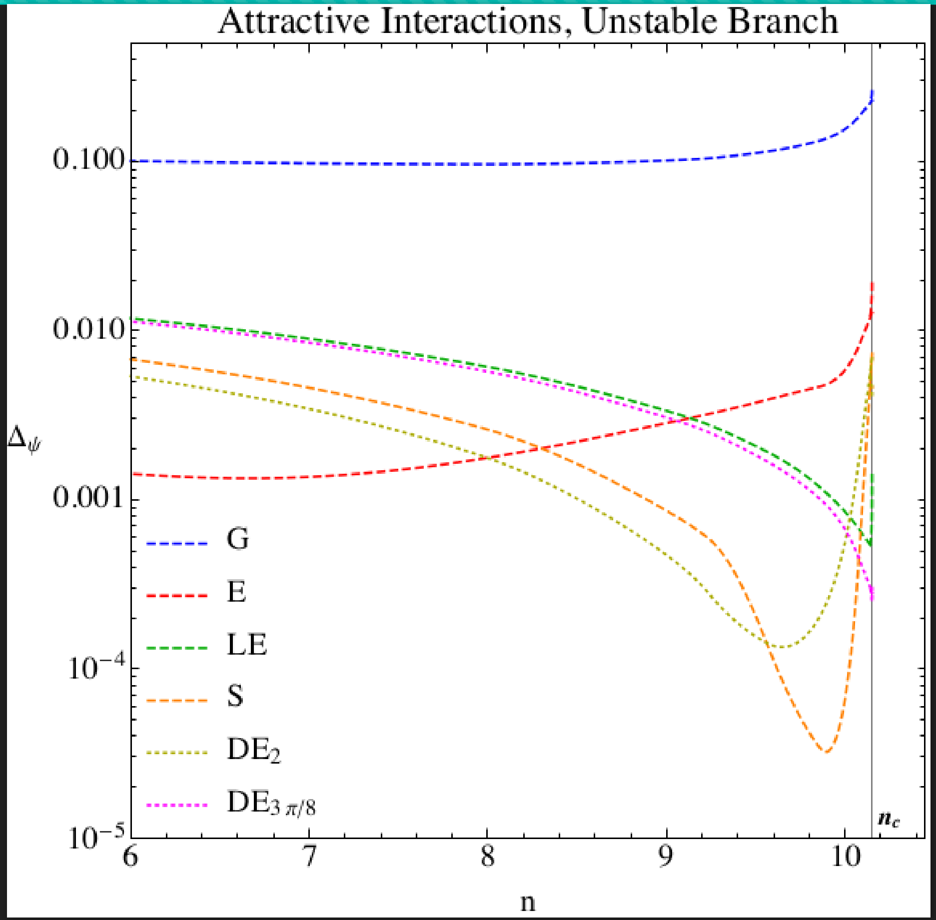
Deviations – attractive, stable



- Relevant for dilute boson stars
- Notable results
  - Lin+exp and double exp. counterpart have smallest deviations
  - Gauss and exp poor fits

[1] arXiv:1809.08598

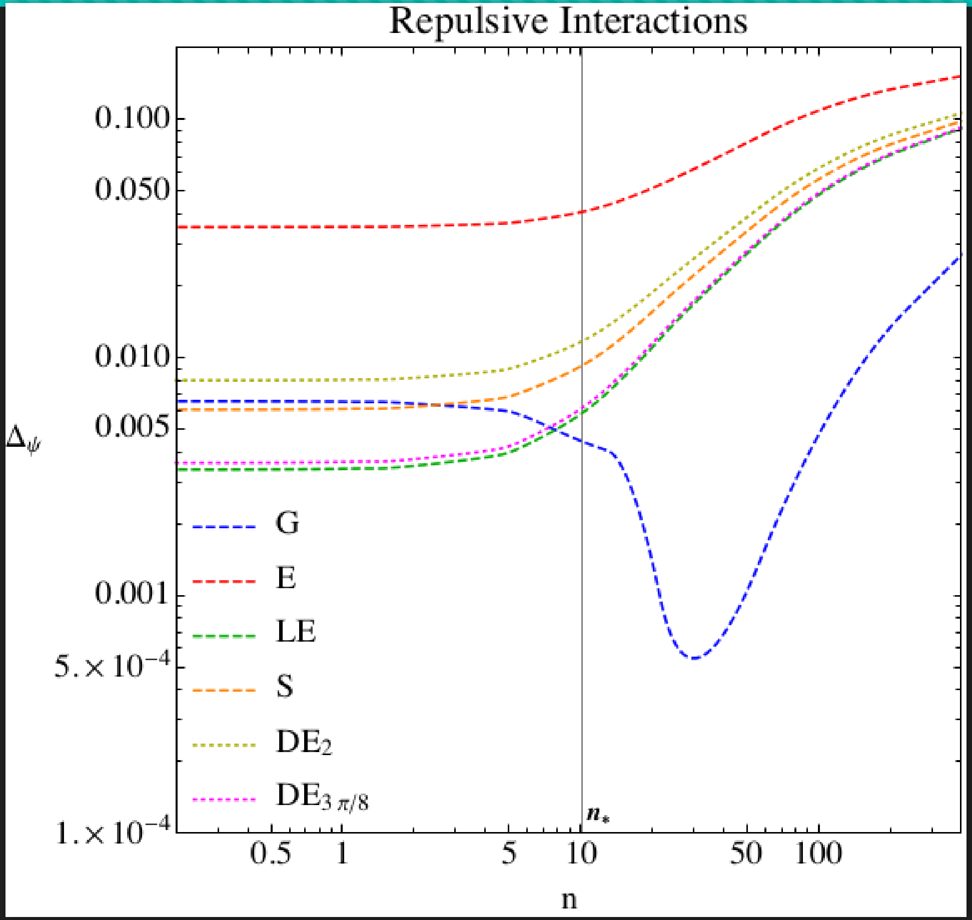
Deviations – attractive, unstable



- Relevant for collapsing boson stars
- Notable results
  - Exp best fit for smaller masses
  - Sech and double exp counterpart best fit for larger masses
  - Gauss poor fit

[1] arXiv:1809.08598

Deviations - repulsive



- Relevant for axion-like particles
- Notable results
  - Turning point at n_* (approaches Thomas-Fermi limit)
  - Lin+exp and double exp counterpart best fits below n_*
  - Gauss best fit above $n > n_*$

[1] arXiv:1809.08598

Concluding remarks

- Have made quantitative comparison of various ansatze for dilute boson stars
  - Varied results – important to know when to use a given ansatz

- Can use best ansatz for time-dependent problems
  - Collapse [10] arXiv:1403.3358, [11] arXiv:1604.05904, [12] arXiv:1608.06911
  - Collisions [13] arXiv:1608.00547, [14] arXiv:1701.01476
  - Decay [15] arXiv:1512.01709, [16] arXiv:1609.05182, [17] arXiv:1705.05385

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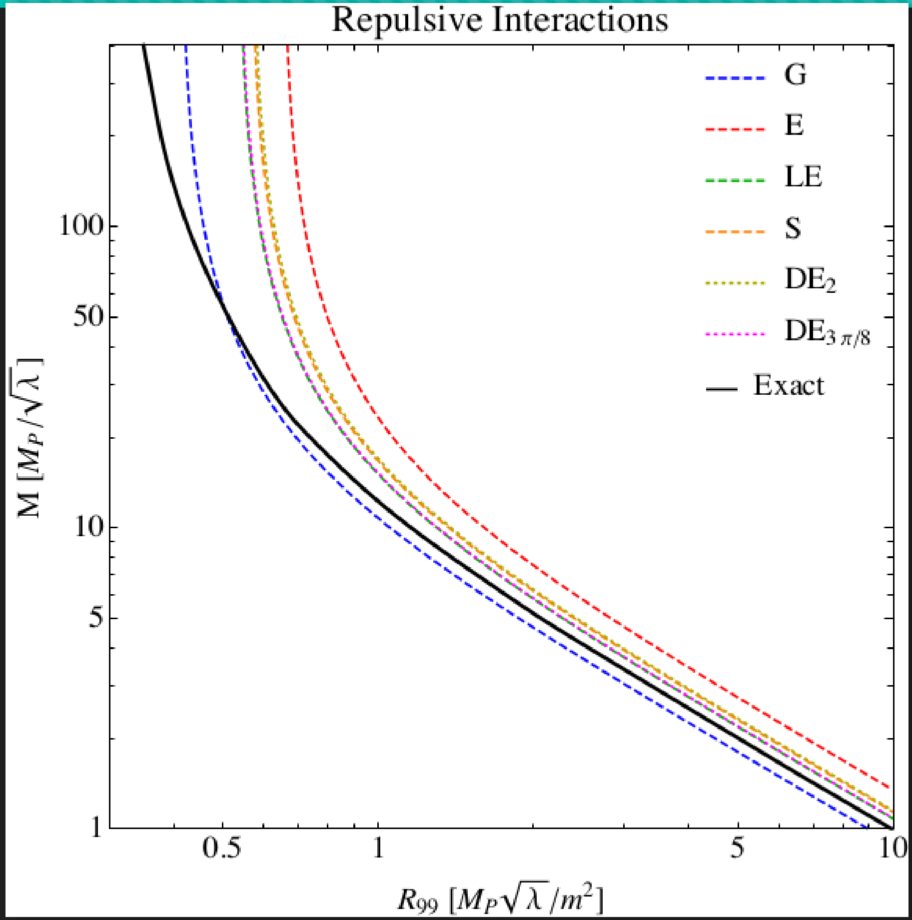
EXTRA: M vs. R99 - repulsive



- Notable results
  - Thomas-Fermi (TF) limit – R99 becomes independent of mass
  - Gauss best fit for large M, lin+exp best fit for small M

[1] arXiv:1809.08598

EXTRA: GPP formalism

- Nonrelativistic limit of Einstein Klein-Gordon formalism

$$S = \int dt\, d^3r \left[i\psi \frac{\partial \psi^*}{\partial t} - i\psi^* \frac{\partial \psi}{\partial t} + \frac{|\nabla\psi|^2}{2m} + \frac{1}{2} V_g\, |\psi|^2 + \frac{\lambda}{16\, m^2}\, |\psi|^4 \right]$$

- Action yields GPP eq.

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi + V_g\, \psi + \frac{\lambda}{8\, m^2}\, |\psi|^2\, \psi \qquad \nabla^2 V_g = 4\pi\, G\, m^2\, |\psi|^2$$

- Energy and chemical potential

$$\mu\, N = \int d^3r \left(\frac{|\nabla\psi|^2}{2m} + V_g\, |\psi|^2 + \frac{\lambda}{8\, m^2}\, |\psi|^4 \right) \qquad E = \int d^3r \left(\frac{|\nabla\psi|^2}{2m} + \frac{1}{2} V_g\, |\psi|^2 + \frac{\lambda}{16\, m^2}\, |\psi|^4 \right)$$

[6] Cambridge/books - Pethick & Smith

EXTRA: Numerical method

- Scaling of EOM

$$\psi = \frac{m^{5/2}}{M_P\,|\lambda|}\,\tilde{\psi} \qquad V_g = \mu + \frac{m^3}{{M_P}^2\,|\lambda|}\,\tilde{V} \qquad r = \frac{\sqrt{|\lambda|}\,M_P}{m^2}\,\tilde{r}$$

- Employ shooting method – proper boundary conditions

$$\tilde{\nabla}^2\tilde{\psi} = 2\,\tilde{V}\,\tilde{\psi} + \frac{\mathrm{sgn}(\lambda)}{4}\tilde{\psi}^3 \qquad \tilde{\nabla}^2\tilde{V} = 4\pi\,\tilde{\psi}^2$$

$$\tilde{\psi}(0) \to 0 \quad \text{and} \quad \tilde{V} \to \text{constant} \quad \text{as} \quad \tilde{r} \to \infty$$

- Various checks
  - Ratio of E/μ at critical mass for attractive
  - Self-consistency check for numerical μ

$$E(\tilde{\psi}) = \frac{m^2}{M_P\,|\lambda|^{3/2}} \int d^3\tilde{r}\left[\frac{1}{2}|\tilde{\nabla}\tilde{\psi}|^2 + \frac{1}{2}\left(\tilde{V} + \tilde{\mu}\right)\tilde{\psi}^2 + \frac{\mathrm{sgn}(\lambda)}{16}\tilde{\psi}^4\right] \qquad \tilde{\mu} = -\lim_{\tilde{r}\to\tilde{r}_0}\tilde{V}$$

[1] arXiv:1809.08598

EXTRA: Ruffini-Bonazzola

- Boson star described by Einstein Klein-Gordon eq.
- Can take weak gravity, weakly bound limit

$$\delta = \frac{8\,\pi\,m^2}{M_P{}^2\,|\lambda|} \ll 1 \qquad \Delta = \sqrt{1 - \frac{\mu_0{}^2}{m^2}} \ll 1 \qquad \mu = -(m - \mu_0)$$

- To leading order in δ (infrared limit) – RB and GPP formalism equiv.

[1] arXiv:1809.08598
[2] PhysRev.187.1767

EXTRA: Thomas Fermi (TF) limit

- Kinetic energy becomes negligible
- Get exact solution to GPP eq.
- Upper and lower bounds on particle number in TF limit

$$X(y) = \frac{2\sqrt{\delta^{3/2} N \lambda}}{\pi} \sqrt{\frac{\sin(p_0\, y)}{p_0\, y}}$$

$$\frac{M_P}{m\sqrt{\lambda}} \ll N_{\mathrm{TF}} \ll \sqrt{\frac{\pi\,\lambda}{128} \frac{{M_P}^3}{m^3}} \qquad \lambda \gg \frac{m^2}{M_P^2}$$

[1] arXiv:1809.08598
[19] Rev.Mod.Phys.71.463

EXTRA: Decays

- Wish to know rate of decay – can then find lifetime of boson star
- Decay rate differs depending on ansatz used
- 3-to-1 decay: 3 bound axions to 1 free axion
  - Particle number not conserved – real scalar field
  - Bound state not definite momentum eigenstate but average momentum conserved
  - Decay rate depends on binding energy
  - Can find lifetime in terms of boson mass and star mass

[15] arXiv:1512.01709
[16] arXiv:1609.05182
[17] arXiv:1705.05385

EXTRA: Collapse

- Time-dependent variational method
  - Formulation of Pethick and Smith
  - Time-dependent variational parameter
  - Wavefunction multiplied by phase – has Hubble parameter like behavior
- Wish to know outcome of collapse
  - Become stable again or form black hole?
  - Need to include φ^6 interactions to stabilize

[6] Cambridge/books - Pethick & Smith
[10] arXiv:1403.3358
[11] arXiv:1604.05904
[12] arXiv:1608.06911