

Numerical calculation of high-order QED contributions to the electron anomalous magnetic moment

Sergey Volkov

SINP MSU, Dubna branch (Russia)

DLNP JINR, Dubna (Russia)

ACAT-2019

AMM of the electron (theory and experiment)

The measured value [2011]:

$$a_e = 0.00115965218073(28)$$

The most accurate prediction (T. Kinoshita et al. [2019]):

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n \geq 1} \left(\frac{\alpha}{\pi} \right)^n a_e^{(2n)},$$

$$a_e^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_e / m_\mu) + A_2^{(2n)}(m_e / m_\tau) + A_3^{(2n)}(m_e / m_\mu, m_e / m_\tau)$$

$$a_e = 0.001159652181606(11)(12)(229)$$

$$(\alpha^{-1} = 137.035999046(27) - \text{independent from } a_e)$$

Uncertainties come from:

$$A_1^{(10)}, a_e(hadronic) + a_e(electroweak), \alpha$$

T. Aoyama, T. Kinoshita, M. Nio, Theory of the Anomalous Magnetic Moment of the Electron, Atoms 2019, 7(1), 28.

$A_1^{(2n)}$ calculations are still important...

Universal QED contributions

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n \geq 1} \left(\frac{\alpha}{\pi} \right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e / m_\mu) + A_2^{(2n)}(m_e / m_\tau) + A_3^{(2n)}(m_e / m_\mu, m_e / m_\tau)$$

- J. Schwinger [1948], analytically: $A_1^{(2)} = 0.5$
- R. Karplus, N. Kroll [1949] – with a mistake
 - A. Petermann [1957], C. Sommerfield [1958], analytically:
 $A_1^{(4)} = -0.328478966\dots$
- ~1970...~1975, 3 loops, numerically:
 1. M. Levine, J. Wright.
 2. R. Carroll, Y. Yao.
 3. T. Kinoshita, P. Cvitanović.
 - T. Kinoshita, P. Cvitanović [1974]: $A_1^{(6)} = 1.195 \pm 0.026$
- E. Remiddi, S. Laporta et al., ~1965...1996, analytically: $A_1^{(6)} = 1.181241456\dots$
- T. Kinoshita, M. Nio et al., numerically, 2015: $A_1^{(8)} = -1.91298(84)$
- S. Laporta, semi-analytically, 2017: $A_1^{(8)} = -1.9122457649\dots$
- T. Kinoshita, M. Nio et al., numerically, 2019: $A_1^{(10)} = 6.737(159)$

The method

- Subtraction procedure for removing both IR and UV divergences in Feynman-parametric space for each individual Feynman diagram
- Diagram-specific importance sampling Monte Carlo integration algorithm for diagrams without lepton loops

The subtraction procedure

- FULLY AUTOMATED AT ANY ORDER OF THE PERTURBATION SERIES.

- UV and IR divergences are eliminated point-by-point in Feynman-parametric space for each individual Feynman diagram. No regularization is required.
- Subtraction by a forest formula with linear operators. Each operator transforms Feynman amplitude of some UV-divergent subdiagram G' (in momentum space) to the polynomial with the degree that is less or equal to $\omega(G')$.
- The subtraction is equivalent to the on-shell renormalization => no residual renormalizations, no calculations of renormalization constants, no other manipulations.

Operators

$$\Gamma_\mu(p, q) = \text{[Diagram: A circle with a wavy line labeled } q \text{ attached to the bottom. An incoming arrow on the left is labeled } p-q/2 \text{ and an outgoing arrow on the right is labeled } p+q/2 \text{.]}$$

$$\Sigma(p) = \text{---}\xrightarrow{p}\text{---}\text{---}\text{---}$$

- A – projector of AMM

$$\bar{u}_2 \Gamma_\mu(p, q) u_1 = \bar{u}_2 (f(q^2) \gamma_\mu - g(q^2) \sigma_{\mu\nu} q^\nu / (2m) + h(q^2) q_\mu) u_1$$

$$\sigma_{\mu\nu} = (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) / 2, \quad (p - q/2)^2 = (p + q/2)^2 = m^2$$

$$(\hat{p} - \hat{q}/2 - m)u_1 = (\hat{p} + \hat{q}/2 - m)u_2 = 0$$

$$A\Gamma_\mu = \gamma_\mu \lim_{q^2 \rightarrow 0} g(q^2)$$

- L – on-shell renormalization for vertex-like subdiagrams

$$L\Gamma_\mu = \gamma_\mu (a(m^2) + b(m^2)m + c(m^2)m^2)$$

can produce additional IR divergences

- U – intermediate operator

$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2)p_\mu + c(p^2)\hat{p}p_\mu + d(p^2)(\hat{p}\gamma_\mu - \gamma_\mu\hat{p})$$

$$\Sigma(p) = r(p^2) + s(p^2)\hat{p}$$

$$U\Gamma_\mu = \gamma_\mu a(m^2)$$

$$U\Sigma = r(m^2) + s(m^2)\hat{p}$$

IR-safe!

U preserves the Ward identity!

For the other types of divergent subgraphs, U =Taylor expansion at 0 up to ω order.

Forest formula for AMM

A set of subgraphs of a diagram is called a **forest** if any two elements of this set don't overlap.

$F[G]$ – the set of all forests of UV-divergent subgraphs in G that contain G .

$I[G]$ – the set of all vertex-like UV-divergent subgraphs in G that contains the vertex that is incident to the external photon line of G .

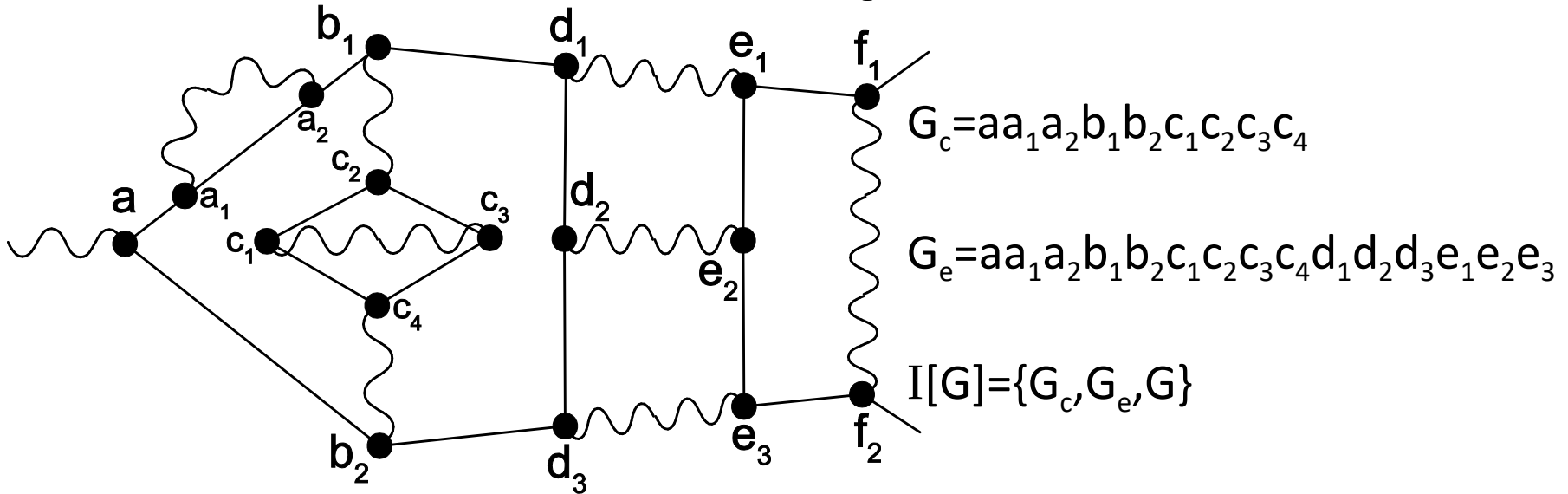
$$\tilde{f}_G = \sum_{\substack{F=\{G_1, \dots, G_n\} \in F[G] \\ G' \in I[G] \cap F}} (-1)^{n-1} K_{G_1}^{G'} \dots K_{G_n}^{G'} f_G$$

$$K_{G''}^{G'} = \begin{cases} A_{G'} & \text{for } G' = G'' \\ U_{G''} & \text{for } G'' \notin I[G], \text{ or } G'' \subseteq G' \text{ and } G'' \neq G' \\ L_{G''} & \text{for } G'' \in I[G], G' \subseteq G'', G'' \neq G, G'' \neq G' \\ (L_{G''} - U_{G''}) & \text{for } G'' = G, G' \neq G \end{cases}$$

\bar{f}_G = coefficient before γ_μ in \tilde{f}_G

$$a_e = \sum_G \bar{f}_G$$

Example



Other UV-divergent subgraphs:

electron self-energy – a_1a_2 , vertex-like – $c_1c_2c_3$, $c_1c_3c_4$,

photon self-energy – $c_1c_2c_3c_4$,

photon-photon scattering – $G_d = aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3$

$$\tilde{f}_G = \left[A_G (1 - U_{G_e}) (1 - U_{G_c}) - (L_G - U_G) A_{G_e} (1 - U_{G_c}) - (L_G - U_G) (1 - L_{G_e}) A_{G_c} \right] \cdot (1 - U_{G_d}) (1 - U_{c_1c_2c_3c_4}) (1 - U_{c_1c_2c_3} - U_{c_1c_3c_4}) (1 - U_{a_1a_2}) f_G$$

Importance sampling Monte Carlo

- Integral: $\int_{\Omega} f(x) dx$
- Probability density function: $g(x)$
- Approximation: $(1/N) \sum_{1 \leq j \leq N} (f(x_j)/g(x_j))$
- Variance: $V(f,g) = \int_{\Omega} (f(x)^2/g(x)) dx - (\int_{\Omega} f(x) dx)^2$
- Error estimation: $\sigma^2 \approx V(f,g)/N$
- The goal is to minimize $V(f,g)$ by choosing $g(x)$.

**NON-ADAPTIVE MONTE CARLO WORKS FINE
FOR HIGH-ORDER CALCULATIONS IN QFT!!!**

Diagram-specific probability density functions

- Integral: $\int_{z_1, \dots, z_M > 0} f(z_1, \dots, z_M) \delta(z_1 + \dots + z_M - 1) dz$
- Hepp sectors: $z_{j_1} \geq z_{j_2} \geq \dots \geq z_{j_M}$
- Density: $C \cdot \frac{\prod_{l=2}^M (z_{j_l} / z_{j_{l-1}})^{\text{Deg}(\{j_l, j_{l+1}, \dots, j_M\})}}{z_1 \cdot z_2 \cdot \dots \cdot z_M},$

Deg is defined on subsets of $\{1, \dots, M\}$

(the idea of E.Speer, J. Math. Phys. 9, 1404 (1968))

- My ideas are:
 - 1) how to calculate *Deg*(s) for each set s
(taking into account the infrared behavior etc.)
 - 2) how to generate samples fastly

Obtaining $Deg(s)$

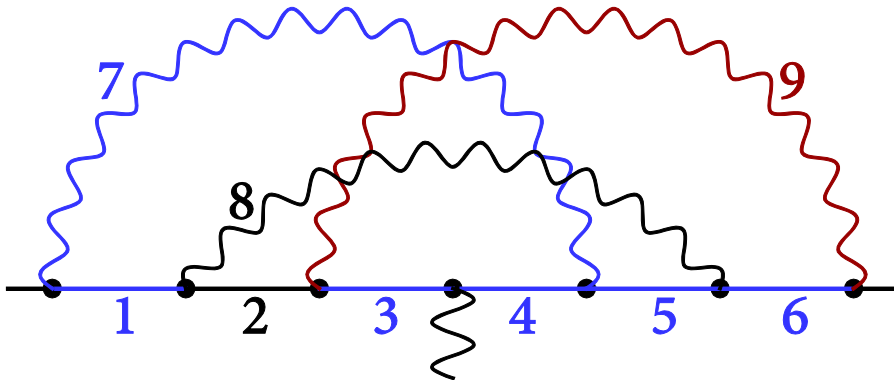
- Sector: $z_{j_1} \geq z_{j_2} \geq \dots \geq z_{j_M}$
- Density: $C \cdot \frac{\prod_{l=2}^M (z_{j_l} / z_{j_{l-1}})^{Deg(\{j_l, j_{l+1}, \dots, j_M\})}}{z_1 \cdot z_2 \cdot \dots \cdot z_M},$

• The rules are constructed using ultraviolet degrees of divergence (with the sign '-') of **I-closures** of sets

(the full description taking into account divergent subdiagrams is in [arXiv:1705.05800](https://arxiv.org/abs/1705.05800))

• $I\text{Clos}(s) = s \cup s'$, where s' is the set of all photon lines for which the electron path connecting their ends is contained in s

Example: $I\text{Clos}(\{1,3,4,5,6,7\}) = \{1,3,4,5,6,7,9\}$

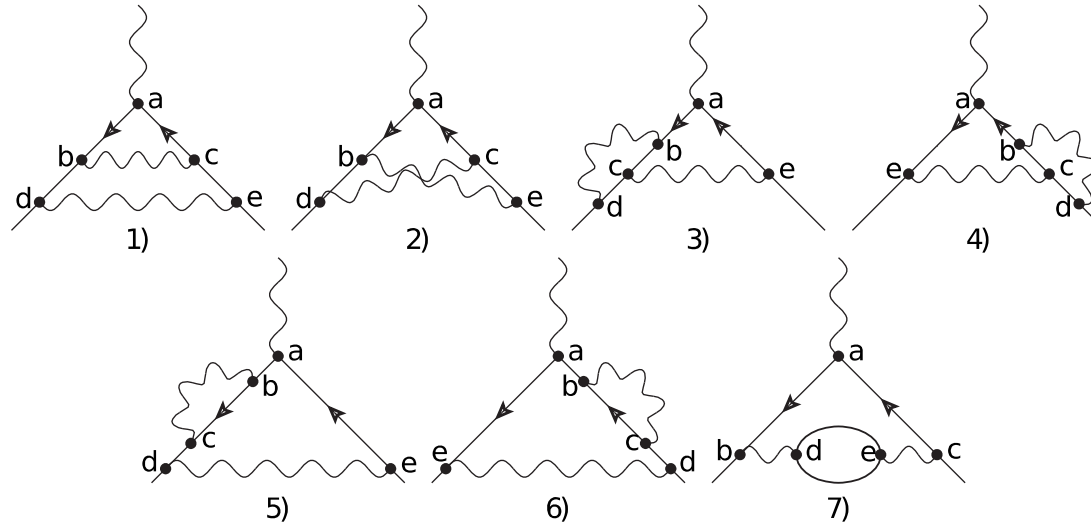


Realization and numerical results

- Monte Carlo integration on Intel-compatible CPUs, NVidia GPUs (Tesla K80, Tesla V100)
- 2 loops: all Feynman diagrams (with electron loops: old, 2015)
- 3 loops: all Feynman diagrams (with electron loops: old, 2015)
- 4 loops: diagrams without electron loops (GPU NVidia Tesla V100, supercomputer «Govorun», JINR, Dubna, Russia)
- 5 loops: diagrams without electron loops (GPU NVidia Tesla V100, supercomputer «Govorun», JINR, Dubna, Russia)
- 6 loops: ladder diagram (NVidia Tesla K80, Google Cloud)

$A_1^{(4)}$: all Feynman diagrams

NVidia Tesla K80, Google Cloud



$$2015: A_1^{(4)} = -0.328513(87)$$

$$2018: A_1^{(4)}[\text{no lepton loops}] = -0.3441651(34)$$

Analytical, 1957:

$$A_1^{(4)} = -0.328478966...$$

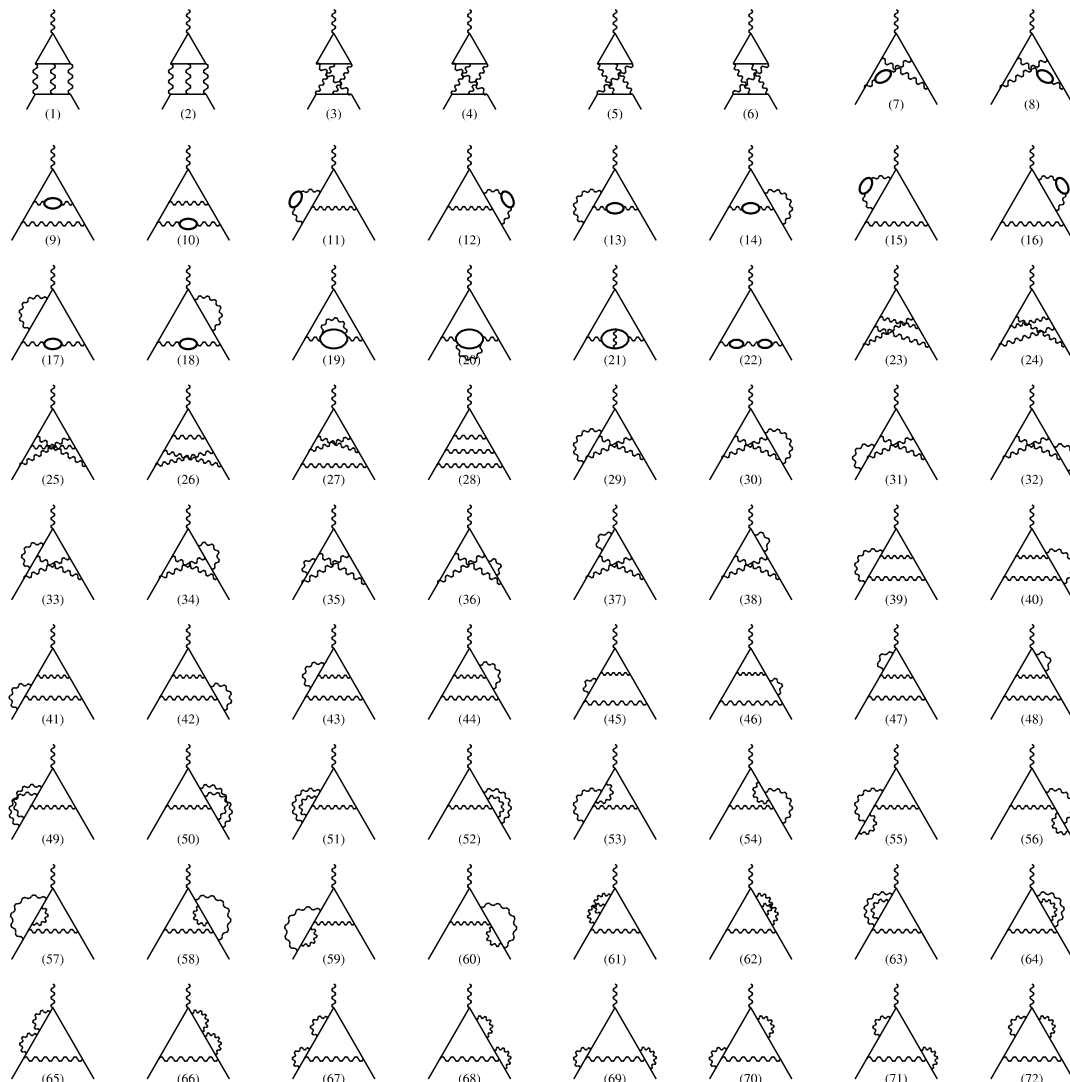
$$A_1^{(4)}[\text{no lepton loops}] = -0.3441663...$$

← old: 2015, personal computer

#	My value	Analytical value (Petermann, 1957)
1	0.77747774(18)	0.77747802
2	-0.4676475(17)	-0.46764544
3,4	-0.0640193(19)	$-0.564021 - (1/2)\log(\lambda^2/m^2)$
5,6	-0.5899758(14)	$-0.089978 + (1/2)\log(\lambda^2/m^2)$
7	0.0156895(25)	0.0156874

$A_1^{(6)}$: all Feynman diagrams

$A_1^{(6)}$ [no lepton loops] = 0.90485(10) Analytical (1996): 0.904979



3-loop Feynman diagrams for electron's AMM. Plot courtesy of F.Jegerlehner

Comparison with known analytical values

#	My value	Analyt. val.	Reference
1-6	0.3708(14)	0.3710	[10]
7-10	0.04989(20)	0.05015	[4,5]
11-12,15-16	-0.08782(15)	-0.08798	[2,4]
13-14,17-18	-0.11230(17)	-0.11234	[3,4]
19-21	0.05288(13)	0.05287	[1]
22	0.002548(20)	0.002559	[1]
23-24	1.861914(17)	1.861908	[11]
25	-0.0267956(78)	-0.026799	[12]
26-27	-3.176700(22)	-3.176685	[8]
28	1.790285(19)	1.790278	[8]
29-30	-1.757945(15)	-1.757936	[12]
33-34,37-38	0.455517(26)	0.455452	[8,11]
31-32,35-36	1.541644(37)	1.541649	[7,9]
39-40	-0.334691(14)	-0.334695	[11]
41-48	-0.402749(46)	-0.402717	[6,7]
49-68	0.533289(54)	0.533355	[6-9,11,12]
69-72	0.421080(43)	0.421171	[6,7,9]

old: 2015

[1] J. Mignaco, E. Remiddi, IL Nuovo Cimento, V. LX A, N. 4, 519 (1969).
 [2] R. Barbieri, M. Caffo, E. Remiddi, Lettere al Nuovo Cimento, V. 5, N. 11, 769 (1972).
 [3] D. Billi, M. Caffo, E. Remiddi, Lettere al Nuovo Cimento, V. 4, N. 14, 657 (1972).
 [4] R. Barbieri, E. Remiddi, Physics Letters, V. 49B, N. 5, 468 (1974).
 [5] R. Barbieri, M. Caffo, E. Remiddi, Ref.TH.1802-CERN (1974).
 [6] M. Levine, R. Roskies, Phys. Rev. D, V. 9, N. 2, 421 (1974).

[7] M. Levine, R. Perisho, R. Roskies, Phys. Rev. D, V. 13, N. 4, 997 (1976).
 [8] R. Barbieri, M. Caffo, E. Remiddi et al., Nuclear Physics B 144, 329 (1978).
 [9] M. Levine, E. Remiddi, R. Roskies, Phys. Rev. D, V. 20, N. 8, 2068 (1979).
 [10] S. Laporta, E. Remiddi, Physics Letters B 265, 182 (1991).
 [11] S. Laporta, Physics Letters B 343, 421 (1995).
 [12] S. Laporta, E. Remiddi, Physics Letters B 379, 283 (1996).

$A_1^{(8)}$: contribution of diagrams without electron loops

My result: **-2.1790(22)** 758 GPU-hours (NVidia Tesla V100, supercomputer «Govorun», JINR)

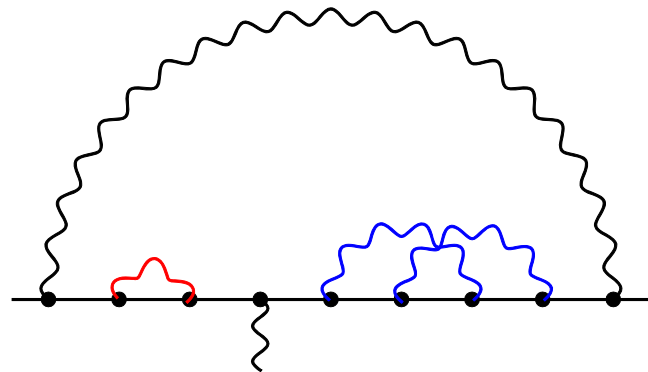
Laporta, 2017: -2.1768660277... T.Kinoshita, M.Nio et al., 2012: -2.1755(20) (not recent)

- 269 Feynman diagrams
- 10-dimensional integrals
- 78 classes of diagrams for comparison with the direct subtraction on the mass shell
- 6 gauge-invariant classes (k,m,n)

(k,m,n):

m and **n** photon lines to the **right** and to the **left** from the external photon (or vice versa),
k photon lines with ends on different sides

Example of a diagram from (1,2,1):



Class	Value	Laporta, 2017
(1,3,0)	-1.96956(93)	-1.97107...
(2,2,0)	-0.1439(12)	-0.14248...
(1,2,1)	-0.6224(10)	-0.62192...
(3,1,0)	-1.04093(90)	-1.04054...
(2,1,1)	1.08594(76)	1.08669...
(4,0,0)	0.51185(34)	0.51246...

$A_1^{(10)}$: diagrams without electron loops

T. Aoyama, T. Kinoshita, M. Nio, 2019 (90% confidence): **7.668(159)**

My result (1σ): **6.782(113)** 25797 GPU-hours, NVidia Tesla V100, supercomputer «Govorun», JINR

Calc 1: 6.739(132) 19515 GPU-hours, CURAND MRG32k3a generator

Calc 2: 6.905(220) 6282 GPU-hours, CURAND Philox_4x32_10 generator

- 3213 Feynman diagrams
- 13-dimensional integrals
- 807 classes of diagrams for comparison with the direct subtraction on the mass shell
- 9 gauge-invariant classes (k,m,n)
- 500 GB of the integrands code (compiled)
- $1.9 \cdot 10^{14}$ Monte Carlo samples

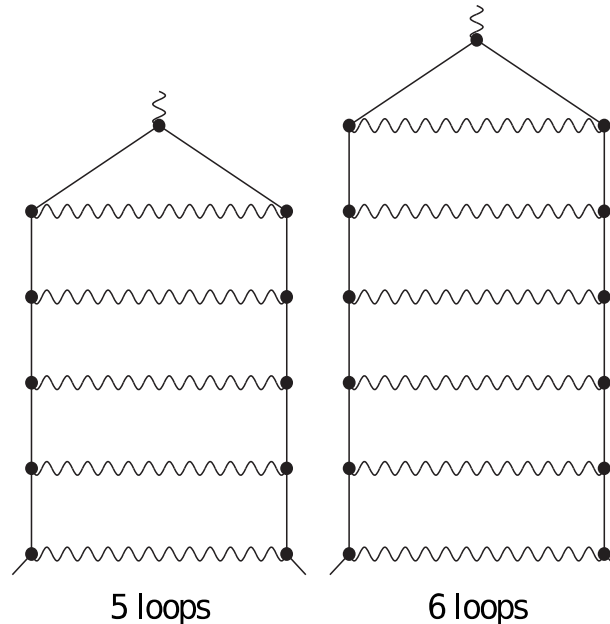
Class	Value	Calc 1	Calc 2	N _{diag}
(1,4,0)	6.172(42)	6.158(49)	6.209(80)	706
(2,3,0)	-0.724(54)	-0.746(63)	-0.66(10)	706
(1,3,1)	0.895(43)	0.854(50)	1.007(82)	148
(3,2,0)	-0.396(43)	-0.399(51)	-0.390(85)	558
(2,2,1)	-2.160(46)	-2.133(53)	-2.236(90)	370
(4,1,0)	-1.017(26)	-1.028(31)	-0.984(51)	336
(1,2,2)	0.301(25)	0.312(30)	0.267(50)	55
(3,1,1)	2.624(30)	2.628(35)	2.614(58)	261
(5,0,0)	1.0898(80)	1.0929(94)	1.081(15)	73

Ladder diagrams: 5 and 6 loops

(NVidia Tesla K80 (1 GPU), Google Cloud)

loops	My value	Analytical value	N_{samples}	time
5	11.6530(58)	11.6592...	$29 \cdot 10^9$	5 hours
6	34.31(20)	34.367	10^{10}	8 hours

All analytical values are from M. Caffo, S. Turrini, E. Remiddi, Nuclear Physics B141 (1978) 302-310.



Thank you for your attention!

volkoff_sergey@mail.ru
sergey.volkov.1811@gmail.com

ЖЭТФ, т. 149, вып. 6, стр. 1164-1191 (2016)

J. Exp. Theor. Phys. 122, 1008 (2016)

arXiv:1507.06435 (short version)

subtraction procedure

Phys. Rev. D 96, 096018 (2017)

arXiv:1705.05800

Monte Carlo integration method

Phys. Rev. D 98, 076018 (2018)

arXiv:1807.05281

realization on GPU, 4-loop results for gauge-invariant classes, ...