Towards an efficient method to compute two-loop scalar amplitudes

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Outline

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1. Motivations
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3. “Standard” 1-loop functions
4. Conclusion

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Motivations

Why two-loop?

- Quality of the experimental data at LHC
- Significance of the NNLO corrections

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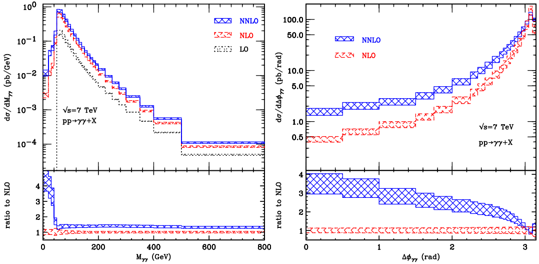

- Purely analytical methods Gerhmann, Remidi, Dixon, Dhur, Smirnov, Henn, Sokatchev, ...
- Semi analytical method Czakon, Kosower, Freitas, Gluza, Riemann, ...
- Purely numerical method Soper, Heinrich, Weinzierl, Smirnov, de Doncker, Yuasa, ...

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State of the art!

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The standard calculation of two-loop, $N = 3$ and 4-point functions $^{(2)}I_N^4$ in the general complex mass case relies on multidimensional numerical integration: very high computing cost!

Structure of two-loop amplitude

After the integration over the loop momenta

$$^{(2)}I_N^n\left(\{p_j\};\mathcal{T}\right) = (-1)^I\,\Gamma(I-n)\int_0^1 \prod_{i=1}^{I} d\tau_i\,\delta\left(1-\sum_{l=1}^{I}\tau_l\right)\frac{[\det A]^{I-3\,n/2}}{\mathcal{F}^{I-n}}$$

where

$$\mathcal{F} = \left[\sum_{j,l=1}^{2} r_j.r_l\,\mathrm{Cof}[A]_{jl} - \det(A)\,C - i\,\lambda\right]$$

$$\Xi = \left[K^T.A.K + 2\,K^T.r + C + i\,\lambda\right] \qquad K^T = (k_1\;\;k_2)$$

Reparametrisation

- S_1 set of labels of propagators involving only k_1, $\rho_1 \equiv \sum_{i \in S_1} \tau_i$
- S_2 set of labels of propagators involving only k_2, $\rho_2 \equiv \sum_{i \in S_2} \tau_i$
- S_3 set of labels of propagators involving both k_1 and k_2, $\rho_3 \equiv \sum_{i \in S_3} \tau_i$

The ρ_j's thus fulfil the constrain

$$\rho_1 + \rho_2 + \rho_3 = \sum_{j=1}^{I} \tau_j = 1$$

The elements of the matrix A read:

$$A_{12} = \rho_3, \quad A_{11} = \rho_1 + \rho_3, \quad \text{and} \quad A_{22} = \rho_2 + \rho_3$$

$$\det(A) = \rho_1\,\rho_2 + \rho_2\,\rho_3 + \rho_3\,\rho_1$$

Reparametrisation

Whenever $|S_j| \geq 2$, introduce $|S_j|$ parameters u_{k_j} $(k_j \in S_j)$ so that:

$$\tau_{k_j} = \rho_j \, u_{k_j} \quad \text{with the constraint} \quad \sum_{k_j \in S_j} u_{k_j} = 1$$

In case some $|S_j| = 1$, no corresponding u parameter is introduced

$$^{(2)}I_N^n\left(\{p_j\}; \mathcal{T}\right) = \int_{(R^+)^3} \left[\prod_{k=1}^{3} d\rho_k \, \rho_k^{|S_k|-1} \right] \delta\left(1 - \sum_{l=1}^{3} \rho_l\right) \times \left[\rho_1 \, \rho_2 + \rho_2 \, \rho_3 + \rho_3 \, \rho_1\right]^{l - \frac{3\,n}{2}} \; ^{(1)}\widetilde{I}_{N'}^{n'}$$

Reparametrisation

The N'-point function of "generalised one-loop type"

$$ {}^{(1)}\widetilde{I}^{n'}_{N'} = \int_{(R^+)^{N'-1}} \prod_{k=1}^{3} \prod_{j \in S_k} du_j \, \delta\left(1 - \sum_{l \in S_k} u_j\right) \left[D(\{u_k\}) - i\,\lambda \right]^{n'/2 - N'} $$

$$ D(\{u_k\}) = \mathcal{F}(\{u_k\}, \{\rho_l\}, \{p_j\}, \mathcal{T}) = \left(U^T.G.U - 2\,V^T.U - C \right) $$

with $U^T = (u_1\; u_2 \cdots u_{N'-1})$ and $N' = I - 2,\; n' = 2\,n - 4$

$$ \left. \begin{array}{lll} G & : & (N'-1) \times (N'-1) \quad \text{matrix} \\ V & : & (N'-1) \quad \text{dim. vector} \\ C & : & \text{scalar} \end{array} \right\} (\{\rho_i\}, \{p_j\}; \mathcal{T}) $$

General formula

$$ {}^{(2)}I_N^n \sim \int_{\Sigma_2} \prod_{i=1}^{3} d\rho_i \, \underbrace{P(\{\rho_i\})} \, \underbrace{{}^{(1)}\widetilde{I}_{N'}^{n'}(\{\rho_i\})} $$

Weighting functions in ρ_i

"generalised" 1L (N')-point fcts where "generalised" means

- integration domain over Feynman param. = other than simplex
- kinematics = more general than accessible for 1L processes @ colliders

Strategy

The strategy is to compute analytically the “generalised” 1-loop functions and perform numerically the two remaining integration over the ρ_i’s.

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How to compute the “generalised” 1-loop functions?

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- develop a new method

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The method
Description

Extensive use of the Stokes-like identity

$$\frac{1}{D^{\alpha+1}(\{u_k\})} = \frac{1}{2\,\alpha\,\Delta_d}\left[\frac{d-2\,\alpha}{D^{\alpha}(\{u_k\})} - \nabla_u^T \cdot \left(\frac{U - G^{-1}.V}{D^{\alpha}(\{u_k\})}\right)\right]$$

d : the number of independent integration variables

$$\Delta_d = V^T.G^{-1}.V + C$$

The idea is to adjust the power of the denominator in the l.h.s in such way that only the boundary term remains.

In the case of the four point function $d = 3$.

$$d - 2\alpha = 0 \rightarrow \alpha = 3/2, \quad \text{i. e.} \quad \alpha + 1 = 5/2$$

However in four point function, D is raised to the power 2 not 5/2.

To shift the power of the denominator

$$\int_0^\infty \frac{d\xi}{(D + \xi^\nu)^\mu} = \frac{1}{\nu} B\left(\frac{1}{\nu}, \mu - \frac{1}{\nu}\right) \frac{1}{D^{\mu - 1/\nu}} \tag{1}$$

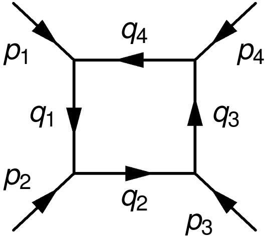
where $B(x, y)$ is the Euler beta function defined by

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x + y)}$$

This integral is convergent for $\mu > 1/\nu > 0$.

As proof of concept, (re)calculation of “standard" 1-loop functions yet with systematic analytic continuation to arbitrary kinematics i.e. not only those restricted to 1-loop collider processes
($\Leftrightarrow$ extension of prior results)

Four-point function
Definition



$$I_4^4 = \int_0^1 \prod_{i=1}^{4} dx_i \, \delta(1 - \sum_{i=1}^{4} x_i) \left(-\frac{1}{2} X^T \mathcal{S} X - i \lambda \right)^{-2}$$

$$\mathcal{S}_{i\,j} = (q_i - q_j)^2 - m_i^2 - m_j^2, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix}$$

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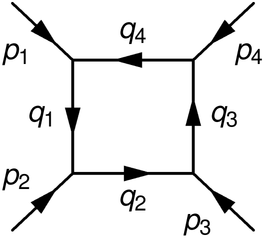


$$I_4^4 = \int_0^1 \prod_{i=1}^{4} dx_i \, \delta(1 - \sum_{i=1}^{4} x_i) \left(-\frac{1}{2} X^T \mathcal{S} X - i \lambda \right)^{-2}$$

Assumption

The imaginary part of the denominator keeps a constant sign in the simplex

The method

Processing

$$I_4^4 = \sum_{i \in S_4} \sum_{j \in S_4 \setminus \{i\}} \sum_{k \in S_4 \setminus \{i,j\}} \frac{\bar{b}_i}{\det(G)} \frac{\bar{b}_j^{\{i\}}}{\det(G^{\{i\}})} \frac{\bar{b}_k^{\{i,j\}}}{\det(G^{\{i,j\}})} \times L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk})$$

with

$$L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) = \kappa \int_0^{+\infty} \frac{d\xi}{(\xi^2 - \Delta_3 - i\lambda)} \int_0^{+\infty} \frac{d\rho}{(\rho^2 + \xi^2 - \Delta_2^{\{i\}} - i\lambda)} \times \int_0^{+\infty} \frac{d\sigma}{(\sigma^2 + \rho^2 + \xi^2 - \Delta_1^{\{i,j\}} - i\lambda)\,(\sigma^2 + \rho^2 + \xi^2 + \tilde{D}_{ijk} - i\lambda)^{1/2}}$$

The method

Processing

$\mathcal{S}^{\{i\}}$: a 3×3 matrix made from $\mathcal{S}$ by removing the row and column i

$$\tilde{D}_{ijk} = 2\, m_l^2 \quad \text{with } l = S_4 \setminus \{i\, j\, k\}$$

$$\bar{b}_i = \det(\mathcal{S}) \sum_{j \in S} \mathcal{S}^{-1}_{i\,j}$$

$$\Delta_1^{\{ij\}} = -\frac{\det(\mathcal{S}^{\{i,j\}})}{\det(G^{\{i,j\}})}$$

$$\Delta_2^{\{i\}} = \frac{\det(\mathcal{S}^{\{i\}})}{\det(G^{\{i\}})}$$

$$\Delta_3 = -\frac{\det(\mathcal{S})}{\det(G)}$$

The method

Integration

$$
\left.\begin{array}{lcccc}
P_{ijk} & = & \tilde{D}_{ijk} & + & \Delta_1^{\{i,j\}} \\
R_{ij} & = & \Delta_2^{\{i\}} & - & \Delta_1^{\{i,j\}} \\
Q_i & = & \Delta_3 & - & \Delta_2^{\{i\}} \\
T & = & -\Delta_3 & &
\end{array}\right\} \Leftrightarrow \left\{\begin{array}{lcr}
P_{ijk} + R_{ij} + Q_i + T & = & \tilde{D}_{ijk} \\
R_{ij} + Q_i + T & = & -\Delta_1^{\{i,j\}} \\
Q_i + T & = & -\Delta_2^{\{i\}} \\
T & = & -\Delta_3
\end{array}\right.
$$

The method

Results

case $\mathrm{Im}(\Delta_3) > 0$, $\mathrm{Im}(\Delta_2^i) > 0$, $\mathrm{Im}(\Delta_1^{ij}) > 0$, $\mathrm{Im}(\tilde{D}_{ijk}) < 0$

$$
\begin{aligned}
& L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \\
& \quad = - \int_0^1 \frac{du}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \\
& \qquad \left[\ln\left(u^2 \, P_{ijk} + (R_{ij} + Q_i + T) \right) - \ln\left(\frac{(R_{ij} + Q_i)}{Q_i} (Q_i + T) \right) \right. \\
& \qquad - \ln\left(u^2 \, (P_{ijk} + R_{ij} + Q_i) + T \right) + \ln\left(T \, \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \, \frac{(R_{ij} + Q_i)}{Q_i} \right) \\
& \qquad + \ln\left(u^2 \, Q_i + T \right) - \ln\left(\frac{T}{P_{ijk}} (P_{ijk} + R_{ij}) \right) \\
& \qquad \left. + \eta\left(\frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta\left(T \, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i} \right) \right]
\end{aligned}
$$

The method

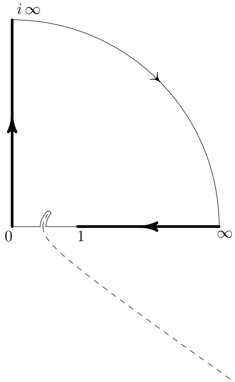
Results

case $\mathrm{Im}(\Delta_3) < 0$, $\mathrm{Im}(\Delta_2^i) > 0$, $\mathrm{Im}(\Delta_1^{ij}) > 0$, $\mathrm{Im}(\tilde{D}_{ijk}) < 0$

$$
\begin{aligned}
& L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \\
&= - \Bigg\{ \int_{\widehat{(0,1)}^+} \frac{du}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \\
&\qquad \left[- \ln\left(u^2 \, (P_{ijk} + R_{ij} + Q_i) + T\right) + \ln\left(T \, \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \, \frac{(R_{ij} + Q_i)}{Q_i}\right) \right] \\
&\quad + \int_0^1 \frac{du}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \\
&\qquad \Bigg[\ln\left(u^2 \, P_{ijk} + (R_{ij} + Q_i + T)\right) - \ln\left(\frac{(R_{ij} + Q_i)}{Q_i} \, (Q_i + T)\right) \\
&\qquad + \ln\left(u^2 \, Q_i + T\right) - \ln\left(T \, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}\right) \\
&\qquad + \eta\left(\frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T)\right) - \eta\left(T \, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i}\right) \Bigg] \\
&\quad - \int_{\Gamma^+} \frac{du}{u^2 \, P_{ijk} \, Q_i - R_{ij} \, T} \, \eta\left(T \, \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \, \frac{R_{ij}}{Q_i}, \frac{R_{ij} + Q_i}{R_{ij}}\right) \Bigg\}
\end{aligned}
$$

The method
Contour deformation

$$\int_{\widehat{(0,1)}^+} \frac{du}{u^2\, P_{ijk}\, Q_i \,-\, R_{ij}\, T} \left[-\ln\left(u^2\,(P_{ijk}+R_{ij}+Q_i)+T\right) + \ln\left(T\,\frac{(P_{ijk}+R_{ij})}{P_{ijk}}\,\frac{(R_{ij}+Q_i)}{Q_i}\right)\right]$$



The method

IR case

No problem to extend the method for IR/UV divergent case ($n = 4 - 2\,\varepsilon$)

For infrared divergent cases, one or several sectors have $\Delta_2^i = 0$ (sub-leading Landau singularities), two scenarios :

- $\tilde{D}_{ijk} \neq 0$ (soft divergence)
- $\tilde{D}_{ijk} = 0$ (soft and collinear or collinear divergence)

Conclusion

Method with nice features

- Two dimensional integration numerically, whatever N is.
- Valid outside the physical domain, it runs smoothly with all the traps of complex mass cases
- Expressed in term of the (reduced) kinematical matrix and the (reduced) Gram matrix

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Numerical implementation of the “one-loop” part

Fortran95 code, checks with looptools

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Numerical implementation of the “one-loop” part

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Drawback

This method generates more dilogarithms than other methods (HV), it can be improved!

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Ongoing step

Extend to the case where the volume of the “one loop” Feynman parameters is not a simplex

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Next steps I

Apply method to ${}^{(1)}\widetilde{I}^{4}_{N+1}$ for all 2-loop topologies
so as to build 2-loop library in general complex mass case.

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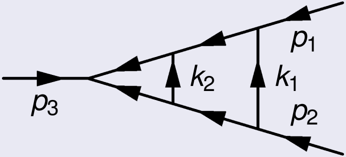
Next steps II

Extension to tensorial integrals

A specific example

Massive 2-loop planar VTX



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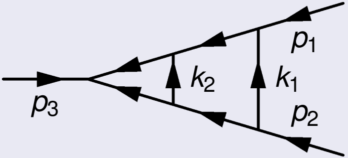

$$ {}^{(2)}I_3^n\left(\{p_j\};\mathcal{T}_l\right) = -(4\pi)^{-n}\,\Gamma(6-n)\int_0^1 \prod_{k=1}^{6} dx_k\,\delta\left(1-\sum_{l=1}^{6} x_l\right) $$

$$ \left[x_2\,(x_1+x_3+x_4)+(x_5+x_6)\,(x_1+x_3+x_4+x_2)\right]^{6-\frac{3\,n}{2}} $$

$$ \times\left[\mathcal{F}(\{x_k\},\{p_j\},\mathcal{T}_l)-i\,\lambda\right]^{n-6} $$

$$\rho_1 = x_5 + x_6, \quad \rho_2 = x_1 + x_3 + x_4, \quad \rho_1 = \rho\,\xi, \quad \rho_2 = (1-\xi)\,\rho$$
$$x_5 = \rho_1\,u_1, \quad x_6 = \rho_1\,(1-u_1)$$
$$x_3 = \rho_2\,u_2, \quad x_4 = \rho_2\,u_3, \quad x_1 = \rho_2\,(1-u_2-u_3)$$

such that:

$$\delta\left(1 - \sum_{l=1}^{6} x_l\right) = \delta(1 - x_2 - \rho)$$
$$\det(M) = \rho\,[1 - \rho + \rho\,\xi\,(1-\xi)]$$

A specific example
Reparametrization

$$\rho_1 = x_5 + x_6, \quad \rho_2 = x_1 + x_3 + x_4, \quad \rho_1 = \rho\,\xi, \quad \rho_2 = (1-\xi)\,\rho$$
$$x_5 = \rho_1\,u_1, \quad x_6 = \rho_1\,(1-u_1)$$
$$x_3 = \rho_2\,u_2, \quad x_4 = \rho_2\,u_3, \quad x_1 = \rho_2\,(1-u_2-u_3)$$

such that:

$$\delta\left(1 - \sum_{l=1}^{6} x_l\right) = \delta(1 - x_2 - \rho)$$
$$\det(M) = \rho\,\left[1 - \rho + \rho\,\xi\,(1-\xi)\right]$$

No U.V. divergences : $n = 4$

A specific example

Reparametrization

After reparametrization

$$
{}^{(2)}I_3^4\left(\{p_j\};\mathcal{T}_l\right) = -\,(4\,\pi)^{-4}\int_0^1\int_0^1 d\rho\,d\xi\,\rho^2\,\xi\,(1-\xi)^2
$$

$$
\times\int_0^1 du_1\int_0^1 du_2\int_0^{1-u_2} du_3\,\frac{1}{(D-i\,\lambda)^2}
$$

$$
D = \left(U^T.G.U - 2\,V^T.U - C\right) \quad \text{with} \quad U^T = (u_1\ u_2\ u_3)
$$

$$
\left.\begin{array}{lcl}
G & : & 3\times 3 \quad \text{matrix} \\
V & : & 3 \quad \text{dim. vector} \\
C & : & \text{scalar}
\end{array}\right\}\left(\{\rho,\xi\},\{p_j\};\mathcal{T}_l\right)
$$