

# Renormalization of gauge theories at five loops



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recent work with John Gracey, Ian Jack,  
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and earlier work with  
J. Möller, C. Studerus

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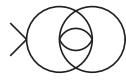
# Motivation

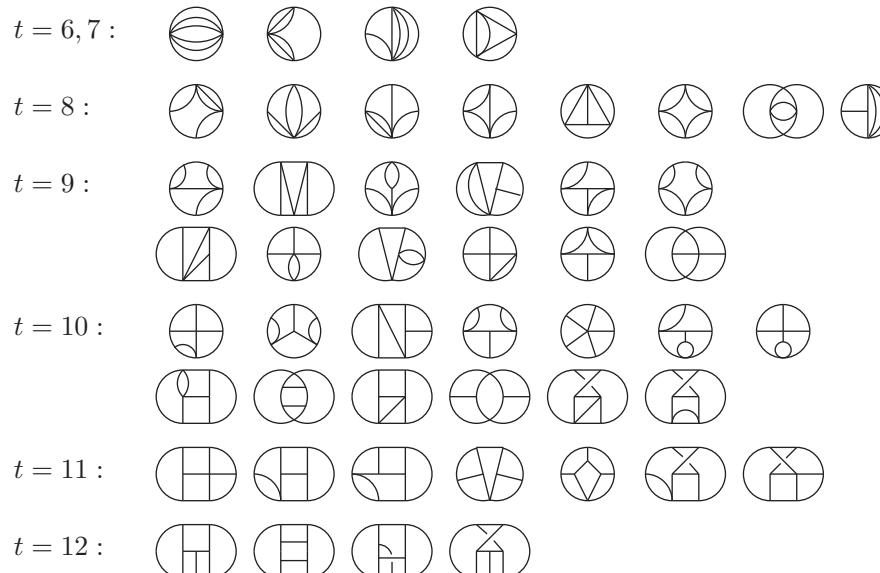
- LHC: era of precision QFT
  - ▷ large SM backgrounds
  - ▷ dominant effects: mainly QCD
  - ▷ high precision required for BSM searches
- theory working horse: renormalizable QFTs
  - ▷ perturbative expansions  $\Rightarrow$  Feynman diagrams
  - ▷ higher-loop effects important (e.g.  $g_\mu - 2$ ;  $m_q$ ;  $H$  production; ...)
- lots of machinery developed recently
  - ▷ high automatization of complicated perturbative calculations
  - ▷ algebraic handling of Feynman diagrams
  - ▷ reduction of Feynman integrals to masters
  - ▷ numerical and/or algebraic determination of masters

# Strategy

- for higher-loop precision, need to regularize and renormalize theory
  - ▷ work in dimensional reg.  $d^4x \rightarrow d^d x$  and in  $\overline{\text{MS}}$  scheme
  - ▷ evaluate all independent RCs: absorb *divergences*
  - ▷ e.g. QCD fields:  $\psi_b = \sqrt{Z_2}\psi_r$ ,  $A_b = \sqrt{Z_3}A_r$ ,  $c_b = \sqrt{Z_3^c}c_r$
  - ▷ e.g. QCD couplings:  $m_b = Z_m m_r$ ,  $g_b = \mu^\epsilon Z_g g_r$
  - ▷ equivalently, def anomalous dimensions:  $\gamma_i = -\partial_{\ln \mu^2} \ln Z_i$
- keep gauge group general
- generate all Fy diagrams; perform algebra: group-, Lorentz-, ... [Nogueira QGRAF; Vermaseren FORM]
- project all Fy integrals to unique set of scalar massive vacuum ints
  - ▷ exact decomposition of propagators ( $m \in \{0, m, M\}$ ) [Chetyrkin/Misiak/Münz 1998]
  - ▷  $\frac{1}{(k-p)^2+m^2} = \frac{1}{k^2+M^2} + \frac{2kp-p^2+M^2-m^2}{(k^2+M^2)((k-p)^2+m^2)}$
  - ▷ recursively lower degree of UV div
  - ▷ other IR regularization schemes: e.g. (local/global) R\* [Chetyrkin 1984]
- map all integrals to minimal set: IBP [Chetyrkin/Tkachov 1981; Laporta 2000]
- evaluate this minimal set (analytically/numerically to high accuracy) [with Luthe since 2011]

# Reduction

- motivation: 5-loop  $\Phi^4$ :  1 integral; YM:  1T integrals ( $2^{14}6^{10}$ )
- complexity reduction via IBP (integration by parts, in  $d$  dim) [Chetyrkin/Tkachov 1981]
  - systematically use  $0 = \int d^d k \partial_{k_\mu} f_\mu(k)$  [Laporta, Baikov, Gröbner]
  - key idea: lexicographic ordering among all loop integrals [Laporta 2000]
- arrive at rep in terms of irreducible ( $\equiv$  master) integrals:  $\sum_i \frac{\text{poly}_i(d, \xi)}{\text{poly}_i(d)} \text{Master}_i(d)$ 
  - e.g. 1234-loop:  $1+1+3+(10+3)= 18$  (fully massive) master ints [Laporta 2002]
- now consider fully massive 5-loop tadpoles
  - Euclidean space-time
  - same mass in all propagators  $\Rightarrow 1/(q_i^2 + 1)$
- arrive at 48 unique 5-loop sectors (+19 factorized ones not shown)



# 5-loop Masters

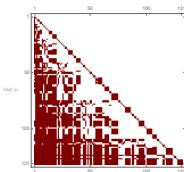
- a (small) IBP reduction reveals that some sectors contain multiple master integrals
  - ▷ need in addition 62 (+3 factorized ones) masters with 'dots'. some examples:
  - ▷ recall that at 1/2/3/4-loop there were 0/0/0/3 masters with 'dots'
- how to evaluate these 48+62 (+19+3) zero-scale master integrals as  $fct(d)$ ?  
various methods, e.g.
  - ▷ explicit integration in x-space
  - ▷ differential eqs (in mass ratio); solve iteratively with HPLs
  - ▷ solve dimensional recurrences, regarding  $d \in \mathbb{C}$  [Lee 2009]
  - ▷ explicit solution of low-order difference equations:  $PF_{P-1}$  etc.
  - ▷ numerical solution of difference equations via factorial series [Laporta 2000]
- Mathematical structure
  - ▷ interested in the coefficients of an  $\epsilon$  expansion
  - ▷ e.g. harmonic sums  $S(N)$  [Vermaseren 1998; Blümlein/Kurth 1998]
  - ▷ e.g. harmonic polylogarithms HPL(x) [Remiddi/Vermaseren 2000]
  - ▷ e.g. elliptic multiple zetas emZV [Levin/Brown 2007]
  - ▷ if solution numerical: use some PSLQ

# Generalities

- parametric representation
  - ▷ write LC of propagals as  $\sum x_i D_i = k^T M k + 2P \cdot k + Q$
  - ▷ get Symanzik polys as  $\mathcal{U} = \det M$  and  $\mathcal{F} = (\det M)(P^T M^{-1} P + Q)$
- simplification for fully massive integrals without external legs
  - ▷  $P = 0$  and  $Q = \sum x_i \Rightarrow \mathcal{F} = \mathcal{U}Q = \mathcal{U}$
- $L$ -loop massive tadpole in  $d$  dimensions, having  $N$  lines with powers  $a_1, \dots, a_N$

$$\frac{I_a}{J^L} = \frac{\Gamma(A - Ld/2)}{[\Gamma(1 - d/2)]^L} \int_0^\infty d^N x \delta(1 - X) \frac{p_a(x)}{[\mathcal{U}(x)]^{d/2}}$$

- normalization by  $J = \int d^d k / (k^2 + 1) = \pi^{d/2} \Gamma(1 - d/2)$
- numerators  $p_a(x) = \prod_{i=1}^N x_i^{a_i - 1} / \Gamma(a_i)$
- dimensional shifts via  $I_a^{(d)} = \mathcal{U}(x_i \rightarrow \partial_{m_i^2}) I_a^{(d+2)}$ 
  - ▷ can be exploited to solve for (some) masters
  - ▷ at 1 loop: large- $d$  asymptotics to fix fcts
  - ▷  $> 1$  loop: solve up to periodic fct; compute special points in fixed dims



[Tarasov 1996]

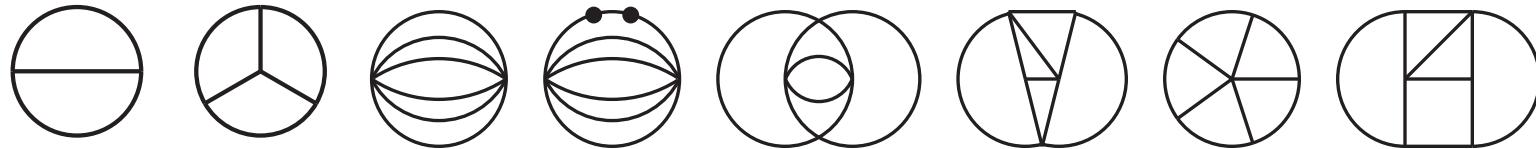
[Tarasov 2000]

[Lee 2009 ff]

# Evaluation

- perform IBP reduction with symbolic power  $\textcolor{red}{x}$  on one line
- derive difference equation for generalized master  $I(\textcolor{red}{x}) \equiv \int \frac{1}{D_1^{\textcolor{red}{x}} D_2 \dots D_N}$ 
  - ▷ generic form:  $\sum_{j=0}^R p_j(\textcolor{red}{x}) I(\textcolor{red}{x} + j) = F(\textcolor{red}{x})$
- typically, want  $I(1)$ ; solve the difference equation
  - ▷ explicitly (e.g. if 1st order; or if soln nested sum)
  - ▷ numerically (very general setup) [Laporta 2000]
- solve via factorial series  $I(x) = I_0(x) + \sum_{j=1}^R I_j(x)$ ,
  - ▷ where  $I_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$
- need boundary condition for fixing, say,  $a_j(0)$ : use decoupling at large  $x$ 
  - ▷  $I(x) = \int_{k_1} g(k_1)/(k_1^2 + 1)^x \Rightarrow I(x) \sim (1)^x x^{-d/2} g(0)$
- deep expansions ( $\epsilon^{20}$ ) at 5 loops (132 masters) at high precision (>250 digits) [with Luthe 2016]

## Sample results (4d)



$$I_{28686.1.1} = +(-3)\epsilon^0 + \left(-\frac{3}{2}\right)\epsilon^1 + \left(\frac{13}{24}\right)\epsilon^2 + \left(-\frac{1267}{1440}\right)\epsilon^3 + \left(-\frac{4193}{3456}\right)\epsilon^4 + \\ + 135.95072868792871461956492733702218574897992953584\epsilon^5 + \dots$$

$$I_{28686.1.3} = +(0)\epsilon^0 + \left(\frac{3}{2}\right)\epsilon^1 + \left(-\frac{1}{2}\right)\epsilon^2 + \left(-\frac{443}{360}\right)\epsilon^3 + \left(\frac{95}{216}\right)\epsilon^4 + \\ - 38.292059175062436961881799538284449799148385376441\epsilon^5 + \dots$$

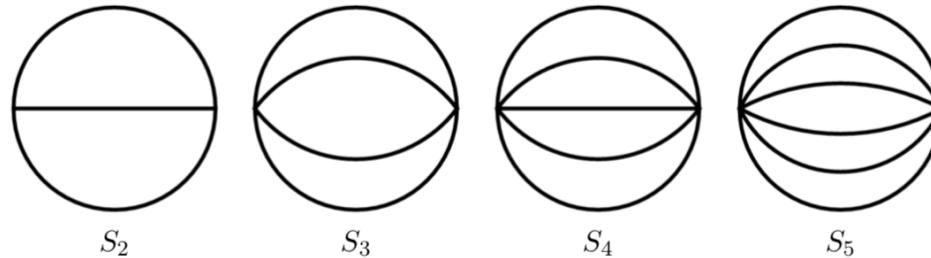
$$I_{30862.1.1} = +\left(-\frac{3}{5}\right)\epsilon^0 + \left(-\frac{27}{10}\right)\epsilon^1 + \left(-\frac{4\zeta_3}{5} - \frac{421}{60}\right)\epsilon^2 + \left(-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24}\right)\epsilon^3 + \left(\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48}\right)\epsilon^4 + \\ + 1143.1838307558764599466030303839590323268318605888\epsilon^5 + \dots$$

$$I_{30231.1.1} = +(0)\epsilon^0 + (0)\epsilon^1 + \left(\frac{3\zeta_3}{5}\right)\epsilon^2 + \left(\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5\right)\epsilon^3 + \left(-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5}\right)\epsilon^4 - \\ - 531.32391547725635267943444561495368318398901378435\epsilon^5 + \dots$$

$$I_{32596.1.1} = +(0)\epsilon^0 + (0)\epsilon^1 + (0)\epsilon^2 + (0)\epsilon^3 + (-14\zeta_7)\epsilon^4 + \quad [\text{Wheel: Broadhurst 1985}] \\ + 235.07729596783467131454388080950411779239347239580\epsilon^5 + \dots$$

$$I_{32279.3.1} = +(0)\epsilon^0 + (0)\epsilon^1 + (0)\epsilon^2 + (0)\epsilon^3 + \left(-\frac{441\zeta_7}{40}\right)\epsilon^4 + \quad [\text{Zigzag: Broadhurst/Kreimer 1995; Brown/Schnetz 2012}] \\ + 181.78223928612340820790788236018642961198741994209\epsilon^5 + \dots$$

## Nice subsets: sunsets



- much-studied class;  $S_{L>3}$  should be elliptic [Laporta 2002]
- at 2 loops: hypergeom soln (for all  $d$ ) [Davydychev/Tausk 1993]
- at 2 loops: expansions contain  $\text{HPL}(e^{2\pi i/6})$ ,  $\text{HPL}(e^{2\pi i/3})$  [see Broadhurst 1998]

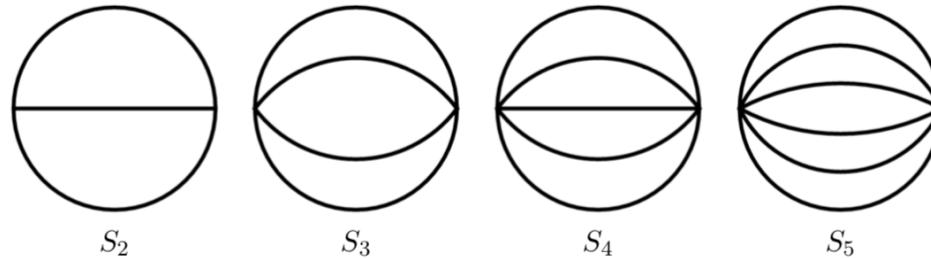
$$S_2/J^2 \stackrel{2-2\epsilon}{=} 6H_2 \epsilon^2 - 12H_3 \epsilon^3 + 24H_4 \epsilon^4 - 48H_5 \epsilon^5 + 96H_6 \epsilon^6 + \dots$$

$$S_2/J^2 \stackrel{1-2\epsilon}{=} \frac{3}{4} \left[ \frac{4}{9} - G_1 \epsilon + G_2 \epsilon^2 - G_3 \epsilon^3 + G_4 \epsilon^4 - G_5 \epsilon^5 + \dots \right]$$

- ▷  $H_n = h_n + h_1 \hat{C}_{n-1} \left( 1 - \frac{3^{\epsilon/2} \Gamma(1-\epsilon)}{\Gamma^2(1-\epsilon/2)} \right)$  where  $h_n = {}_{n+1}F_n \left( \frac{1}{2}, \dots, \frac{1}{2}; \frac{3}{2}, \dots, \frac{3}{2}; \frac{3}{4} \right)$
- ▷ e.g.  $3H_1 = 3h_1 = 2\pi/\sqrt{3}$ ,  $3H_2 = \sqrt{3} \text{Cl}_2(2\pi/3)$  with  $\text{Cl}_n(x) = \text{Im Li}_n(e^{ix})$
- ▷  $G_n = g_n + \frac{16}{9} \hat{C}_n \left( \frac{\pi 3^\epsilon \Gamma(1-2\epsilon)}{\Gamma^2(1/2-\epsilon)} - 1 \right)$  where  $g_n = {}_{n+2}F_{n+1} \left( \frac{3}{2}, 1, \dots, 1; 2, \dots, 2; \frac{3}{4} \right)$
- ▷ e.g.  $9G_1 = -32 \ln \frac{3}{2}$ ,  $9G_2 = 16(3\text{Li}_2(\frac{1}{4}) - \zeta_2 + 3 \ln^2 2 - \ln^2 \frac{3}{2})$

- at 2 loops: dim-shift rel  $\hat{S}_2^{(d+2)} = \frac{3d}{4(d-1)} [\hat{S}_2^{(d)} - 1]$

## Nice subsets: sunsets



- at 3 loops: expansions contain  $\text{HPL}(1)$

$$\begin{aligned} S_3/J^3 &\stackrel{2-2\epsilon}{=} 7\zeta_3\epsilon^3 + [48A_4 - 51\zeta_4]\epsilon^4 + [288A_5 + 306\zeta_4 \ln 2 - 465\zeta_5/2]\epsilon^5 \\ &\quad + [1728A_6 + 720s_6 - 918\zeta_4 \ln^2 2 - 1134\zeta_6 - 305\zeta_3^2]\epsilon^6 + \dots \end{aligned}$$

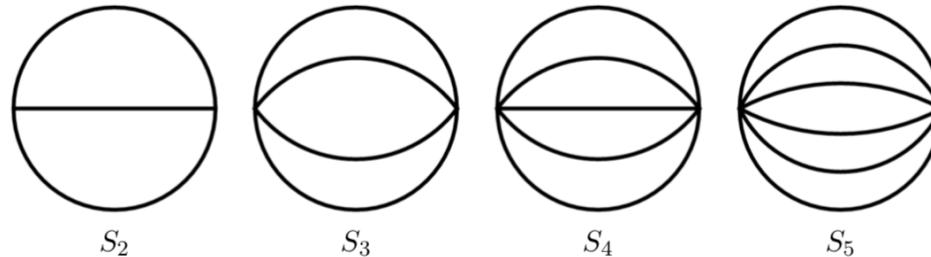
$$S_3/J^3 \stackrel{1-2\epsilon}{=} \frac{1}{4} + \frac{3}{2}[\ln 2]\epsilon + \frac{3}{2}[4 \ln 2 - \ln^2 2 - \zeta_2]\epsilon^2 + \dots$$

▷ where  $A_n = \text{Li}_n(\frac{1}{2}) + \text{LC}(\zeta_2, \ln 2)$  and  $s_6 = S_{-5,-1}(\infty) \approx 0.98744\dots$

- at 3 loops: dim-shift rel  $\hat{S}_3^{(d+2)} = \frac{2d^2}{3(d-1)(3d-4)(3d-2)}[8(d-2)\hat{S}_3^{(d)} - (11d-16)]$
- at 4 loops: numerical expansion

$$\begin{aligned} S_4/J^4 &\stackrel{4-2\epsilon}{=} -\frac{5}{2} - \frac{5}{3}\epsilon - \frac{5}{144}\epsilon^2 - \frac{625}{864}\epsilon^3 \\ &\quad - 16.146469213466433695263636425983589656582094409714\epsilon^4 \\ &\quad - 302.68724953271679694373481692172522196605049592725\epsilon^5 + \dots \end{aligned}$$

## Nice subsets: sunsets



- at 5 loops: numerical expansion

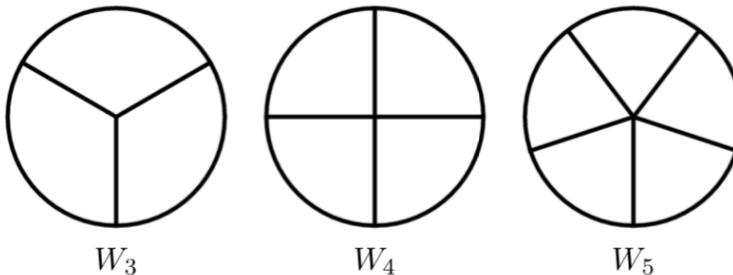
$$\begin{aligned}
 S_5/J^5 &\stackrel{4-2\epsilon}{=} -3 - \frac{3}{2}\epsilon + \frac{13}{24}\epsilon^2 - \frac{1267}{1440}\epsilon^3 - \frac{4193}{3456}\epsilon^4 \\
 &+ 135.95072868792871461956492733702218574897992953584\epsilon^5 \\
 &+ 2603.8178589306305816799512636574088079352965131038\epsilon^6 + \dots
 \end{aligned}$$

- interesting structure arises in d=2: leading term is a Bessel moment

$$\tilde{S}_L = \frac{S_L^{(2d)} / (\epsilon J)^L}{\Gamma(L+2)} = \frac{2^L}{\Gamma(L+2)} \int_0^\infty dx x [K_0(x)]^{L+1}$$

- ▷ known evaluations are  $\tilde{S}_1 = 1/2$  ,  $\tilde{S}_2 = H_2$  ,  $\tilde{S}_3 = 7\zeta_3/24$
- ▷ and  $\tilde{S}_\infty = \exp(-2\gamma_E)$

## Nice subsets: wheels



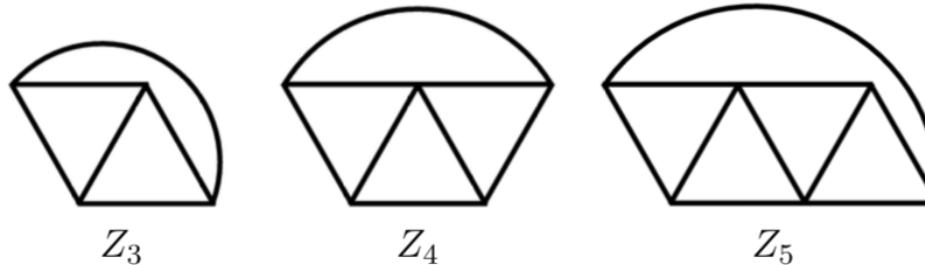
- leading terms known (at d=4) to all loop orders [Broadhurst 1985]

$$\frac{W_L}{J^L} = \frac{(-1)^L \Gamma(2L - 1)}{\Gamma(L) \Gamma(L + 1)} \zeta(2L - 3) \epsilon^{L-1} + \mathcal{O}(\epsilon^L)$$

- can use this as check on numerics; for  $W_3$  and  $W_4$  OK to tens of thousands digits
- at 5 loops  $W_5/J^5 = -14\zeta_7\epsilon^4 + \mathcal{O}(\epsilon^6)$  to 250 digits (plus many more orders numerically)

$$\begin{aligned}
 W_5/J^5 &\stackrel{4-2\epsilon}{=} -14.116889883346919575757165697897154634398089847913 \epsilon^4 \\
 &\quad + 235.07729596783467131454388080950411779239347239580 \epsilon^5 \\
 &\quad - 2267.7386832930084122962994480580205855487545413738 \epsilon^6 + \dots
 \end{aligned}$$

## Nice subsets: zigzags



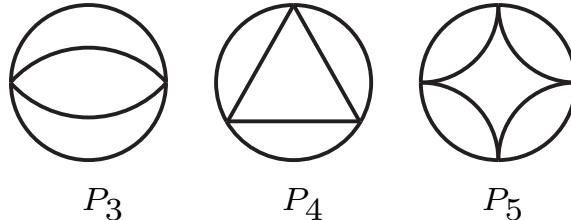
- note  $Z_3 = W_3$  and  $Z_4 = W_4$
- leading terms known (at d=4) to all loop orders [Kazakov 1984; Broadhurst 1995; Brown/Schnetz 2012]

$$\frac{Z_L}{J^L} = \frac{W_L}{J^L} \frac{4}{L} \left( 1 + \frac{(-1)^L - 1}{2^{2L-3}} \right) + \mathcal{O}(\epsilon^L)$$

- can use this as check on numerics
- at 5 loops  $Z_5/J^5 = -\frac{441}{40}\zeta_7\epsilon^4$  to 250 digits (plus many more orders numerically)

$$\begin{aligned} Z_5/J^5 &\stackrel{4-2\epsilon}{=} -11.117050783135699165908767987094009274588495755231 \epsilon^4 \\ &\quad + 181.78223928612340820790788236018642961198741994209 \epsilon^5 \\ &\quad - 1725.9996137403520805951673992442117288607704957542 \epsilon^6 + \dots \end{aligned}$$

## Nice subsets: strings of bubbles



- $P_3 (= S_3)$  and  $P_4$  contain HPL(1) only, to all orders in  $\epsilon$  (proof via differential eq in  $M/m$ )

$$P_4/J^4 \stackrel{2-2\epsilon}{=} 3\zeta_3\epsilon^4 + [48A_4 + 30\zeta_3 - 57\zeta_4]\epsilon^5 + [96A_4 + 288A_5 + 306\zeta_4 \ln 2 - 276\zeta_3 - 78\zeta_4 + 39\zeta_5/2]\epsilon^6 + \dots$$

- PSLQ-fitted numerical result for  $P_5$  contains HPL(1) only

$$\begin{aligned} P_5/J^5 &\stackrel{4-2\epsilon}{=} -\frac{6}{5} - \frac{23}{5}\epsilon - \frac{91}{10}\epsilon^2 + \left[ \frac{28\zeta_3}{5} + \frac{697}{20} \right] \epsilon^3 + \left[ \frac{84\zeta_2^2}{25} - \frac{1066\zeta_3}{5} + \frac{22189}{40} \right] \epsilon^4 \\ &\quad + \left[ -1536A_4 + \frac{16482\zeta_2^2}{25} - \frac{12001\zeta_3}{5} - \frac{7716\zeta_5}{5} + \frac{354017}{80} \right] \epsilon^5 + \dots \end{aligned}$$

- dim-shift relations are closed within this set:

$$\begin{aligned} \hat{P}_3^{(d+2)} &= \frac{2d^2}{3(d-1)(3d-4)(3d-2)} \left[ 8(d-2)\hat{P}_3^{(d)} - (11d-16) \right] \\ \hat{P}_4^{(d+2)} &= \frac{d^3}{32(d-1)^3(2d-3)} \left[ \frac{6(d-3)(3d-7)(3d-5)}{(d-2)(2d-5)} \hat{P}_4^{(d)} - \frac{295d^3-1614d^2+2855d-1620}{(2d-5)(3d-4)} \hat{P}_3^{(d)} + \frac{2(19d^2-53d+36)}{(3d-4)} \right] \\ \hat{P}_5^{(d+2)} &= r_5(d)\hat{P}_5^{(d)} + r_4(d)\hat{P}_4^{(d)} + r_3(d)\hat{P}_3^{(d)} + r_1(d) \quad , \quad \text{etc.} \end{aligned}$$

- general  $L$ -loop  $\epsilon$ -expansions possible?

# d=4: Quantum Chromodynamics

- drive renormalization program of QCD to five loops [with Luthe/Maier/Marquard]
- long-term effort of three independent groups [also: Chetyrkin/Baikov; Ueda/Vermaseren/Vogt]
- notation: strong coupling constant  $a = \frac{C_A g^2(\mu)}{16\pi^2}$ 
  - ▷ color:  $n_f \equiv \frac{T_F N_f}{C_A}$  ,  $c_f \equiv \frac{C_F}{C_A}$  ,  $d_{FF} \equiv \frac{[sTr(T^a T^b T^c T^d)]^2}{N_A T_F^2 C_A^2}$  , ...
- result: 5-loop QCD  $\beta$ -function [with Luthe/Maier/Marquard]
  - ▷  $\beta = \partial_{\ln \mu^2} a = -a \left[ \epsilon + \frac{11-4n_f}{3} a + b_1 a^2 + b_2 a^3 + b_3 a^4 + b_4 a^5 + \dots \right]$
  - ▷  $3^5 b_4 = n_f^4 \left[ c_1 c_f + c_2 \right]$  [Gracey 1996]
  - ▷  $+ n_f^3 \left[ c_3 c_f^2 + c_4 c_f + c_5 d_{FF} + c_6 \right]$  [LMMS 2016]
  - ▷  $+ n_f^2 \left[ \dots \right] + n_f \left[ \dots \right] + \left[ c_{22} d_{AA} + c_{23} \right]$  [Herzog/Ruijl/Ueda/Vermaseren/Vogt 2017]
  - ▷ the  $n_f^4$  term agrees exactly with known result
  - ▷ all  $c_i$  in terms of Zeta values,  
e.g.  $c_6 = -3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)$
  - ▷ last row confirmed [LMMS 2017]

# d=4: Quantum Chromodynamics

- drive renormalization program of QCD to five loops [with Luthe/Maier/Marquard]
- notation: strong coupling constant  $a = \frac{C_A g^2(\mu)}{16\pi^2}$ 
  - ▷ color:  $n_f \equiv \frac{T_F N_f}{C_A}$  ,  $c_f \equiv \frac{C_F}{C_A}$  ,  $d_{FF} \equiv \frac{[sTr(T^a T^b T^c T^d)]^2}{N_A T_F^2 C_A^2}$
- result: 5-loop QCD quark mass anomalous dimension [LMMS 2016]
  - ▷  $\gamma_m = \partial_{\ln \mu^2} \ln m_q = -a c_f \left[ 3 + \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4 + \dots \right]$
  - ▷ as usual,  $\gamma_1 = (9c_f + 97)/6 - 10/3 n_f$  , ... ,
  - ▷ and now  $\gamma_4 = n_f^4 [c_1] + n_f^3 [c_2 c_f + c_3] + n_f^2 [c_4 c_f^2 + c_5 c_f + c_6 d_{FF} + c_7] + \dots$
  - ▷ know all  $c_i$  analytically (Zetas only), for all  $n_f$  and color structures
  - ▷ the  $n_f^4$  and  $n_f^3$  terms agree exactly with known results [Gracey 1996]
- completion of 5-loop renormalization program
  - ▷ besides  $\beta$  and  $\gamma_m$ , have also  $Z_{\psi\psi}$ ,  $Z_{cc}$  and  $Z_{ccg}$  (Fy gauge +  $\xi^1$ ) [LMMS 2017]
  - ▷ all other RCs follow from these five, due to gauge invariance
  - ▷ full gauge dependence now also available [Chetyrkin/Falcioni/Herzog/Vermaseren 2017]

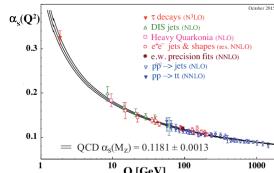
# Conclusions

- recent advance in methods allows high-loop renormalization of quantum field theories
  - ▷ highly automated computer-algebra approaches
  - ▷ improved IBP algorithms (finite fields, ...)
  - ▷ new insight into functional content of Feynman integrals
- difference equations (+ lots of CPU) are powerful enough to achieve 5 loops
  - ▷ non-trivial algorithmic fine-tuning
  - ▷ more? memory seems to become an issue
- applications to a host of QFTs in various fields
  - ▷ CM theory / effective models
  - ▷ mathematical physics/ SUSY
  - ▷ phenomenology / QCD+SM
  - ▷ ...
- applications in various space-time dimensions
  - ▷ 2d, 3d, 4d, ..., fractional, ...

# Apparent convergence

- one might be concerned about 'asymptotic series' behavior at 5 loops
  - e.g. QM: 1d anharmonic oscillator  $\hat{H} \sim \hat{p}^2 + \hat{x}^2 + g\hat{x}^4$
  - ground state energy  $E_0 = \sum e_n g^n$ , with  $|\frac{e_{n+1}}{e_n}| \stackrel{n \gg 1}{\approx} \frac{3n}{4}$ , diverges [Bender/Wu 1973]
  - (however defines unique function  $E_0(g)$  via Borel; compare  $\frac{1}{1-x} = \sum x^n$ )

- now QCD.  $N_c = 3$ ,  $\alpha_s = \frac{g^2(Q)}{4\pi} =$ 
  - Beta function:



$\approx 0.1$  (at EW scale)

[see also Baikov/Chetyrkin/Kühn 2016]

$$\beta_{N_f=3} = -0.17 \alpha_s^2 [1 + 0.57 \alpha_s + 0.45 \alpha_s^2 + 0.68 \alpha_s^3 + 0.58 \alpha_s^4 + \dots]$$

$$\beta_{N_f=4} = -0.16 \alpha_s^2 [1 + 0.49 \alpha_s + 0.31 \alpha_s^2 + 0.49 \alpha_s^3 + 0.28 \alpha_s^4 + \dots]$$

$$\beta_{N_f=5} = -0.15 \alpha_s^2 [1 + 0.40 \alpha_s + 0.15 \alpha_s^2 + 0.32 \alpha_s^3 + 0.08 \alpha_s^4 + \dots]$$

$$\beta_{N_f=6} = -0.13 \alpha_s^2 [1 + 0.30 \alpha_s - 0.03 \alpha_s^2 + 0.18 \alpha_s^3 + 0.002 \alpha_s^4 + \dots]$$

- Quark mass anomalous dimension:

[see also Baikov/Chetyrkin/Kühn 2014]

$$\gamma_{N_f=3} = -0.32 \alpha_s [1 + 1.21 \alpha_s + 1.26 \alpha_s^2 + 1.43 \alpha_s^3 + 2.04 \alpha_s^4 + \dots]$$

$$\gamma_{N_f=4} = -0.32 \alpha_s [1 + 1.16 \alpha_s + 1.01 \alpha_s^2 + 0.88 \alpha_s^3 + 1.15 \alpha_s^4 + \dots]$$

$$\gamma_{N_f=5} = -0.32 \alpha_s [1 + 1.12 \alpha_s + 0.75 \alpha_s^2 + 0.36 \alpha_s^3 + 0.43 \alpha_s^4 + \dots]$$

$$\gamma_{N_f=6} = -0.32 \alpha_s [1 + 1.07 \alpha_s + 0.49 \alpha_s^2 - 0.15 \alpha_s^3 - 0.10 \alpha_s^4 + \dots]$$

# Fractional dimensions

- difference eqs carry full information on  $d$ 
  - ▷ expand massive tadpoles e.g. around  $d = \frac{10}{3} - 2\epsilon$ ?
- motivation: renormalization in critical dimension (where  $[g]=0$ )
  - ▷ e.g. O(N) scalar  $\phi^n$  theory:  $\partial_\mu \phi \partial^\mu \phi + g \phi^n$  [Gracey 2017]
  - ▷ critical dim  $D_n = \frac{2n}{n-2} \Rightarrow D_{\{3,4,5,6,7,\dots,\infty\}} = \{6, 4, \frac{10}{3}, 3, \frac{14}{5}, \dots, 2\}$
  - ▷ near FP (nontriv zero of Beta fct), RG fcts carry info on phase transitions
- study RG fcts in non-integer dimensions
  - ▷ renormalization 'as usual', dim. reg. natural (angular ints?!)
  - ▷ fewer diagrams (e.g.  $\phi^5$ : LO 2pt diag has 3 loops)
  - ▷ numbers:  $\frac{p}{q} \rightarrow \Gamma(\frac{p}{q})$  at LO;  $\zeta(s) \rightarrow$  Dirichlet  $\beta$  fct at NLO ( $\in HPL(i)$ ) [Hager 2002]
- sample results in  $d = \frac{10}{3} - 2\epsilon$  [Luthe]
  - ▷ as usual we normalize by  $1/J^{loop}$ ; here,  $J \sim \Gamma(1 - \frac{d}{2}) = -\frac{3}{2}\Gamma(\frac{1}{3}) + \dots$
  - ▷ 2-loop sunset =  $-2.4882241714632542542 \epsilon^0 - 8.6116306893649818141 \epsilon^1 + \dots$
  - ▷ 3-loop merc. =  $-0.0305966721641989027 \epsilon^0 + 0.0149523122719722312 \epsilon^1 + \dots$
  - ▷ 4-loop non-pl =  $+0.0006880032418228675 \epsilon^0 + 0.0023853718027957011 \epsilon^1 + \dots$
  - ▷ etc.