

Algorithm to find an all-order in the running coupling solution to an equation of the DGLAP type

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Evolution equations

The IDE in which the splitting function $P(x)$ participates is

$$u \frac{d}{du} f(x, u) = \frac{\alpha(u)}{2\pi} \int_x^1 \frac{dy}{y} f(y, u) P\left(\frac{x}{y}\right). \quad (1)$$

We calculate Mellin N -moment of both the parts of this equation and obtain the relation

$$\begin{aligned} u \frac{d}{du} \int_0^1 dx \ x^{N-1} f(x, u) &= \frac{\alpha(u)}{2\pi} \int_0^1 dx \ x^{N-1} \int_x^1 \frac{dy}{y} f(y, u) P\left(\frac{x}{y}\right) \\ &= \frac{\alpha(u)}{2\pi} \int_0^1 dy \ f(y, u) \frac{1}{y} \int_0^y dx \ x^{N-1} P\left(\frac{x}{y}\right) = \\ &\quad \frac{\alpha(u)}{2\pi} \int_0^1 dy \ y^{N-1} f(y, u) \int_0^1 dx \ x^{N-1} P(x) \\ &= \frac{\alpha(u)}{2\pi} \gamma(N, \alpha(u)) M[f(y, u), y](N), \end{aligned} \quad (2)$$

Anomalous dimension

We define $\gamma(N, \alpha(u))$ as

$$\int_0^1 dx \ x^{N-1} P(x) = \gamma(N, \alpha(u)), \quad \gamma(1, \alpha(u)) = 1. \quad (3)$$

Thus, the RGE

$$u \frac{d}{du} M[f(x, u), x](N) = \frac{\alpha(u)}{2\pi} \gamma(N, \alpha(u)) M[f(x, u), x](N). \quad (4)$$

may be re-written in the form of IDE (1).

Complete form of Equations in QCD

$$u \frac{d}{du} \Delta_{ij}(x, u) = \frac{\alpha(u)}{2\pi} \int_x^1 \frac{dy}{y} \Delta_{ij}(y, u) P_{qq} \left(\frac{x}{y} \right) \quad (5)$$

$$\begin{aligned} u \frac{d}{du} \Sigma(x, u) = & \frac{\alpha(u)}{2\pi} \int_x^1 \frac{dy}{y} \left[\Sigma(y, u) P_{qq} \left(\frac{x}{y} \right) + \right. \\ & \left. + (2f) G(y, u) P_{qG} \left(\frac{x}{y} \right) \right] \end{aligned} \quad (6)$$

$$\begin{aligned} u \frac{d}{du} G(x, u) = & \frac{\alpha(u)}{2\pi} \int_x^1 \frac{dy}{y} \left[\Sigma(y, u) P_{Gq} \left(\frac{x}{y} \right) + \right. \\ & \left. + G(y, u) P_{GG} \left(\frac{x}{y} \right) \right] \end{aligned} \quad (7)$$

with $\Delta_{ij}(x, u) = q_i(x, u) - q_j(x, u)$ and $\Sigma(x, u) = \sum_i [q_i(x, u) + \bar{q}_i(x, u)]$,

DGLAP equation at small x for Integrated gluon distribution

$$\begin{aligned} u \frac{d}{du} G(x, u) &= \frac{\alpha(u)}{2\pi} \int_x^1 \frac{dy}{y} G(y, u) P_{GG} \left(\frac{x}{y}, \alpha(u) \right), \\ u \frac{d}{du} G(N, u) &= \frac{\alpha(u)}{2\pi} \gamma(N, \alpha(u)) G(N, u), \\ G(N, u) &= \int_0^1 dx \ x^{N-1} G(x, u), \\ \gamma(N, \alpha(u)) &= \frac{\alpha(u)}{2\pi} \int_0^1 dx \ x^{N-1} P_{GG}(x, \alpha(u)). \end{aligned} \tag{8}$$

DGLAP equation at small x for Unintegrated gluon distribution

Integrated gluon distribution $G(x, u)$ is related to unintegrated gluon distribution $\varphi(x, k_\perp^2)$ via the integral relation

$$G\left(x, \frac{Q^2}{\mu^2}\right) = \int_0^{Q^2} dk_\perp^2 \varphi\left(x, \frac{k_\perp^2}{\mu^2}\right). \quad (9)$$

$$\begin{aligned} Q^2 \frac{d}{dQ^2} \int_0^{Q^2} dk_\perp^2 \varphi\left(N, \frac{k_\perp^2}{\mu^2}\right) &= \frac{\alpha}{2\pi} \gamma(N, \alpha) \int_0^{Q^2} dk_\perp^2 \varphi\left(N, \frac{k_\perp^2}{\mu^2}\right), \Rightarrow \\ Q^2 \frac{d}{dQ^2} Q^2 \varphi\left(N, \frac{Q^2}{\mu^2}\right) &= \frac{\alpha}{2\pi} \gamma(N, \alpha) Q^2 \varphi\left(N, \frac{Q^2}{\mu^2}\right). \end{aligned}$$

Solution to DGLAP IDE at LO for small x

$$\begin{aligned} u \frac{d}{du} \phi(x, u) &= \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} \phi(y, u) P_{GG} \left(\frac{x}{y}, \alpha \right) \Rightarrow \\ u \frac{d}{du} \int_{a-i\infty}^{a+i\infty} dN x^{-N} \phi(N, u) &= \\ \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} \int_{a-i\infty}^{a+i\infty} dN y^{-N} \phi(N, u) P_{GG} \left(\frac{x}{y}, \alpha \right) &\Rightarrow \\ \int_{a-i\infty}^{a+i\infty} dN x^{-N} \phi_1(N) u^{\frac{\alpha}{2\pi} \gamma(N, \alpha)} \gamma(N, \alpha) &= \\ = \int_x^1 \frac{dy}{y} \int_{a-i\infty}^{a+i\infty} dN y^{-N} \phi_1(N) u^{\frac{\alpha}{2\pi} \gamma(N, \alpha)} P_{GG} \left(\frac{x}{y}, \alpha \right) &\Rightarrow \quad (10) \end{aligned}$$

$$\int_{a-i\infty}^{a+i\infty} dN x^{-N} \phi_1(N) u^{\frac{\alpha}{2\pi} \gamma(N, \alpha)} \left[\gamma(N, \alpha) - x^N \int_x^1 \frac{dy}{y} y^{-N} P_{GG} \left(\frac{x}{y}, \alpha \right) \right] = 0$$

A method to solve the DGLAP equation analytically

The integral in the bracket may be transformed to

$$\begin{aligned} \int_x^1 \frac{dy}{y} y^{-N} P_{GG} \left(\frac{x}{y}, \alpha \right) &= \int_1^{1/x} \frac{dy}{y} y^N P_{GG}(xy, \alpha) \\ &= x^{-N} \int_x^1 \frac{dy}{y} y^N P_{GG}(y, \alpha). \end{aligned}$$

The DGLAP IDE may be written in such a form

$$\begin{aligned} \int_{a-i\infty}^{a+i\infty} dN x^{-N} \phi_1(N) u^{\frac{\alpha}{2\pi}\gamma(N,\alpha)} \left[\gamma(N, \alpha) - \int_x^1 \frac{dy}{y} y^N P_{GG}(y, \alpha) \right] \\ = \int_{a-i\infty}^{a+i\infty} dN x^{-N} \phi_1(N) u^{\frac{\alpha}{2\pi}\gamma(N,\alpha)} \int_0^x \frac{dy}{y} y^N P_{GG}(y, \alpha) = 0 \end{aligned}$$

For the future use we introduce the notation

$$T(N, x, \alpha) \equiv \int_x^1 \frac{dy}{y} y^N P_{GG}(y, \alpha). \quad (11)$$

Solution to DGLAP equation in a simple toy model

We may consider the splitting function of gluons in the form

$$P_{GG}(z, \alpha) = 2z$$

With this simple splitting function we may illustrate the main idea of the method.

$$T(N, x, \alpha) = \int_x^1 \frac{dy}{y} y^N P_{GG}(y, \alpha) = \int_x^1 \frac{dy}{y} y^N (2y) = \frac{2}{N+1} - \frac{2x^{N+1}}{N+1}.$$

We have from Eqs. above

$$\gamma(N, \alpha) = T(N, 0, \alpha) = \frac{2}{N+1}.$$

Thus, DGLAP in this case

$$\int_{a-i\infty}^{a+i\infty} dN x^{-N} \phi_1(N) u^{\frac{\alpha}{2\pi}\gamma(N, \alpha)} \frac{x^{N+1}}{N+1} = x \int_{a-i\infty}^{a+i\infty} dN \frac{\phi_1(N) u^{\frac{\alpha}{2\pi}\gamma(N, \alpha)}}{N+1} = 0,$$

Simple example

As an example, we may obtain the form of unintegrated gluon distribution $\phi(x, u)$ for the simplest case when $\phi_1(N) = 1/(N + 1)$, that is,

$$\begin{aligned}\phi(x, u) &= \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \phi_1(N) u^{\frac{\alpha}{\pi} \frac{1}{N+1}} x^{-N} = \\ &\quad \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1} u^{\frac{\alpha}{\pi} \frac{1}{N+1}} = \\ &\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\alpha}{\pi} \ln u \right)^k \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{(N+1)^{k+1}} = \\ &x \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\alpha}{\pi} \ln u \right)^k \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N-1}}{(N+1)^{k+1}} \\ &= x \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\alpha}{\pi} \ln u \right)^k \frac{(-\ln x)^k}{k!} = x I_0 \left(2 \sqrt{\frac{\alpha}{\pi} \ln u \ln \frac{1}{x}} \right),\end{aligned}$$

Equations dual to DGLAP-type IDE

We may give a simple example of the dual integro-differential equations.

$$u \frac{d}{du} \phi(x, u) = \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} \phi(y, u) P_{GG} \left(\frac{x}{y}, \alpha \right),$$

and gamma is given

$$P_{GG}(z, \alpha) = 2z$$

with $\beta = 0$. The result is the Bessel function $I_0 \left(2\sqrt{\frac{\alpha}{\pi} \ln u \ln \frac{1}{x}} \right)$.

Now we do the complex diffeomorphism in the complex plane of variable N and go to another variable M which is related to original variable N by the duality transformation $\gamma(N) = M$ in the integral

$$\phi(x, u) = \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1} u^{\frac{1}{N+1}}$$

If we define the inverse function $\chi(M) \equiv N$, than we model the duality conditions $\chi(\gamma(N)) = N$ and $\gamma(\chi(M)) = M$.

Equations dual to DGLAP-type IDE

We may write explicitly for this example

$$\gamma(N) = \frac{1}{N+1} = M \Rightarrow N = \frac{1}{M} - 1 \equiv \chi(M),$$

$$\chi(\gamma(N)) = \frac{1}{\gamma(N)} - 1 = (N+1) - 1 = N,$$

$$\gamma(\chi(M)) = \frac{1}{\chi(M)+1} = \frac{1}{\left(\frac{1}{M}-1\right)+1} = M.$$

$$\begin{aligned}\phi(x, u) &= \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1} u^{\frac{1}{N+1}} = \oint_C dMN'(M) \frac{x^{-\chi(M)}}{N+1} u^M = \\ &- \oint_C dM \frac{x^{-\chi(M)}}{M} u^M = \oint_C dM \frac{x^{1-1/M}}{M} u^M = -x \oint_C dM \frac{x^{-1/M}}{M} u^M = \\ &- \sum_{k=0}^{\infty} \frac{1}{k!} (-\ln x)^k \oint_C dM \frac{u^M}{M^{k+1}} = x \sum_{k=0}^{\infty} \frac{1}{k!} (-\ln x)^k \frac{(\ln u)^k}{k!} =\end{aligned}$$

Equations dual to DGLAP-type IDE

$$= x \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} (\ln u \ln x)^k = x I_0 \left(2 \sqrt{\ln u \ln \frac{1}{x}} \right).$$

We may write the gluonic pdf in a dual form

$$\phi(x, u) = \oint_{C_1} dN \phi(N, u) x^{-N} = \oint_{C_2} dM \phi(x, M) u^M = x I_0 \left(2 \sqrt{\ln u \ln \frac{1}{x}} \right),$$

The functions $\phi(N, u)$ and $\phi(x, M)$ satisfy to different differential equations,

$$u \frac{d}{du} \phi(N, u) = \gamma(N, \alpha) \phi(N, u),$$

$$x \frac{d}{dx} \phi(x, M) = \chi(M, \alpha) \phi(x, M),$$

The solutions to these equations are simple,

$$\phi(N, u) = \phi_1(N) u^{\gamma(N)}, \quad \phi(x, M) = \phi_2(M) x^{\chi(M)}$$

Equations dual to DGLAP-type IDE

These couple of integro-differential equations are called dual because they contain the same information about the same function. Functions $\phi_1(N)$ and $\phi_2(M)$ are not independent but they are related by a diffeomorphic transformation. The first equation is produced by DGLAP after the taking the Mellin moments with respect to the variable x . The second equation is produced by BFKL. Now we use the experience.

$$\begin{aligned}\phi(x, u) &= \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1} u^{\frac{1}{N+1}} = \\ &\quad \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{e^{-N \ln x + \frac{\ln u}{N+1}}}{N+1} \equiv \\ &\quad \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{e^{-M(N) \ln x \ln u}}{N+1} = \oint_C dMN'(M) \frac{e^{-M(N) \ln x \ln u}}{N(M)+1} = \\ &\quad x I_0 \left(2 \sqrt{\ln u \ln \frac{1}{x}} \right)\end{aligned}$$

Running coupling

$$u \frac{d}{du} G(N, u) = \frac{\alpha(u)}{2\pi} \gamma(N, \alpha(u)) G(N, u) \Rightarrow$$

$$\frac{d}{d\alpha} G(N, \alpha) = \frac{\alpha}{2\pi} \frac{\gamma(N, \alpha)}{\beta(\alpha)} G(N, \alpha)$$

$$G(N, \alpha) = \phi_1(N) \exp F(\alpha, N) \Rightarrow$$

$$G(x, u) = \oint_C dN \phi_1(N) \exp [F(\alpha, N) - N \ln x]$$

Integrals obtained are inverse Laplace transformations of Jacobians of diffeomorphic change of variables in the complex plane.