Three loop QCD corrections to heavy quark form factors

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ACAT 2019, Saas Fee, Switzerland

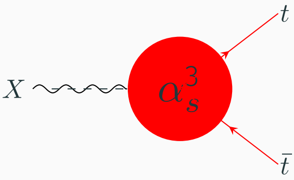
The need for accuracy

The Standard Model is the most successful theory in describing the elementary particles and their fundamental interactions, due to combined effort from both the magnificent experiments like Tevatron, HERA, LHC etc. and precise theory predictions, namely perturbative calculations.

Tevatron	LHC	FCC/ILC
the top quark	the Higgs boson	BSM physics?!
fundamental laws $\sim 10\%$	fundamental laws $\sim 5\%$	more precision!
agreement with NLO theory predictions	agreement with NNLO theory predictions	More precise theory predictions needed!

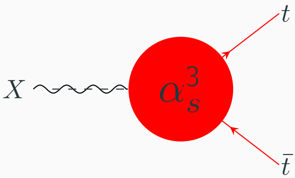
In this talk, we present a brief overview on the generic procedure to compute top pair production at electron-positron colliders.

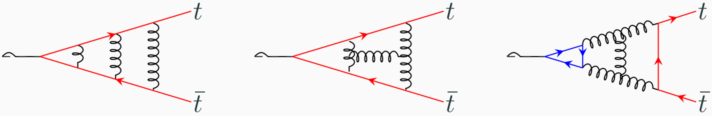
Specifically, what we compute



In this talk, we present a brief overview on the generic procedure to compute top pair production at electron-positron colliders.

Specifically, what we compute





+ many more

Plan of this talk

1. Heavy quark form factors

2. QCD corrections at three loops

3. Conclusion

Top-quark phenomenology

✓ **Top quark is heavy**

- The heaviest SM particle - probes the Higgs sector most
  - plays unique role in understanding the EW symmetry breaking
- New physics potential : perfect place to manifest it

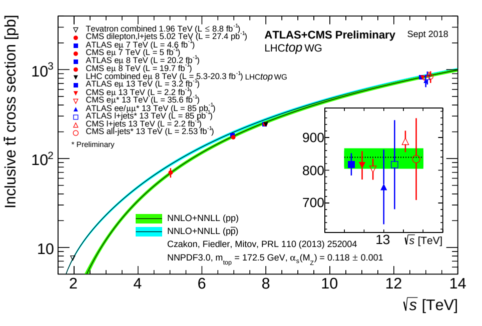
✓ **Top quark is short-lived**

- (Mostly) decays before hadronization -the only 'free' quark -
  Spin properties, Interaction vertices, Precise description of mass

✓ **Sudakov behavior of the massive form factors**

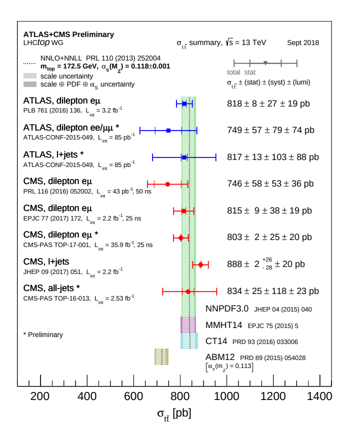
- The form factors are basic building blocks for many physical quantities.
  Also, they exhibit Sudakov behavior in the asymptotic limit.

Top @ LHC

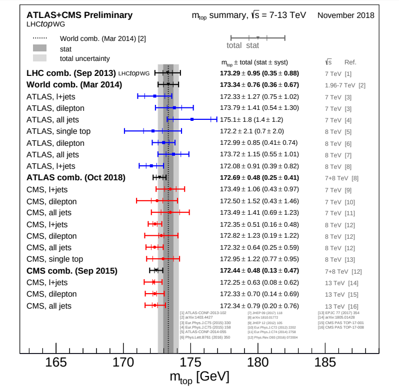


ATLAS+CMS Sep'18

TOP @ LHC



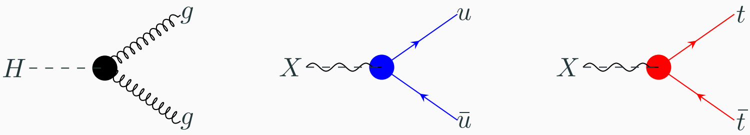
ATLAS+CMS Preliminary
LHCtop WG
σ_tt̄ summary, √s = 13 TeV
Sept 2018
NNLO+NNLL PRL 110 (2013) 252004
m_top = 172.5 GeV, α_s(M_Z) = 0.118±0.001
scale uncertainty
scale ⊕ PDF ⊕ α_S uncertainty
total stat
σ_tt̄ ± (stat) ± (syst) ± (lumi)
ATLAS, dilepton eμ
PLB 761 (2016) 136, L_int = 3.2 fb-1
818 ± 8 ± 27 ± 19 pb
ATLAS, dilepton ee/μμ *
ATLAS-CONF-2015-049, L_int = 85 pb-1
749 ± 57 ± 79 ± 74 pb
ATLAS, l+jets *
ATLAS-CONF-2015-049, L_int = 85 pb-1
817 ± 13 ± 103 ± 88 pb
CMS, dilepton eμ
PRL 116 (2016) 052002, L_int = 43 pb-1, 50 ns
746 ± 58 ± 53 ± 36 pb
CMS, dilepton eμ
EPJC 77 (2017) 172, L_int = 2.2 fb-1, 25 ns
815 ± 9 ± 38 ± 19 pb
CMS, dilepton eμ *
CMS-PAS TOP-17-001, L_int = 35.9 fb-1, 25 ns
803 ± 2 ± 25 ± 20 pb
CMS, l+jets
JHEP 09 (2017) 051, L_int = 2.2 fb-1
888 ± 2 +26 -28 ± 20 pb
CMS, all-jets *
CMS-PAS TOP-16-013, L_int = 2.53 fb-1
834 ± 25 ± 118 ± 23 pb
NNPDF3.0 JHEP 04 (2015) 040
MMHT14 EPJC 75 (2015) 5
CT14 PRD 93 (2016) 033006
ABM12 PRD 89 (2015) 054028
[α_s(m_Z) = 0.113]
* Preliminary
200
400
600
800
1000
1200
1400
σ_tt̄ [pb]



ATLAS+CMS Preliminary
LHCtopWG
m_top summary, √s = 7-13 TeV
November 2018
World comb. (Mar 2014) [2]
stat
total uncertainty
total stat
m_top ± total (stat ± syst)
√s Ref.
LHC comb. (Sep 2013) LHCtopWG 173.29 ± 0.95 (0.35 ± 0.88) 7 TeV [1]
World comb. (Mar 2014) 173.34 ± 0.76 (0.36 ± 0.67) 1.96-7 TeV [2]
ATLAS, l+jets 172.33 ± 1.27 (0.75 ± 1.02) 7 TeV [3]
ATLAS, dilepton 173.79 ± 1.41 (0.54 ± 1.30) 7 TeV [3]
ATLAS, all jets 175.1± 1.8 (1.4 ± 1.2) 7 TeV [4]
ATLAS, single top 172.2 ± 2.1 (0.7 ± 2.0) 8 TeV [5]
ATLAS, dilepton 172.99 ± 0.85 (0.41± 0.74) 8 TeV [6]
ATLAS, all jets 173.72 ± 1.15 (0.55 ± 1.01) 8 TeV [7]
ATLAS, l+jets 172.08 ± 0.91 (0.39 ± 0.82) 8 TeV [8]
ATLAS comb. (Oct 2018) 172.69 ± 0.48 (0.25 ± 0.41) 7+8 TeV [8]
CMS, l+jets 173.49 ± 1.06 (0.43 ± 0.97) 7 TeV [9]
CMS, dilepton 172.50 ± 1.52 (0.43 ± 1.46) 7 TeV [10]
CMS, all jets 173.49 ± 1.41 (0.69 ± 1.23) 7 TeV [11]
CMS, l+jets 172.35 ± 0.51 (0.16 ± 0.48) 8 TeV [12]
CMS, dilepton 172.82 ± 1.23 (0.19 ± 1.22) 8 TeV [12]
CMS, all jets 172.32 ± 0.64 (0.25 ± 0.59) 8 TeV [12]
CMS, single top 172.95 ± 1.22 (0.77 ± 0.95) 8 TeV [13]
CMS comb. (Sep 2015) 172.44 ± 0.48 (0.13 ± 0.47) 7+8 TeV [12]
CMS, l+jets 172.25 ± 0.63 (0.08 ± 0.62) 13 TeV [14]
CMS, dilepton 172.33 ± 0.70 (0.14 ± 0.69) 13 TeV [15]
CMS, all jets 172.34 ± 0.79 (0.20 ± 0.76) 13 TeV [16]
165
170
175
180
185
m_top [GeV]

Form factors

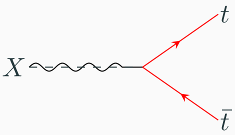
- ✓ The form factors are basic building blocks for many physical quantities
- ✓ They exhibit a universal infrared behavior - leads to information on anomalous dimensions
- ✓ The massive cusp anomalous dimension controls the infrared structure of massive form factors - studying the form factors helps in better understanding of the massive cusp
- ✓ Another important motive is to study high energy behavior of the massive form factors



Heavy quark form factors

The process

We consider decay of a color neutral massive particle to a pair of heavy quark of mass m.



Notation

$$X(q) \to t(q_1) + \bar{t}(q_2)$$

$$X = V, A, S, P$$

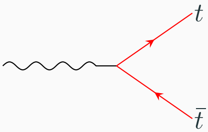
$$s = \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

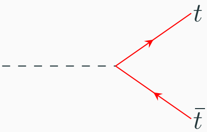
The general structure

Vector and Axial Vector

V: $-i\delta_{ij} v_Q \left(\gamma^\mu \, F_{V,1} + \frac{i}{2m} \sigma^{\mu\nu} q_\nu \, F_{V,2} \right)$

A: $-i\delta_{ij} a_Q \left(\gamma^\mu \gamma_5 \, F_{A,1} + \frac{1}{2m} q^\mu \gamma_5 \, F_{A,2} \right)$





Scalar and Pseudo Scalar

$-\frac{m}{v} \delta_{ij} \left[s_Q F_S + i p_Q \gamma_5 F_P \right]$

The form factors are expanded in the strong coupling constant as

$$F_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n F_I^{(n)}$$

To obtain $F_I^{(n)} \Rightarrow$ appropriate projector on the amplitudes

$$P_{V,i} = \frac{i}{v_Q} \frac{\not{q}_2 - m}{m} \left(\gamma_\mu g^1_{V,i} + \frac{1}{2m}(q_{2\mu} - q_{1\mu}) g^2_{V,i}\right) \frac{\not{q}_1 + m}{m} \,,$$

$$P_{A,i} = \frac{i}{a_Q} \frac{\not{q}_2 - m}{m} \left(\gamma_\mu \gamma_5 g^1_{A,i} + \frac{1}{2m}(q_{1\mu} + q_{2\mu}) \gamma_5 g^2_{A,i}\right) \frac{\not{q}_1 + m}{m} \,,$$

$$P_S = \frac{v}{2m s_Q} \frac{\not{q}_2 - m}{m} \left(g_S\right) \frac{\not{q}_1 + m}{m} \,, \; P_P = \frac{v}{2m p_Q} \frac{\not{q}_2 - m}{m} \left(i \gamma_5 g_P\right) \frac{\not{q}_1 + m}{m} \,,$$

$g_I \equiv g_I(s, d)$ and are determined by applying the projectors on the generic Lorentz structure.

One-loop and beyond

$F_{V,i}^{(1)}, F_{A,i}^{(1)}$ [Arbuzov, Bardin, Leike ’92; Djouadi, Lampe, Zerwas ’95]

$F_{S}^{(1)}, F_{P}^{(1)}$ [Braaten, Leveille ’80; Sakai ’80; Drees, Hikasa ’90]

$F_{V,i}^{(2)}, F_{A,i}^{(2)}$ [Altarelli, Lampe ’93; Ravindran, van Neerven ’98; Catani, Seymour ’99]

$F_{S}^{(2)}, F_{P}^{(2)}$ [Gorishnii *et. al.* ’91; Chetyrkin, Kwiatkowski ’95; Harlander, Steinhauser ’97]

Two-loop

$F_{I}^{(2)}$ [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi ’04,’05]

$F_{V,i}^{(2)}(\mathcal{O}(\epsilon))$ [Gluza, Mitov, Moch, Riemann ’09]

$F_{I}^{(2)}(\mathcal{O}(\epsilon^2))$ [Ablinger, Behring, Blümlein, Falcioni, Freitas, Marquard, Rana, Schneider ’17]

Three-loop

$F_{V,i}^{(3)}|_{\text{large N}}$ [Henn, Smirnov, Smirnov, Steinhauser ’16]

$F_{V,i}^{(3)}, F_{A,i}^{(3)}, F_{S}^{(3)}, F_{P}^{(3)}|_{\text{large N+full}\, n_l}$ [Lee, Smirnov, Smirnov, Steinhauser ’18]

$F_{V,i}^{(3)}, F_{A,i}^{(3)}, F_{S}^{(3)}, F_{P}^{(3)}|_{\text{large N+full}\, n_l}$ [Ablinger, Blümlein, Marquard, Rana, Schneider ’18]

QCD corrections at three loops

J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, NR and C. Schneider,
`Heavy quark form factors at two loops`,
Phys.Rev. D97 (2018) 094022 (arXiv:1712.09889 [hep-ph]).

J. Ablinger, J. Blümlein, P. Marquard, NR and C. Schneider,
`Heavy Quark Form Factors at Three Loops in the Planar Limit`,
Phys.Lett. B782 (2018) 528-532 (arXiv:1804.07313 [hep-ph]).

J. Ablinger, J. Blümlein, P. Marquard, NR and C. Schneider,
`Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable`,
Nucl.Phys. B939 (2019) 253-291 (arXiv:1810.12261 [hep-ph]).

The generic procedure

$\boxed{d = 4 - 2\epsilon}$

- Diagrammatic approach -> `QGRAF` [Nogueira] to generate diagrams
- `FORM` [Vermaseren] for algebraic manipulation :

  Lorentz, Dirac and Color [Ritbergen, Schellekens, Vermaseren] *algebra*
- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l.p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

- Identity relations among scalar integrals : *IBPs, LIs & SRs*
- Algebraic linear system of equations relating the integrals

$$\Downarrow$$

Master integrals (MIs)

- `Crusher` [Marquard, Seidel] for reduction to master integrals
- Computation of MIs : *Differential eqns.*

Computing the master integrals

A scalar integral can be expressed as

$$J(\nu_1, \ldots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon\gamma_E}\right)^3 \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{d^d l_3}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

Computing the master integrals

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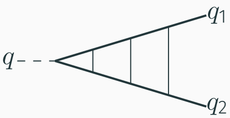
$$J(\nu_1, \ldots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon \gamma_E} \right)^3 \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{d^d l_3}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

For example



$$\begin{aligned} &l_1^2 - m^2, (l_1 - q)^2 - m^2, (l_1 - l_2)^2, \\ &l_2^2 - m^2, (l_2 - q)^2 - m^2, (l_2 - l_3)^2, \\ &l_3^2 - m^2, (l_3 - q)^2 - m^2, (l_1 - q_1)^2 \end{aligned}$$

To evaluate the integral $\rightarrow$ Feynman parametrization, Mellin-Barnes ...
Or method of differential/difference equations!

Using differential equations

$$\boxed{\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}}$$

The integral is a function of d, q^2 and m^2.

$$J(1,1,1,1,1,1,1,1,1) \equiv f(d,\, q^2,\, m^2) \equiv f(d,\, x)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.$ x and solve it.

Using differential equations

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The idea is to obtain a differential eqn. for the integral $w.r.t.$ x and solve it.

$$\frac{d}{dx} J_i = \text{some combinations of integrals}$$

$$\Downarrow \text{ IBP identities}$$

$$= \sum_j c_{ij} J_j$$

c_{ij}'s are rational function of d and x.

Using differential equations

$$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

The integral is a function of d, q^2 and m^2.

$$J(1,1,1,1,1,1,1,1,1) \equiv f(d,\, q^2,\, m^2) \equiv f(d,\, x)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.$ x and solve it.

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_x \mathbb{J} = \mathbb{A}(d,x)\mathbb{J}$$

Canonical Basis

The choice of MIs is not unique!
The idea is to find a basis such that the deqs decouple as $\epsilon \to 0$

[Kotikov; Henn]

$$d_x \tilde{\mathbb{J}} = \epsilon \tilde{\mathbb{A}}(x) \tilde{\mathbb{J}}$$

Now one can perform Laurent series expansion in ϵ and trivially solve the deqs as at each order the homogeneous solutions are constants.

Algorithm to find such basis by [Lee]
and implementation in `Epsilon`, `Canonica` ...

Drawback : Finding such a basis is not always straightforward.

Generic Basis

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

Generic Basis

$$
d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}
$$

To solve such a system, it would be best to organize it in such a way that it diagonalizes, or at least it takes a block-triangular form.

$$
d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 0 & 0 & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}
$$

Let's consider the 12^{th} blob from below

$$\frac{d}{dx}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} + \begin{pmatrix} R_1(\epsilon, x) \\ R_2(\epsilon, x) \\ R_3(\epsilon, x) \end{pmatrix},$$

$$\begin{aligned} c_{11} &= \frac{(7 + 6x + 7x^2 - 2d(1 + x + x^2))}{x(1+x)^2}, \\ c_{12} &= \frac{(-4 + d)(-10 + 3d)}{2(-3 + d)^2(1 + x)^2}, \\ c_{13} &= \frac{(d^2(15 + 8x + 15x^2) + 8(20 + 9x + 20x^2) - 2d(49 + 24x + 49x^2))}{4(-3 + d)^2 x(1 + x)^2}, \ldots \end{aligned}$$

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$$\begin{aligned}
c_{11} &= \frac{(7 + 6x + 7x^2 - 2d(1 + x + x^2))}{x(1 + x)^2}, \\
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\end{aligned}$$

For these topologies, the integrals can have, at max, a cubic pole in ϵ.

$$J_i = \frac{1}{\epsilon^3}J_i^{-3} + \frac{1}{\epsilon^2}J_i^{-2} + \frac{1}{\epsilon}J_i^{-1} + J_i^0 + \epsilon J_i^1 + \cdots$$

Series expansion and compare each order of ϵ!

Let's consider the 12^{th} blob from below

$$\frac{d}{dx}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} + \begin{pmatrix} R_1(\epsilon, x) \\ R_2(\epsilon, x) \\ R_3(\epsilon, x) \end{pmatrix},$$

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Each order in ϵ-expansion gives a much simpler form

$$\frac{d}{dx}\begin{pmatrix} J_1^{-3} \\ J_2^{-3} \\ J_3^{-3} \end{pmatrix} = \begin{bmatrix} \frac{1}{x} + \frac{2}{1-x} & 0 & \frac{1}{1+x} - \frac{2}{x} - \frac{3}{1-x} \\ -\frac{1}{x} + \frac{2}{1+x} & \frac{1}{1+x} - \frac{1}{x} - \frac{1}{1-x} & \frac{1}{x} - \frac{2}{1+x} \\ \frac{1}{x} + \frac{2}{1-x} & 0 & \frac{1}{1+x} - \frac{2}{x} - \frac{3}{1-x} \end{bmatrix}\begin{pmatrix} J_1^{-3} \\ J_2^{-3} \\ J_3^{-3} \end{pmatrix} + \begin{pmatrix} R_1^{-3}(x) \\ R_2^{-3}(x) \\ R_3^{-3}(x) \end{pmatrix},$$

The general form of expansion

$$\frac{d}{dx} J_n(x,\epsilon) = \mathcal{C}_{nm}(x,\epsilon) J_n(x,\epsilon) + \mathcal{R}_n(x,\epsilon)$$

Expand in ϵ

$$J_n(x,\epsilon) = \sum_{k=-3}^{\infty} J_n^{(k)}(x)\epsilon^k$$

$$\mathcal{C}_n(x,\epsilon) = \sum_{k=0}^{\infty} \mathcal{C}_n^{(k)}(x)\epsilon^k$$

$$\mathcal{R}_n(x,\epsilon) = \sum_{k=-3}^{\infty} \mathcal{R}_n^{(k)}(x)\epsilon^k$$

$$\frac{d}{dx} J_n^{(k)}(x) = \mathcal{C}_{nm}^{(0)}(x) J_n^{(k)}(x) + \sum_{p=1}^{k+3} \mathcal{C}_{nm}^{(p)}(x) J_n^{(k-p)}(x) + \mathcal{R}_n^{(k)}(x)$$

Algorithm - I : the homogeneous part

- It boils down to solving a system of linear first order diff. eqns.

Algorithm - I : the homogeneous part

- It boils down to solving a system of linear first order diff. eqns.
- A natural first step is to reduce the system to a higher order equation in a single unknown

  `Note that, the inverse operation is trivial!`

Algorithm - I : the homogeneous part

- It boils down to solving a system of linear first order diff. eqns.
- A natural first step is to reduce the system to a higher order equation in a single unknown
  `Note that, the inverse operation is trivial!`
- The classical/naive method to achieve this uncoupling is the cyclic vector algorithm

An example of a 2×2 coupled system

$$\frac{d}{dx}\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} + \begin{pmatrix} R_1(\varepsilon, x) \\ R_2(\varepsilon, x) \end{pmatrix},$$

$$J_2 = p'(x)\frac{dJ_1}{dx} + q'(x)J_1 + r'(x)$$

$$\frac{d^2 J_1}{dx^2} + p(x)\frac{dJ_1}{dx} + q(x)J_1 = r(x)$$

Algorithm - I : the homogeneous part

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- But, it gives a complicated decoupled equation.

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  `Note that, the inverse operation is trivial!`
- The classical/naive method to achieve this uncoupling is the cyclic vector algorithm
- But, it gives a complicated decoupled equation.
- We use the following uncoupling algorithms

`Zürcher` : Block-diagonal form
`Incomplete Zürcher` : Block-triangular form
`Abramov Zima` : Block-triangular form

Implemented in `OreSys`!

Using one of the algorithms to ϵ^{-3} part

$$\left[\frac{d^2}{dx^2} - \frac{2}{1-x}\frac{d}{dx} + \left(\frac{2}{x} - \frac{2}{1+x} - \frac{2}{(1+x)^2}\right)\right] I_1^{-3}(x) = r_1^{-3}(x)$$

$$\left[\frac{d}{dx} - \left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{1+x}\right)\right] I_2^{-3}(x) = r_2^{-3}(x)$$

The 12^{th} blob from below

Using one of the algorithms to ϵ^{-3} part

$$\left[\frac{d^2}{dx^2} - \frac{2}{1-x}\frac{d}{dx} + \left(\frac{2}{x} - \frac{2}{1+x} - \frac{2}{(1+x)^2}\right)\right] I_1^{-3}(x) = r_1^{-3}(x)$$

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Solving for homogeneous part of each diff. eqns.

$$y_1(x) = \frac{x}{1-x^2}\,, \quad y_2(x) = 1 - \frac{2x}{1-x^2} H(0,x)\,; \qquad \mu(x) = \frac{1}{x} - x\,;$$

Using one of the algorithms to ϵ^{-3} part

$$\left[\frac{d^2}{dx^2} - \frac{2}{1-x}\frac{d}{dx} + \left(\frac{2}{x} - \frac{2}{1+x} - \frac{2}{(1+x)^2}\right)\right] I_1^{-3}(x) = r_1^{-3}(x)$$

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Next, use variation of constant to obtain the solution

$$I_1^{-3}(x) = y_1(x)\left[C_1 - \int dx \frac{r_1^{-3}(x) y_2(x)}{W(y_1,y_2)}\right] + y_2(x)\left[C_2 + \int dx \frac{r_1^{-3}(x) y_1(x)}{W(y_1,y_2)}\right]$$

$$I_2^{-3}(x) = \frac{1}{\mu(x)}\left(C_3 + \int dx \mu(x) r_2^{-3}(x)\right)$$

$W(y_1, y_2)$ is the Wronskian.

Finally $J_i^{-3}(x) = f(\{I_1^{-3}(x), I_2^{-3}(x)\})$!

Algorithm - II : the nonhomogeneous part

- Structure of homogeneous part is same at each order in ϵ-expansion
- Hence the homogeneous solutions and uncoupling procedure are similar for each order
- Start with the ϵ^{-3} part
- Find the best uncoupling for the sub-system and solve for the corresponding homogeneous solutions
- Now at each order in ϵ, find the nonhomogeneous parts keeping the uncoupling structure fixed
- Solve order by order in ϵ using variation of constant

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Automatization using

`Sigma` [Schneider] , `OreSys` [Gerhold]
and `HarmonicSums` [Ablinger, Blümlein, Schneider]

The results are obtained in terms of HPLs and Cyclotomic HPLs.

Boundary conditions

Boundary conditions are fixed by imposing regularity of the integrals in the limit of vanishing space-like momentum $q^2 \to 0$ $i.e.$ $x \to 1$.

- In the planar limit, the integrals are regular in $y = 1 - x$, and hence can be expanded as follows

$$J_i(y) = \sum_{n=0}^{\infty} \sum_{j=-3}^{r} \epsilon^j C_{i,j}(n) y^n$$

- $q^2 \to 0$ reduces them to known two-point integrals, providing $C_{i,j}(0)$
- The differential equations now can be solved to obtain $C_{i,j}(n)$ for sufficiently high order (n) in y

Iterated integrals and Harmonic polylogarithms (HPLs)

Given a set of integration kernels $K_i(t)$, one can define

$$\mathcal{I}(i_n, \ldots i_1, x) = \int_{x_0}^{x} K_{i_n}(t)\mathcal{I}(i_{n-1}, \ldots i_1, t)dt$$

Classic example is $\mathrm{Li}_n(x)$. We have five kernels K_m

$$\Big\{-1, 0, 1, \{6,0\}, \{6,1\}\Big\} \equiv \Big\{\frac{1}{1+x}, \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2}\Big\}$$

and correspondingly we define the HPLs as

$$H(m_n, \ldots, m_1, x) = \int_0^x K_{m_n}(t) H(m_{n-1}, \ldots, m_1, t)dt$$

Some important properties :

Shuffle algebra, Scaling invariance and integration-by-parts identities

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New!

$$\Big\{ -1, 0, 1, \{6,0\}, \{6,1\} \Big\} \equiv \Big\{ \frac{1}{1+x}, \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2} \Big\}$$

and correspondingly we define the HPLs as

$$H(m_n, \dots, m_1, x) = \int_0^x K_{m_n}(t) H(m_{n-1}, \dots, m_1, t) dt$$

Some important properties :

Shuffle algebra, Scaling invariance and integration-by-parts identities

We consider a hybrid scheme for UV renormalization.

Heavy quark mass and wave function $(Z_{m,\mathrm{OS}}, Z_{2,\mathrm{OS}})$: On-shell
QCD strong coupling constant (Z_{a_s}) : $\overline{MS}$

The renormalization of F_I for these topologies, is straightforward

$$F_{V,i} = Z_{2,\mathrm{OS}}\hat{F}_{V,i} \qquad F_S = Z_{m,\mathrm{OS}}Z_{2,\mathrm{OS}}\hat{F}_S$$
$$F_{A,i} = Z_{2,\mathrm{OS}}\hat{F}_{A,i} \qquad F_P = Z_{m,\mathrm{OS}}Z_{2,\mathrm{OS}}\hat{F}_P$$

where $\hat{F}_I$ contains the counterterms from mass renormalization.

For Axial-Vector and Pseudo-Scalar currents, for these topologies *i.e.* the non-singlet case, both the γ_5-matrices appear in the same chain of Dirac matrices, which allows us to use anti-commuting γ_5 in D space-time dimensions.

Ward identities

Chiral Ward identity $\boxed{q_\mu \Gamma_A^{\mu,ns} = 2m\Gamma_P^{ns}}$

$$2F_{A,1}^{ns} + \frac{1}{2}\left(-\frac{(1-x)^2}{x}\right) F_{A,2}^{ns} = 2mF_P^{ns}$$

Ward identities

Chiral Ward identity

$$\boxed{q_\mu \Gamma_A^{\mu,ns} = 2m\Gamma_P^{ns}}$$

$$2F_{A,1}^{ns} + \frac{1}{2}\left(-\frac{(1-x)^2}{x}\right) F_{A,2}^{ns} = 2mF_P^{ns}$$

Anomalous Ward identity

$$\boxed{q_\mu \Gamma_A^{\mu,s} = 2m\Gamma_P^{s} - i\tfrac{\alpha_s}{4\pi} T_F \langle G\tilde{G}\rangle_Q}$$

$\langle G\tilde{G}\rangle_Q$ denotes the truncated matrix element of the gluonic operator $G\tilde{G}$ between the vacuum and an on-shell heavy quark pair ($Q\bar{Q}$).

The infrared singularities factorize as a multiplicative factor

[Becher, Neubert '09]

$$F_I(\epsilon, x) = Z(\epsilon, x, \mu) F_I^{fin}(x, \mu)$$

$Z(\epsilon, x, \mu)$ is universal/independent of current
$F_I^{fin}(x, \mu)$ is finite as $\epsilon \to 0$

Renormalization group evolution of $Z(\epsilon, x, \mu)$ provides

$$\begin{aligned} Z(\epsilon, x, \mu) = 1 &+ \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{\Gamma_0}{2\epsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{\epsilon^2}\left(\frac{\Gamma_0^2}{8} - \frac{\beta_0 \Gamma_0}{4}\right) + \frac{1}{\epsilon}\left(\frac{\Gamma_1}{4}\right)\right] \\ &+ \left(\frac{\alpha_s}{4\pi}\right)^3 \left[\frac{1}{\epsilon^3}\left(\frac{\Gamma_0^3}{48} - \frac{\beta_0 \Gamma_0^2}{8} + \frac{\beta_0^2 \Gamma_0}{6}\right) + \frac{1}{\epsilon^2}\left(\frac{\Gamma_0 \Gamma_1}{8} - \frac{\beta_1 \Gamma_0}{6}\right) + \frac{1}{\epsilon}\left(\frac{\Gamma_2}{6}\right)\right] \end{aligned}$$

Γ_n is the n^{th} order massive cusp anomalous dimension.

[Korchemsky, Radyushkin '87, '92; Grozin, Henn, Korchemsky, Marquard '14, '15]

Results & Checks

- Computational : Automation to solve a single scale and first order factorizable system of differential equations.
- Phenomenological : We have obtained color-planar and complete light quark contributions to the three loop form factors.

$$F_{V,1}^{(3)}, F_{V,2}^{(3)}, F_{A,1}^{(3)}, F_{A,2}^{(3)}, F_{S}^{(3)}, F_{P}^{(3)}$$

✓ We agree with the results from Lee $et\,al.$ obtained using different method

✓ The results reproduce the universal infrared structure

✓ Chiral Ward identity is satisfied between $F_{A,i}^{(3)}$ and $F_{P}^{(3)}$

A glimpse of the result

$$
\begin{aligned}
F^{(3)}_{A,1} = {}& C_F n_l^2 T_F^2 \Bigg[\frac{1}{\varepsilon^3}\Big\{\frac{16}{27} - \frac{32}{27}\xi H_0\Big\} + \frac{1}{\varepsilon^2}\Big\{\frac{208}{81} + \xi\Big(- \frac{32}{9}H_{0,0} + \frac{64}{9}H_{-1,0} + \frac{32}{9}\zeta_2\Big) - \frac{16}{81}\Big(47 + 18x \\
& + 47x^2\Big)\eta H_0\Big\} + \frac{1}{\varepsilon}\Big\{\frac{688}{81} + \eta\Big(- \frac{16}{27}\Big(93 + 2x + 93x^2\Big)H_0 - \frac{16}{27}\Big(47 + 18x + 47x^2\Big)H_{0,0} + \frac{32}{27}\Big(47 \\
& + 18x + 47x^2\Big)H_{-1,0} + \frac{8}{27}\Big(121 + 36x + 67x^2\Big)\zeta_2\Big) + \xi\Big(- \frac{32}{3}H_{0,0,0} + \frac{64}{3}H_{0,-1,0} + \frac{64}{3}H_{-1,0,0} \\
& - \frac{128}{3}H_{-1,-1,0} + \Big(- \frac{16}{3}H_0 - \frac{64}{3}H_{-1}\Big)\zeta_2 + \frac{64}{3}\zeta_3\Big)\Big\} + \Big\{\frac{18224}{729} + \eta\Big(- \frac{16}{729}\Big(12475 - 3078x \\
& + 12475x^2\Big)H_0 - \frac{16}{9}\Big(93 + 2x + 93x^2\Big)H_{0,0} + \frac{32}{9}\Big(93 + 2x + 93x^2\Big)H_{-1,0} - \frac{16}{9}\Big(47 + 18x \\
& + 47x^2\Big)H_{0,0,0} + \frac{32}{9}\Big(47 + 18x + 47x^2\Big)H_{0,-1,0} + \frac{32}{9}\Big(47 + 18x + 47x^2\Big)H_{-1,0,0} - \frac{64}{9}\Big(47 \\
& + 18x + 47x^2\Big)H_{-1,-1,0} + \Big(\frac{8}{9}\Big(225 + 4x + 147x^2\Big) - \frac{8}{9}\Big(47 + 18x + 47x^2\Big)H_0 - \frac{32}{9}\Big(47 \\
& + 18x + 47x^2\Big)H_{-1}\Big)\zeta_2 + \frac{16}{27}\Big(319 + 108x + 245x^2\Big)\zeta_3\Big) + \xi\Big(- 32H_{0,0,0,0} + 64H_{0,0,-1,0} \\
& + 64H_{0,-1,0,0} - 128H_{0,-1,-1,0} + 64H_{-1,0,0,0} - 128H_{-1,0,-1,0} - 128H_{-1,-1,0,0} + 256H_{-1,-1,-1,0} \\
& + \Big(- 16H_{0,0} - 64H_{0,-1} + 32H_{-1,0} + 128H_{-1,-1}\Big)\zeta_2 + \frac{384}{5}\zeta_2^2 + \Big(\frac{544H_0}{27} - 128H_{-1}\Big)\zeta_3\Big)\Big\}\Bigg] \\
& + C_F^2 n_l T_F\Bigg[\frac{1}{\varepsilon^3}\Big\{ - \frac{4}{3}\Big(17 + 10x + 17x^2\Big)x_+^2 + \frac{8\eta x_+^2}{3}\Big(1 + 20x + 14x^2 + 20x^3 + x^4\Big)H_0 \\
& - \frac{16}{3}\xi^2 H_{0,0}\Big\} + \frac{1}{\varepsilon^2}\Big\{\frac{1}{9}\Big(- 187 + 706x - 187x^2\Big)x_+^2 + \eta x_+^2\Big(\frac{16}{9}\Big(7 + 167x + 77x^2 + 167x^3 \\
& + 7x^4\Big)H_0 - 8\Big(1 + 20x + 14x^2 + 20x^3 + x^4\Big)H_{-1,0} - 4\Big(1 + 20x + 14x^2 + 20x^3 + x^4\Big)\zeta_2\Big) \\
& + \xi^2\Big(- 24H_{0,0,0} + 16H_{0,-1,0} + 32H_{-1,0,0} + 8H_0\zeta_2\Big) - \frac{4\eta^2 x_+}{9}\Big(65 - 61x + 238x^2 + 130x^3 \\
& + 281x^4 + 83x^5\Big)H_{0,0}\Big\} + \frac{1}{\varepsilon}\Big\{\frac{1}{54}\Big(- 14071 + 13546x - 14071x^2\Big)x_+^2 + \xi^2\Big(- \frac{160}{3}H_{0,-1,-1,0} \\
& - \frac{256}{3}H_{1,0,1,0} + \frac{256}{3}H_{1,0,-1,0} + 144H_{-1,0,0,0} - 96H_{-1,0,-1,0} - 192H_{-1,-1,0,0} - 48H_{-1,0}\zeta_2 \\
& - \frac{128}{3}H_1\zeta_3\Big) + \eta\Big(- \frac{128}{3}x^2\Big(30 - 120x + 30x^2 + x^3 + x^4\Big)x_+^4 H_{0,1,0,0} + \Big(\frac{8}{27}\Big(217 + 4493x \\
& + 857x^2 + 4493x^3 + 217x^4\Big)H_0 + \frac{256}{3}\Big(2 + x + 10x^2 + x^3 + 2x^4\Big)H_{1,0} - \frac{16}{3}\Big(77 + 160x + 676x^2 \\
& + 160x^3 + 77x^4\Big)H_{-1,0} + \frac{80}{3}\Big(1 + 20x + 14x^2 + 20x^3 + x^4\Big)H_{-1,-1,0}\Big)x_+^2 + \Big(- \frac{128}{3}H_{0,1,0,0} \\
& + \Big(\frac{2}{3}\Big(- 735 - 2157x + 994x^2 - 4866x^3 + 1029x^4 + 631x^5\Big) + \frac{8}{3}\Big(- 43 + 321x - 286x^2 + 626x^3 \\
& - 111x^4 + 53x^5\Big)H_{-1}\Big)\zeta_2\Big)x_+^3\Big) + \eta^2\Big(\Big(- \frac{8}{27}\Big(479 - 1436x + 2013x^2 - 7912x^3 + 15099x^4 \\
& + 1732x^5 + 2441x^6\Big)H_{0,0} + \frac{64}{3}\Big(5 + 4x + 59x^2 - 72x^3 + 59x^4 + 4x^5 + 5x^6\Big)H_{0,1,0} - \frac{16}{3}\Big(24 - 24x \\
& + 751x^2 - 1444x^3 + 686x^4 - 124x^5 + 19x^6\Big)H_{0,-1,0} + \frac{8}{3}\Big(53 + 12x + 1253x^2 - 1504x^3 + 1383x^4 \\
& + 212x^5 + 63x^6\Big)H_{-1,0,0} + \frac{8}{3}\Big(9 + 32x - 687x^2 + 1208x^3 - 557x^4 + 232x^5 + 19x^6\Big)\zeta_3\Big)x_+^2
\end{aligned}
$$

1

The functional structure of the form factors

The form factors contain $\mathrm{polynomials}$ of x, HPLs and cyclotomic HPLS.

$$F^{(3)} \equiv x, H[\_\_, x], \zeta_2, \zeta_3, \zeta_5, \mathrm{Li}_4\left(\frac{1}{2}\right), \mathrm{Li}_5\left(\frac{1}{2}\right)$$

The iterative integrals contribute up to weight 6, where 'weight' denotes the number of iteration. For example

$$H[0, 1, \{6, 1\}, x] = \int_0^x \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1 - t_2} \int_0^{t_2} \frac{t_3\ dt_3}{1 - t_3 + t_3^2}$$

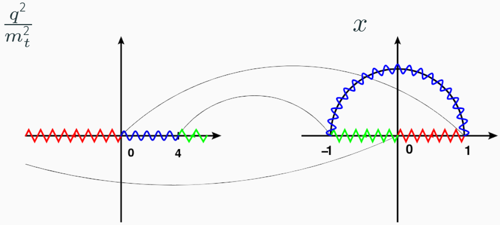
Goal

To obtain a numerical implementation for $x \in [-1, 1]$.

Outcome

Fortran routines `HPOLY.f` and `CPOLY.f`

$$\boxed{\frac{q^2}{m_t^2} = -\frac{(1-x)^2}{x}}$$



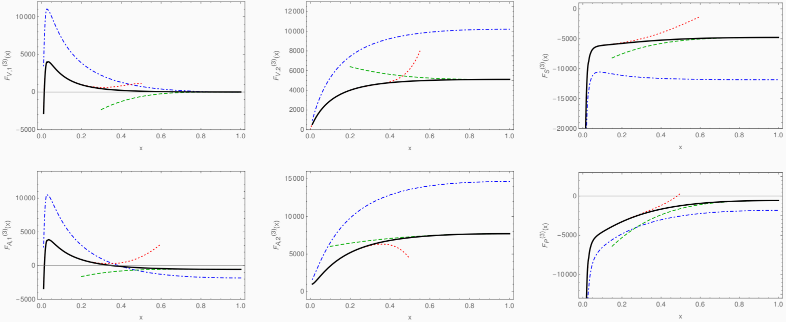
[from a talk by M. Steinhauser]

Low energy region $\quad q^2 \ll m^2 \quad$ or $\quad x \to 1$

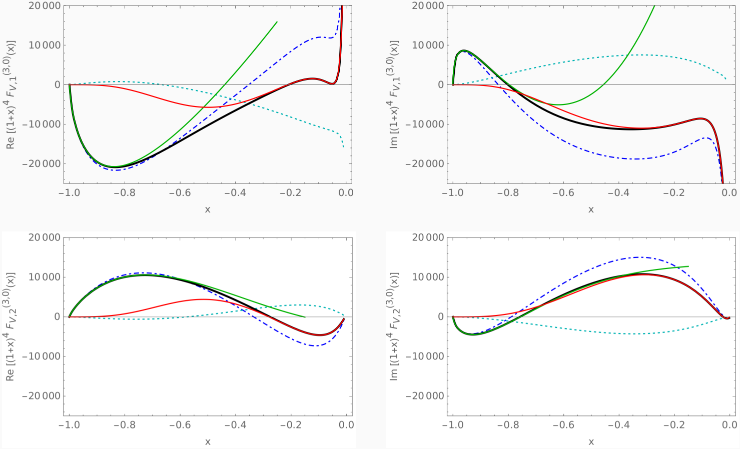
High energy region $\quad q^2 \gg m^2 \quad$ or $\quad x \to 0$

Threshold region $\quad q^2 \sim 4m^2 \quad$ or $\quad x \to -1$

The finite component of the FORM FACTORS and their Expansions in different region



The finite component of the VECTOR FORM FACTORS
threshold Expansions



Conclusion

Summary

- We have computed the master integrals which appear in the three-loop massive form factors in the color-planar limit and for full light quark (n_l) contributions.
- We have solved the first order factorizable system of differential equations by a new method.
- The use of `OreSys` to uncouple the differential equations has made it possible to automatize the procedure.
- The method applies to all first order factorizable systems in any basis.
- The alphabet contains sixth root of unity letters in the real representation.
- Finally, we have obtained the color-planar and complete light quark three-loop contributions to the heavy quark form factors for vector, axial-vector, scalar and pseudo-scalar currents.

Thank You!