Higher-order QED contributions to the lepton anomalous magnetic moments

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This talk is based on collaboration w/
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lepton $g-2$

- Intrinsic magnetic property of a single lepton particle is characterized by a dimensionless number, called g-factor.

$$H = -\vec{\mu} \cdot \vec{B}, \qquad \vec{\mu} = g \frac{e}{2m} \vec{s}$$

- Anomaly, $a \equiv (g-2)/2$, is a consequence of quantum nature of elementary particles. R. Kusch and H. M. Foley 1948, J. Schwinger 1948

- Electron $g-2$ is measured by using a Penning trap:
  University of Washington: H. Dehmelt et al. (1987)
  Harvard University: G. Gabrielse et al. (2006, 2008)
  positron $g-2$ measurement is in preparation.

- Muon $g-2$ is measured by using a muon storage ring:
  Old experiments: CERN(1959-1979), BNL(1984-2006)
  On-going experiments: J-PARC(2009–), Fermilab(2011–)

Both are the state-of-the-art measurements in study of elementary particles

Electron $g-2$ measurement

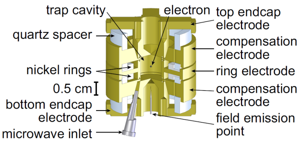


FIG. 4. Cylindrical Penning trap cavity used to confine a single electron and inhibit spontaneous emission.

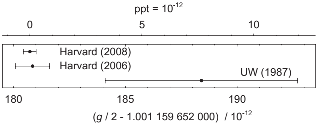


FIG. 1. Measurements [1, 2, 4] of the dimensionless magnetic moment of the electron, $g/2$, which is the electron magnetic moment in Bohr magnetons.

D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse, PRL100(2008)120801;PRA83(2010)052122

Harvard 2008 measurement

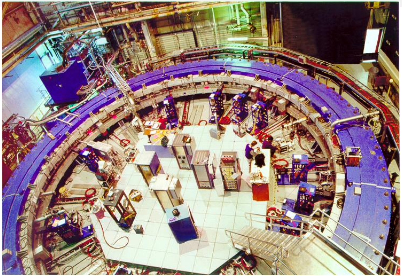
$$a_e \equiv (g_e - 2)/2 = (1\ 159\ 652\ 180.73 \pm 0.28) \times 10^{-12} \qquad [0.24\text{ppb}]$$

Theory needs QED up to 5 loop + hadronic $\mathcal{O}(10^{-12})$ + weak $\mathcal{O}(10^{-14})$:

$$(\alpha/\pi)^5 \sim 0.068 \times 10^{-12}, \qquad \alpha \equiv e^2/(4\pi\epsilon_0\hbar c) = 1/137.03\cdots,$$

where α is the fine-structure constant.

Muon $g-2$ at BNL



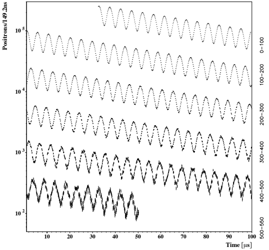


FIG. 2. The positron time spectrum obtained with muon injection for $E > 1.8$ GeV. These data represent 84 million positrons.

BNL final result 2006 G. W. Bennett et al. (Muon g-2), PRD73(2006)072003

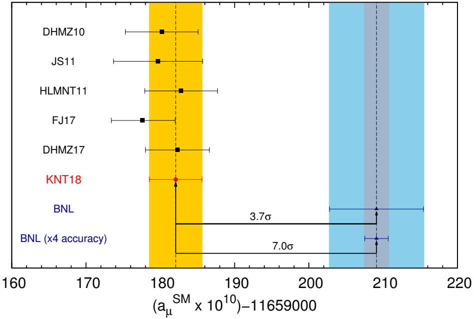
$$a_\mu \equiv (g_\mu - 2)/2 = (116\ 592\ 089 \pm 63) \times 10^{-11} \qquad [0.5\text{ppm}]$$

Theory needs QED up to 5 loop + hadronic $\mathcal{O}(10^{-7})$ + weak $\mathcal{O}(10^{-9})$:

$$(\alpha/\pi)^5 \underbrace{\pi^2 \ln^3(m_\mu/m_e)}_{\sim 1500} \sim 10 \times 10^{-11}$$

because of enhancement due to the electron loop, $m_e \ll m_\mu$.

New Muon $g-2$ experiments



A. Keshavarzi, D. Nomura, and T. Teubner, arXiv:1802.02995

Precision will be reduced from 0.5 ppm to 0.1 ppm.

Theory of lepton $g-2$

The Standard Model contribution to the lepton $g-2$:

$$a_l = \underbrace{a_l(\mathrm{QED})}_{\gamma,\ e,\ \mu,\ \tau} + \underbrace{a_l(\mathrm{weak})}_{W^\pm,\ Z^0} + a_l(\mathrm{hadron})$$

The QED contribution depends on lepton-mass ratios.
For the electron $g-2$, the dimensionless a_l is divided into

$$a_e(\mathrm{QED}) = \underbrace{A_1}_{\gamma,\ e} + \underbrace{A_2(m_e/m_\mu)}_{\gamma,\ e,\ \mu} + \underbrace{A_2(m_e/m_\tau)}_{\gamma,\ e,\ \tau} + \underbrace{A_3(m_e/m_\mu, m_e/m_\tau)}_{\gamma,\ e,\ \mu,\ \tau}.$$

A_1 is the same for any lepton, mass-independent and universal.
Perturbation expansion of QED:

$$A_i = \left(\frac{\alpha}{\pi}\right) A_i^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 A_i^{(4)} + \left(\frac{\alpha}{\pi}\right)^3 A_i^{(6)} + \left(\frac{\alpha}{\pi}\right)^4 A_i^{(8)} + \left(\frac{\alpha}{\pi}\right)^5 A_i^{(10)} + \cdots$$

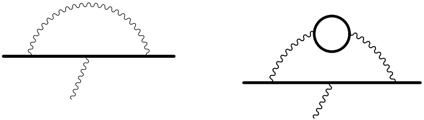
By 2017, all terms up to the 8th order are well known.

QED electron $g-2$

One electron scattering by an external photon C, P, T, and Lorentz invariance guarantee the form of scattering amplitude:

$$e\bar{u}(p+q/2)\left[\gamma^\mu F_1(q^2) + \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2)\right]u(p-q/2)A_\mu(q)$$

$$F_2(q^2=0) \equiv a_e, \qquad F_1(q^2=0) \equiv 1$$



The muon and tau-lepton contribute to a_e very little:

$$a_e(\text{QED:mass-dependent}) = 2.747\ 5719\ (13) \times 10^{-12}$$

from 4th, 6th, 8th and 10th-order diagrams involving fermion loops.

QED mass-independent term A_1

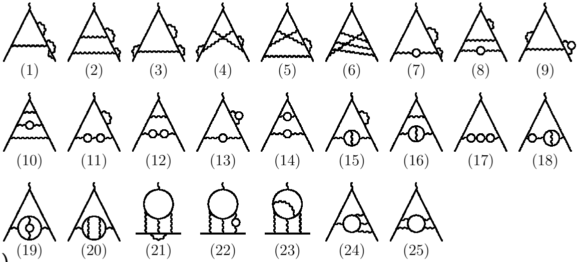
Focus on the mass-independent A_1:

n loops	# of F diagrams	w/ fermion loops	w/o fermion loops
1	1	0	1
2	7	1	6
3	72	22	50
4	891	373	518
5	12,672	6,318	6,354

n loops	$A_1^{(2n)}$	who & when
1	$A_1^{(2)} = \ \ 0.5$	Schwinger 1948
2	$A_1^{(4)} = -0.328\ 478\ 965\cdots$	Petermann 1957, Sommerfield 1958
3	$A_1^{(6)} = \ \ 1.181\ 241\ 456\cdots$	Laporta and Remiddi 1996
4	$A_1^{(8)} = -1.912\ 245\ 764\cdots$	Laporta 2017
5	$A_1^{(10)} = \ 6.737\ (159)$	Aoyama et al. (AHKN) 2018

QED 8th-order $A_1^{(8)}$

891 Feynman vertex diagrams: S. Laporta, PLB772(2017)232, talk on Day 2



History of $A_1^{(8)}$:

year	who	$A_1^{(8)}$	comment
2017	Laporta	$-1.912245764\cdots$	near analytic, 1100 digits
2015	AHKN	$-1.91298\ (84)$	latest numerical
2008	AHKN	$-1.9144\ (35)$	two integrals revised
2005	KN	$-1.7283\ (35)$	light-by-light revised
1990	Kinoshita	$-1.43\ (14)$	improved
1981	K & Lindquist	$-0.8\ (2.5)$	1st result

More on 8th-order terms

- More on the mass-independent term $A_1^{(8)}$: P. Marquard,et al.

  1) Alternative semi-analytic result: $A_1^{(8)} = -1.87\ (12)$ arXiv:1708.07138

     Consistent with Laporta's -1.912 and AHKN's -1.913.

  2) Alternative numerical work on the contribution from 518 diagrams w/o fermion loops:

Laporta	$-2.176\ 866\ 02 \cdots$	S. Laporta, PLB772(2017)232
AHKN	$-2.177\ 33\ (82)$	AKHN, PRD91(2015)033006
Volkov	$-2.1790\ (22)$	S. Volkov, PRD98(2018)076018, talk on Day 2

- Mass-dependent terms $A_2^{(8)}$ and $A_3^{(8)}$:

  1) Numerical calculation: AKHN, PRL109(2012)111807

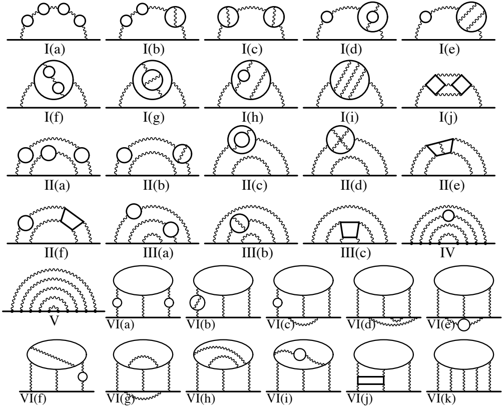
     Change the loop fermion mass from m_e to $m_\mu(m_\tau)$. Easy.

  2) Analytic calculation: A. Kurz et al. PRD93(2016)053017, NPB879(2014)1

     An additional small expansion parameter $m_e/m_\mu\ (m_\tau) \ll 1$.

Don't worry about the 8th order any more. It's **CORRECT**.

QED 10th-order vertex diagrams



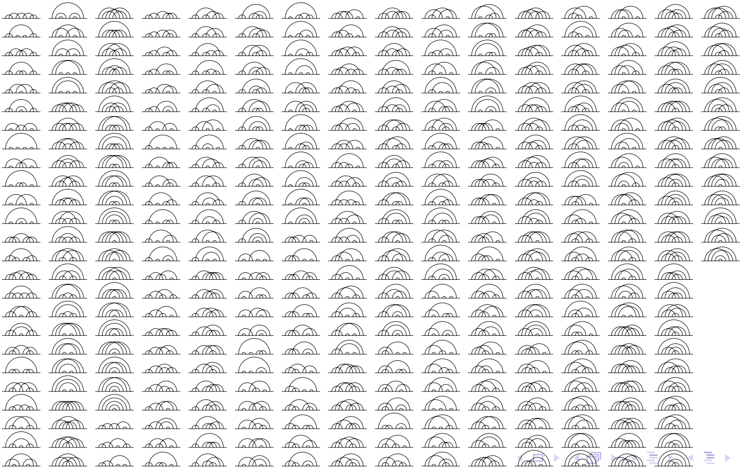
12,672 Feynman vertex diagrams divided into 32 subsets:

- 6,354 vertex diagrams w/o a fermion loop, Set V. difficult
- 6,318 diagrams w/ closed fermion loops, Set I-IV, IV. easier

10th-order Set V

The hardest diagrams to evaluate belong to Set V.
Ward-Takahashi concatenation:

$6354/9 = 706 \rightarrow 389$, because of time-reversal symmetry.



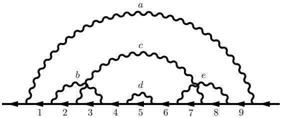
Numerical approach to QED Feynman diagrams

Uniqueness of Kinoshita's approach to QED $g-2$:

P. Cvitanovic and T. Kinoshita (1974)

- Ward-Takahashi sum of vertex diagrams
- Feynman parameter space

Momentum space, 20 dim. v.s. Feynman parameter space, 13 dim.



· $\Lambda^{\nu}(p,q)$ ... sum of 9 vertex diagrams
· $\Sigma(p)$ .... a self-energy diagram
· q............. momentum of an external photon
· $p \pm q/2$ ... momenta of external on-shell electrons

$$\Lambda^{\nu}(p,q) \approx q_{\mu} \left[\frac{\partial \Lambda^{\mu}(p,q)}{\partial q_{\nu}} \right]_{q=0} - \frac{\partial \Sigma(p)}{\partial p_{\nu}}$$

The r.h.s. is to be calculated instead of the l.h.s.

Feynman parametric amplitude

Loop momenta are exactly and analytically integrated out.
The bare amplitude of a n-loop self-energy like diagram $\mathcal{G}$ is

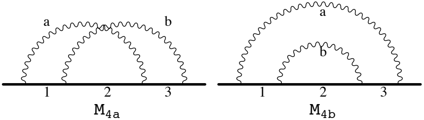
$$M_{\mathcal{G}}^{(2n)} = \left(\frac{-1}{4}\right)^n (n-1)! \int (dz)_{\mathcal{G}} \left[\frac{1}{n-1}\left(\frac{E_0+C_0}{U^2V^{n-1}} + \frac{E_1+C_1}{U^3V^{n-2}} + \cdots + \frac{E_{n-2}+C_{n-2}}{U^nV}\right) + \left(\frac{N_0+Z_0}{U^2V^n} + \frac{N_1+Z_1}{U^3V^{n-1}} + \cdots + \frac{N_{n-1}+Z_{n-1}}{U^{n+1}V}\right)\right] .$$

E_i and C_i are from $\partial\Lambda/\partial q$. N_i and Z_i are from $\partial\Sigma/\partial p$.
All are expressed by the building blocks:

- z_i... Feynman parameter of line i.
- B_{ij}... "correlation function" of lines i and line j, determined by and only by the topology of a diagram.
- A_i... scalar current of the external momentum p on line i.

UV and IR counter terms: 4th-order example

Divergence structures of the WT-sum is same as that of the self-energy diagram.



On-shell renormalization defines the contributions:

$$a_{4a} \equiv \underbrace{M_{4a}}_{\text{unrenorm.}} - \underbrace{2L_2}_{\text{vertex renorm.}} \times \underbrace{M_2}_{\text{Schwinger's } a_e} ,$$

$$a_{4b} \equiv \underbrace{M_{4b}}_{\text{unrenorm.}} - \underbrace{dm_2}_{\text{mass renorm.}} \times M_{2*} - \underbrace{B_2}_{\text{wave func. renorm.}} \times M_2$$

UV divergences arise in M_{4a}, M_{4b}, L_2, dm_2, B_2 .
IR divergences arise in a_{4a}, a_{4b}, M_{4b}, M_{2*}, L_2, B_2.
Only M_2 is finite.

UV and IR separation by K-operation

Both a_{4a} and a_{4b} are IR divergent, but the sum $a_{4a} + a_{4b}$ is finite.

Thanks to the Kinoshita-Lee-Nauenberg IR cancellation theorem.

Express the finite contribution in terms of the finite quantities:

$$a_{4a} + a_{4b} = \Delta M_{4a} + \Delta M_{4b} - \Delta LB_2,$$

where

$$\Delta M_{4a} \equiv M_{4a} - 2L_2^{\mathrm{UV}} M_2,$$

$$\Delta M_{4b} \equiv M_{4b} \underbrace{- dm_2^{\mathrm{UV}} M_{2*} - B_2^{\mathrm{UV}} M_2}_{\text{UV subtraction}} \quad \underbrace{- L_2^{\mathrm{R}} M_2 - dm_2^{\mathrm{R}} M_{2*}}_{\text{IR subtraction}}$$

$$L_2 = L_2^{\mathrm{UV}} + L_2^{\mathrm{R}}, \quad B_2 = B_2^{\mathrm{UV}} + B_2^{\mathrm{R}}, \quad dm_2 = dm_2^{\mathrm{UV}} + dm_2^{\mathrm{R}},$$

$$\Delta LB_2 \equiv L_2^{\mathrm{R}} + B_2^{\mathrm{R}}$$

UV terms are determined by K-operation.

IR terms are determined as residues of the UV terms.

The K-operation is a simple power counting rule of Feynman parameters.

Easy to implement it as a manipulation in a computer code.

Contribution from the 10th order Set V

Do the same separation for 389 Set V self-energy-like diagrams.
The integrands of ΔM_{X001}–ΔM_{X389} are automatically generated.
The residual finite renormalization term is obtained as

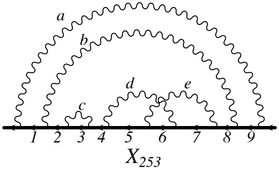
$$
\begin{aligned}
A_1^{(10)}[\text{Set V}] = {} & \Delta M_{10} + \Delta M_8 \left(-7\Delta LB_2\right) + \Delta M_6 \left\{-5\Delta LB_4 + 20(\Delta LB_2)^2\right\} \\
& + \Delta M_4 \left\{-3\Delta LB_6 + 24\Delta LB_4 \ \Delta LB_2 - 28(\Delta LB_2)^3\right\} \\
& + \Delta M_4 \left(2\Delta dm_4 \ \Delta L_{2^*}\right) \\
& + M_2 \left\{-\Delta LB_8 + 8\Delta LB_6 \ \Delta LB_2 - 28\Delta LB_4 \ (\Delta LB_2)^2 + 4(\Delta LB_4)^2 + 14(\Delta LB_2)^4\right\} \\
& + M_2 \ \Delta dm_6 \left(2\Delta L_{2^*}\right) \\
& + M_2 \ \Delta dm_4 \left(-16\Delta LB_2 \ \Delta L_{2^*} - 2\Delta dm_{2^*} \ \Delta L_{2^*} + \Delta L_{4^*}\right),
\end{aligned}
$$

where

$$\Delta M_{10} = \sum_{\mathcal{G}=X001}^{X389} \Delta M_{\mathcal{G}} \ .$$

Each $\Delta M_{\mathcal{G}}$ is to be numerically evaluated.

Automatic code generation for Set V



X253 represents 18 vertex diagrams
6354 vertex diagrams → 389 integrals

automation

About 72,000 lines

Diagram information
X253: "abccdedeba"

GencodeN

Fortran Programs
ΔM(X253)

1. Amplitude M(X253)
2. UV subtraction terms
   $M(X253)^R = M(X253)$ — (23 UV terms)
3. IR subtraction terms
   $\Delta M(X253) = M(X253)^R$ — (91 IR terms)

When they are numerically integrated by VEAGS, quadruple precision of real numbers is used.

HOKUSAI-BigWaterfall 2017-, 2.5 Pflops
HOKUSAI-GreatWave 2015-, 1 Pflops
RICC 2009-2017, 96Tflops
RSCC 2004-2009, 12Tflops RIKEN Wako

Tools used for automatic code generation

`GencodeN` .... our code generator

- Identify diagram information by `Perl` program
- Create building blocks for a diagram
  linear algebra, eg. inversion of a matrix, by `Maple`
- Create an bare amplitude
  γ-trace calculation by `FORM`
- Create UV and IR quantities
  symbolic manipulation by `FORM`
- Control the whole process by `Perl` program

All 389 integrands had been constructed by 2006.

Numerical integration

Very difficult numerical integration

- Round-off problem

  IR cancellation causes the problem.

  Need very fast quadruple precision arithmetics

  A clue is the library of double-double and quad-double arithmetic.

  D. Bailey et al. 2000, 2007

- Sharp-peak problem

  An integrand has very sharp peaks at the surface volume of
  a 13 dimensional hypercube.

  Need robust algorithm of multi-dimensional integration

  A clue is `VEGAS` algorithm. G. P. Lepage 1978

  Monte-Carlo integration, error decreases slowly by $1/\sqrt{N}$

Huge computational resources are required.

HOKUSAI GW & BW at RIKEN

神奈川沖浪裏

北斎

下野黒髪山きりふりの滝

1 PFLOPS Fujitsu PRIMEHPC FX100 (34560 cores)
April 2015-
Cutting edge supercomputer
Compatibility with the K computer
Availability for highly parallelized programs

HOKUSAI GREAT WAVE

HOKUSAI BIG WATERFALL

2.58 PFLOPS IA Cluster of Xeon Gold 6148 (33600cores)
October 2017-
Raising HPC environment of RIKEN
Popular architecture
High versatility

Cross-check for integrals of Set V

- Reshuffle integration variables of the 389 integrals.
- 2017 calculation is therefore independent from 2015 calculation.
- Numerical results with different mappings are in good agreement.

Integrals showing relatively large discrepancies:

integral	2017 result	2015 result	difference
X100	−15.232(17)	−15.292(33)	0.060
X141	−12.496(17)	−12.557(35)	0.060
X113	−4.443(17)	−4.385(32)	−0.058
X325	11.539(17)	11.596(34)	−0.056
X146	−2.246(17)	−2.299(34)	0.053
X236	2.107(21)	2.056(18)	0.051
X153	14.845(17)	14.894(34)	−0.048
X251	−1.343(20)	−1.391(08)	0.047
X044	4.365(16)	4.412(28)	−0.047
X144	23.677(17)	23.724(37)	−0.047
X252	−10.865(17)	−10.909(34)	0.044
X256	−13.996(17)	−14.041(34)	0.044

Independent calculation of Set V

Our 2017 calculation is continued.
S. Volkov announced his preliminary result at his ACAT2019 Day 2 talk.

Latest results of Set V

$$A_1^{(10)}[\text{Set V}] = \begin{cases} 7.668\ (159) & \text{AKN2019} \\ 6.782\ (113) & \text{S. Volkov, ACAT 2019} \end{cases}$$

- Both rely on numerical means.
- Approaches to Feynman diagrams are different. So, independent.
- Difference $-0.89\ (20)$ is about $4.5\ \sigma$.
- Not seriously affect the current precision of lepton $g-2$.
- But becomes serious for the future electron experiments.

New value of $A_1^{(10)}$

New and massive evaluation of Set V leads to

the mass-independent 10th-order $A_1^{(10)}$

$$A_1^{(10)} = 6.675\ (192) \qquad \longrightarrow 6.737\ (159) \quad \text{at present}$$

AKN,PRD97(2018)036001[arXiv:1712.06060] AKN, Atoms 7(1) 28, 2019

If Volkov's new result for Set V is used,

the mass-independent 10th-order $A_1^{(10)}$

$$A_1^{(10)} = 5.851\ (113)$$

QED mass-dependent terms for a_e

Input parameters:

$m_e/m_\mu = 0.483\ 633\ 170\ (11) \times 10^{-2}$ CODATA2014

$m_e/m_\tau = 0.287\ 585\ (19) \times 10^{-3}$ PDG2018

$A_2^{(4)}(m_e/m_\mu)$	$0.519\ 738\ 676\ (24) \times 10^{-6}$	Elend1966
$A_2^{(4)}(m_e/m_\tau)$	$0.183\ 790\ (25) \times 10^{-8}$	Elend1966
$A_3^{(4)}(m_e/m_\mu, m_e/m_\tau)$	0	
$A_2^{(6)}(m_e/m_\mu)$	$-0.737\ 394\ 164\ (24) \times 10^{-5}$	Samuel&Li1991,Li,Mendel&Samuel1993, Laporta1993,Laporta&Remiddi1993
$A_2^{(6)}(m_e/m_\tau)$	$-0.658\ 273\ (79) \times 10^{-7}$	Samuel&Li1991,Li,Mendel&Samuel1993, Laporta1993,Laporta&Remiddi1993
$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)$	$0.1909\ (1) \times 10^{-12}$	Passera2006
$A_2^{(8)}(m_e/m_\mu)$	$0.916\ 197\ 070\ (37) \times 10^{-3}$	Kurz et al.2013,AHKN2012
$A_2^{(8)}(m_e/m_\tau)$	$0.742\ 92\ (12) \times 10^{-5}$	Kurz et al.2013,AHKN2012
$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$	$0.746\ 87\ (28) \times 10^{-6}$	Kurz et al.2013,AHKN2012
$A_2^{(10)}(m_e/m_\mu)$	$-0.003\ 82\ (39)$	AHKN2012,2015
$A_2^{(10)}(m_e/m_\tau)$	$\mathcal{O}(10^{-5})$	
$A_3^{(10)}(m_e/m_\mu, m_e/m_\tau)$	$\mathcal{O}(10^{-5})$	

The fine-structure constant α

To obtain the theory prediction of a_e, we need the non-QED value of the fine-structure constant α.
Two values of α from $h/m(X)$ measurements of the atom interferometer:

$$\alpha^{-1}(\mathrm{Rb}) = 137.035\,998\,995\,(85)$$ R. Bouchendira et al. 2011

$$\alpha^{-1}(\mathrm{Cs}) = 137.035\,999\,046\,(27)$$ R. H. Parker et al. , 2018

through the relation

$$\alpha^{-1}(X) = \left[\frac{2R_\infty}{c} \times \frac{A_r(X)}{A_r(e)} \times \frac{h}{m(X)}\right]^{-1/2} .$$

R_∞ ...	the Rydberg constant
$A_r(X)$ ...	relative atomic mass of a particle X
h ...	Planck constant
c ...	speed of light
$m(X)$ ...	mass of an atom X

Electron $g-2$, theory v.s. experiment

Hadronic and weak contributions to a_e are F. Jegerlehner, arXiv:1705.00263

$$a_e(\text{hadron}) = 1.693\ (12) \times 10^{-12}, \qquad a_e(\text{weak}) = 0.030\ 53\ (23) \times 10^{-12}\ .$$

With $\alpha(\text{Rb})$ or $\alpha(\text{Cs})$, the SM prediction of a_e is

$$a_e(\text{theory} : \alpha(\text{Rb})) = 1\ 159\ 652\ 182.037\ (11)(12)(720) \times 10^{-12}$$
$$a_e(\text{theory} : \alpha(\text{Cs})) = 1\ 159\ 652\ 181.606\ (11)(12)(229) \times 10^{-12}$$

QED 10th, hadron, $\alpha(X)$

The Harvard measurement of a_e: D. Hanneke, S. Fogwell , and G. Gabrielse (2008)

$$a_e(\text{expt.}) = 1\ 159\ 652\ 180.72\ (28) \times 10^{-12}$$

Difference between measurement and theory:

$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Rb})) = (-1.31 \pm 0.77) \times 10^{-12} \qquad 1.7\sigma$$
$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Cs})) = (-0.88 \pm 0.36) \times 10^{-12} \qquad 2.4\sigma$$

Electron $g-2$, theory v.s. experiment w/ Volkov's result

Volkov's new $A_1^{(10)}$[Set V] makes the theoretical prediction smaller by

$$0.886 \left(\frac{\alpha}{\pi}\right)^5 = 0.0599 \times 10^{-12}$$

and reduce the uncertainty due to the tenth-order QED

$$0.011 \times 10^{-12} \quad \longrightarrow \quad 0.0076 \times 10^{-12}$$

Comparison of measurement and theory becomes

$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Rb})) = (-1.25 \pm 0.77) \times 10^{-12} \qquad 1.6\sigma$$
$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Cs})) = (-0.82 \pm 0.36) \times 10^{-12} \qquad 2.3\sigma$$

New expts on a_e and α(Cs) bring comparison to the level of 0.03 $\times 10^{-12}$.
Discrepancy in the tenth-order QED $A_1^{(10)}$ must be resolved.

Search of new physics through α

One more value of α is obtained from the electron $g-2$.
Solve $\alpha(a_e)$ from $a_e(\text{expt.}) = a_e(\text{theory})$:

$$\alpha^{-1}(a_e) = 137.035\,999\,1496\;(13)(14)(330)$$ AKN2018, 2019

QED 10th, hadron, expt.

Difference between two other determinations of α:

$$\alpha^{-1}(a_e) - \alpha^{-1}(\text{Rb}) = (0.155 \pm 0.091) \times 10^{-6} \qquad 1.7\sigma$$
$$\alpha^{-1}(a_e) - \alpha^{-1}(\text{Cs}) = (0.104 \pm 0.043) \times 10^{-6} \qquad 2.4\sigma$$
$$\alpha^{-1}(\text{Rb}) - \alpha^{-1}(\text{Cs}) = (-0.051 \pm 0.089) \times 10^{-6} \qquad 0.6\sigma$$

New physics or misinterpretation of the SM physics?
Misinterpretation in experiment or theory?

CODATA2017 and new SI

$\alpha(a_e)$ and $\alpha(\mathrm{Rb})$ are used to determine **exact** values of some fundamental constants.

In the new SI, the Planck constant h, the elementary charge e, the Boltzmann constant k, and the Avogadro number N_A become defined numbers like the speed of light c:

$$\begin{aligned} h &= 6.626\ 070\ 15 \times 10^{-34}\ \mathrm{Js}, \\ e &= 1.602\ 176\ 634 \times 10^{-19}\ \mathrm{C}, \\ k &= 1.380\ 649 \times 10^{-23}\ \mathrm{JK}^{-1}, \\ N_A &= 6.022\ 140\ 76 \times 10^{23}\ \mathrm{mol}^{-1}. \end{aligned}$$

Definition of kilogram is based on the Planck constant h after the new SI launches in 2019.

Good bye, the International Prototype Kilogram.

P. J. Mohr, D. B. Newell, B. N. Taylor and E. Tiesinga, Metrologia 55(2018)125

QED muon $g-2$

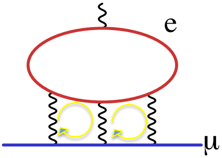
Mass-dependent terms for a_μ

For higher-order terms, the electron-loop contributions are dominant.

$A_2^{(4)}(m_\mu/m_e)$	1.094 258 3093 (76)	Elend1966
$A_2^{(4)}(m_\mu/m_\tau)$	0.000 078 076 (11)	Elend1966
$A_3^{(4)}(m_\mu/m_e, m_\mu/m_\tau)$	0	
$A_2^{(6)}(m_\mu/m_e)$	22.868 379 98 (20)	Samuel&Li1990,1992,1993 Laporta1993,Laporta&Remiddi1993
$A_2^{(6)}(m_\mu/m_\tau)$	0.000 360 671 (94)	Samuel&Li1990,1992, 1993 Laporta1993,Laporta&Remiddi1993
$A_3^{(6)}(m_\mu/m_e, m_\mu/m_\tau)$	0.000 527 738 (75)	Czarnecki&Skzrypek1999
$A_2^{(8)}(m_\mu/m_e)$	132.6852 (60)	AHKN2012, Kurz et al.2016a
$A_2^{(8)}(m_\mu/m_\tau)$	0.042 4941 (53)	AHKN2012, Kurz et al.2016b
$A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$	0.062 722 (10)	AHKN2012, Kurz et al.2016a
$A_2^{(10)}(m_\mu/m_e)$	742.32 (86)	AHKN2012
$A_2^{(10)}(m_\mu/m_\tau)$	−0.0656 (45)	AHKN2012
$A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau)$	2.011 (10)	AHKN2012

The light-by-light contribution to a_μ

The large enhancement factor
comes from the light-by-light diagrams:



$$\frac{2}{3}\pi^2 \ln(m_\mu/m_e) \sim 35$$

A factor π comes from the integration over a Coulomb loop.

S. Karshenboim 1993

Insertion of the 2nd-order VP adds another log factor:

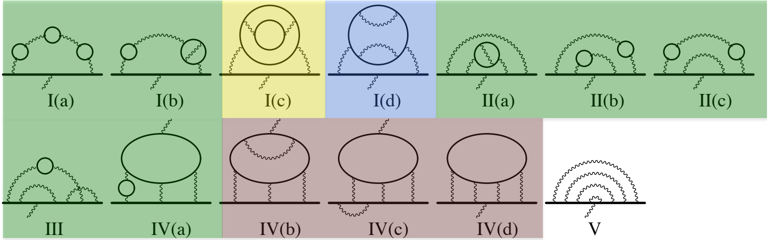
$$\frac{2}{3}\ln(m_\mu/m_e) - \frac{5}{9} \sim 3$$

T. Kinoshita started developing his numerical calculation method of QED to investigate this 6th-order light-by-light diagrams for a_μ.

T. Kinoshita 1967, J. Aldins et al. 1969, 1970

QED 8th-order mass-dependent term $A_2^{(8)}$

Mass-dependence comes from diagrams w/ fermion loops:



AHKN2012, Kurz et al.2016

- I(a), I(b), II(a), II(b), II(c)
  2nd and/or 4th-order VP functions
- I(c) Checked w/ the 1-dimensional integral of the VP function
- I(d) Checked w/ the Padé approximated VP function
- IV(b), IV(c), IV(d)
  Checked w/ the asymptotic expansion

Analytic calculations of $A_2^{(8)}$

Analytic calculation of A_2 and A_3 are a little easier than A_1.
But still tough for the higher orders.

- Expansion parameters: $m_e/m_\mu = 1/206.7\cdots$, $m_\mu/m_\tau = 1/16.8\cdots$
- Different analytic structures: $m_{\text{loop}} \gg m_{\text{external}}$ or $m_{\text{loop}} \ll m_{\text{external}}$

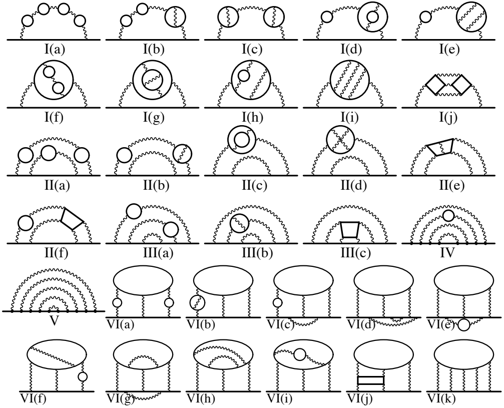
Analytic expansions of the 8th order were obtained in 2014 and in 2016:

$m_{\text{loop}} \ll m_{\text{external}}$ A. Kurz et al. PRD93, 053017(2016)

$m_{\text{loop}} \gg m_{\text{external}}$ A. Kurz et al. PoS LL2014, 051(2014)

- Diagrams involving a light-by-light loop are very difficult to evaluate.
- The analytic expansion results are less accurate than the numerical results.
- Both are in good agreement.

QED 10th-order vertex diagrams



12,672 Feynman vertex diagrams divided into 32 subsets:

- 6,354 vertex diagrams w/o a fermion loop, Set V. difficult
- 6,318 diagrams w/ closed fermion loops, Set I-IV, IV. easier

QED 10th-order mass-dependent term $A_2^{(10)}$

All diagrams have been numerically calculated.

AHKN (2012)

Many consistency checks:

- LO contribution can be estimated with help of renormalization group.
  A. Kataev 1992, 1995, 2006
- Non-relativistic calculation is also useful for VI(k), the light-by-light diagram to which six photons are attached . S. Karshenboim,1993
- Padé approximated 4-loop vacuum-polarization function.
  P. A. Baikov, A. Maier, and P. Marquard, 2013
- Asymptotic expansion of the diagrams w/ 2nd- and/or 4th-order VP functions. S. Laporta,1994

QED contributions to a_μ, muon $g-2$

With $\alpha(\mathrm{Cs})$ and $\alpha(a_e)$ the QED contribution to a_μ is

$$a_\mu(\mathrm{QED} : \alpha(\mathrm{Cs})) = 1\ 165\ 847\ 189.31\ (7)(17)(6)(23)[30] \times 10^{-12}$$
$$a_\mu(\mathrm{QED} : \alpha(a_e)) = 1\ 165\ 847\ 188.42\ (7)(17)(6)(28)[34] \times 10^{-12}$$

lepton-mass ratios, QED 8th, QED 10th, α, combined

All are sufficiently accurate for the current and future measurements of a_μ.
Further numerical improvement on QED 8th and 10th is possible.
Targets are diagrams involving a light-by-light scattering subdiagram.
How about the 12th-order contribution?
The 6th-order light-by-light + three 2nd-order VP insertions:

$$\left(\frac{\alpha}{\pi}\right)^6 \times \underbrace{10}_{\text{ways of insertion}} \times \underbrace{3^3}_{\text{3 VP}} \times \underbrace{20}_{\text{LbyL}} \sim 0.8 \times 10^{-12}\ .$$

The size of the whole 12th-order contribution might be $\mathcal{O}(10^{-12})$.
Need a crude estimate of the 12th-order contribution

Summary

- QED $g-2$ up to the 8th-order contribution has been firmly established in last 3 years.
- QED $g-2$ of the 10th order has been extensively calculated and checked.
- QED $g-2$ is ready for the on-going new experiments of electron-positron $g-2$ and of muon $g-2$.
- QED $g-2$ is served for the new SI. After the new SI launches, the fine-structure constant α is the unique source of uncertainties of other fundamental physical constants.
- QED $g-2$ shows that we are able to compute many and complex Feynman diagrams using analytic/numerical methods with help of powerful computers.