Higher-order QED contributions to the lepton anomalous magnetic moments

Makiko Nio (RIKEN)

This talk is based on collaboration w/
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M. Hayakawa (Nagoya U)
T. Kinoshita (Cornell U and UMass Amherst)

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Intrinsic magnetic property of a single lepton particle is characterized by a dimensionless number, called $g$-factor.

$$H = -\vec{\mu} \cdot \vec{B}, \quad \vec{\mu} = g \frac{e}{2m}\hat{s}$$

Anomaly, $a \equiv (g - 2)/2$, is a consequence of quantum nature of elementary particles. R. Kusch and H. M. Foley 1948, J. Schwinger 1948

Electron $g - 2$ is measured by using a Penning trap:
- University of Washington: H. Dehmelt et al. (1987)

Positron $g - 2$ measurement is in preparation.

Muon $g - 2$ is measured by using a muon storage ring:
- On-going experiments: J-PARC(2009–), Fermilab(2011–)

Both are the state-of-the-art measurements in study of elementary particles
Electron $g - 2$ measurement

![Diagram of cylindrical Penning trap cavity](image)

FIG. 4. Cylindrical Penning trap cavity used to confine a single electron and inhibit spontaneous emission.


Harvard 2008 measurement

\[ a_e \equiv (g_e - 2)/2 = (1\,159\,652\,180.73 \pm 0.28) \times 10^{-12} \quad [0.24\text{ppb}] \]

Theory needs QED up to 5 loop + hadronic $\mathcal{O}(10^{-12})$ + weak $\mathcal{O}(10^{-14})$:

\[ \left(\frac{\alpha}{\pi}\right)^5 \sim 0.068 \times 10^{-12}, \quad \alpha \equiv e^2/(4\pi\epsilon_0\hbar c) = 1/137.03 \cdots, \]

where $\alpha$ is the fine-structure constant.
Muon $g - 2$ at BNL

BNL final result 2006

G. W. Bennett et al. (Muon g-2), PRD73(2006)072003

$$a_\mu \equiv (g_\mu - 2)/2 = (116\,592\,089 \pm 63) \times 10^{-11} \quad [0.5\text{ppm}]$$

Theory needs QED up to 5 loop + hadronic $\mathcal{O}(10^{-7}) +$ weak $\mathcal{O}(10^{-9})$:

$$(\alpha/\pi)^5 \pi^2 \ln^3 (m_\mu/m_e) \sim 10 \times 10^{-11} \quad \sim 1500$$

because of enhancement due to the electron loop, $m_e \ll m_\mu$. 
New Muon $g-2$ experiments


Precision will be reduced from 0.5 ppm to 0.1 ppm.
**Theory of lepton $g - 2$**

The Standard Model contribution to the lepton $g - 2$:

$$a_l = a_l(QED) + a_l(\text{weak}) + a_l(\text{hadron})$$

- $\gamma, e, \mu, \tau$
- $W^\pm, Z^0$

The QED contribution depends on lepton-mass ratios. For the electron $g - 2$, the dimensionless $a_l$ is divided into

$$a_e(QED) = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau).$$

$A_1$ is the same for any lepton, mass-independent and universal.

Perturbation expansion of QED:

$$A_i = \left(\frac{\alpha}{\pi}\right)^2 A_i^{(2)} + \left(\frac{\alpha}{\pi}\right)^4 A_i^{(4)} + \left(\frac{\alpha}{\pi}\right)^6 A_i^{(6)} + \left(\frac{\alpha}{\pi}\right)^8 A_i^{(8)} + \left(\frac{\alpha}{\pi}\right)^{10} A_i^{(10)} + \cdots$$

By 2017, all terms up to the 8th order are well known.
One electron scattering by an external photon C, P, T, and Lorentz invariance guarantee the form of scattering amplitude:

\[ \bar{e}u(p + q/2) \left[ \gamma^\mu F_1(q^2) + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_2(q^2) \right] u(p - q/2) A_\mu(q) \]

\[ F_2(q^2 = 0) \equiv a_e, \quad F_1(q^2 = 0) \equiv 1 \]

The muon and tau-lepton contribute to \( a_e \) very little:

\[ a_e(\text{QED:mass-dependent}) = 2.747 \, 5719 (13) \times 10^{-12} \]

from 4th, 6th, 8th and 10th-order diagrams involving fermion loops.
QED mass-independent term $A_1$

Focus on the mass-independent $A_1$:

<table>
<thead>
<tr>
<th>$n$ loops</th>
<th># of F diagrams</th>
<th>w/ fermion loops</th>
<th>w/o fermion loops</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>72</td>
<td>22</td>
<td>50</td>
</tr>
<tr>
<td>4</td>
<td>891</td>
<td>373</td>
<td>518</td>
</tr>
<tr>
<td>5</td>
<td>12,672</td>
<td>6,318</td>
<td>6,354</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$ loops</th>
<th>$A_1^{(2n)}$</th>
<th>who &amp; when</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_1^{(2)} = 0.5$</td>
<td>Schwinger 1948</td>
</tr>
<tr>
<td>2</td>
<td>$A_1^{(4)} = -0.328 \ 478 \ 965 \cdots$</td>
<td>Petermann 1957, Sommerfield 1958</td>
</tr>
<tr>
<td>3</td>
<td>$A_1^{(6)} = 1.181 \ 241 \ 456 \cdots$</td>
<td>Laporta and Remiddi 1996</td>
</tr>
<tr>
<td>4</td>
<td>$A_1^{(8)} = -1.912 \ 245 \ 764 \cdots$</td>
<td>Laporta 2017</td>
</tr>
<tr>
<td>5</td>
<td>$A_1^{(10)} = 6.737 \ (159)$</td>
<td>Aoyama et al. (AHKN) 2018</td>
</tr>
</tbody>
</table>
# QED 8th-order $A_1^{(8)}$

891 Feynman vertex diagrams: S. Laporta, PLB772(2017)232, talk on Day 2

![Diagram](image)

## History of $A_1^{(8)}$:

<table>
<thead>
<tr>
<th>Year</th>
<th>Who</th>
<th>$A_1^{(8)}$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>2017</td>
<td>Laporta</td>
<td>$-1.912245764 \cdots$</td>
<td>near analytic, 1100 digits</td>
</tr>
<tr>
<td>2015</td>
<td>AHKN</td>
<td>$-1.91298 \ (84)$</td>
<td>latest numerical</td>
</tr>
<tr>
<td>2008</td>
<td>AHKN</td>
<td>$-1.9144 \ (35)$</td>
<td>two integrals revised</td>
</tr>
<tr>
<td>2005</td>
<td>KN</td>
<td>$-1.7283 \ (35)$</td>
<td>light-by-light revised</td>
</tr>
<tr>
<td>1990</td>
<td>Kinoshita</td>
<td>$-1.43 \ (14)$</td>
<td>improved</td>
</tr>
<tr>
<td>1981</td>
<td>K &amp; Lindquist</td>
<td>$-0.8 \ (2.5)$</td>
<td>1st result</td>
</tr>
</tbody>
</table>
More on 8th-order terms

- More on the mass-independent term $A_1^{(8)}$:
  1) Alternative semi-analytic result: $A_1^{(8)} = -1.87 (12)$
      Consistent with Laporta's $-1.912$ and AHKN's $-1.913$.
  2) Alternative numerical work on the contribution from 518 diagrams w/o fermion loops:
      Laporta $-2.176\,866\,02 \cdots$ S. Laporta, PLB772(2017)232
      AHKN $-2.177\,33\,(82)$ AKHN, PRD91(2015)033006
      Volkov $-2.1790\,(22)$ S. Volkov, PRD98(2018)076018, talk on Day 2

- Mass-dependent terms $A_2^{(8)}$ and $A_3^{(8)}$:
  1) Numerical calculation:
      Change the loop fermion mass from $m_e$ to $m_\mu(m_\tau)$. Easy.
  2) Analytic calculation:
      An additional small expansion parameter $m_e/m_\mu(m_\tau) \ll 1$.
      Don’t worry about the 8th order any more. It’s **CORRECT**.
12,672 Feynman vertex diagrams divided into 32 subsets:

- 6,354 vertex diagrams w/o a fermion loop, Set V.
  - difficult

- 6,318 diagrams w/ closed fermion loops, Set I-IV, IV.
  - easier
10th-order Set V

The hardest diagrams to evaluate belong to Set V.

Ward-Takahashi concatenation:

\[ \frac{6354}{9} = 706 \rightarrow 389, \text{ because of time-reversal symmetry.} \]
Numerical approach to QED Feynman diagrams

Uniqueness of Kinoshita’s approach to QED $g - 2$:

P. Cvitanovic and T. Kinoshita (1974)

- Ward-Takahashi sum of vertex diagrams
- Feynman parameter space
  - Momentum space, 20 dim. v.s. Feynman parameter space, 13 dim.

$\Lambda^{\nu}(p, q) \ldots$ sum of 9 vertex diagrams
$\Sigma(p) \ldots \ldots \ldots$ a self-energy diagram
$q \ldots \ldots \ldots$ momentum of an external photon
$p \pm q/2 \ldots$ momenta of external on-shell electrons

$\Lambda^{\nu}(p, q) \approx q_\mu \left[ \frac{\partial \Lambda^{\mu}(p, q)}{\partial q_\nu} \right]_{q=0} - \frac{\partial \Sigma(p)}{\partial p_\nu}$

The r.h.s. is to be calculated instead of the l.h.s.
Feynman parametric amplitude

Loop momenta are exactly and analytically integrated out. The bare amplitude of a $n$-loop self-energy like diagram $G$ is

$$M_G^{(2n)} = \left(\frac{-1}{4}\right)^n (n-1)! \int (dz)_G$$

$$= \frac{1}{n-1} \left[ \frac{E_0 + C_0}{U^2 V^{n-1}} + \frac{E_1 + C_1}{U^3 V^{n-2}} + \cdots + \frac{E_{n-2} + C_{n-2}}{U^n V} \right]$$

$$+ \left\{ \frac{N_0 + Z_0}{U^2 V^n} + \frac{N_1 + Z_1}{U^3 V^{n-1}} + \cdots + \frac{N_{n-1} + Z_{n-1}}{U^{n+1} V} \right\}.$$ 

$E_i$ and $C_i$ are from $\partial \Lambda / \partial q$. $N_i$ and $Z_i$ are from $\partial \Sigma / \partial p$.

All are expressed by the building blocks:
- $z_i$... Feynman parameter of line $i$.
- $B_{ij}$... "correlation function" of lines $i$ and line $j$, determined by and only by the topology of a diagram.
- $A_i$... scalar current of the external momentum $p$ on line $i$. 
UV and IR counter terms: 4th-order example

Divergence structures of the WT-sum is same as that of the self-energy diagram.

![Diagram of WT-sum and self-energy](image)

On-shell renormalization defines the contributions:

\[
a_{4a} \equiv M_{4a} \text{unrenorm.} - 2L_2 \text{vertex renorm.} \times M_2 \text{Schwinger’s } a_e
\]

\[
a_{4b} \equiv M_{4b} \text{unrenorm.} - dm_2 \text{mass renorm.} \times M_{2*} - B_2 \text{wave func. renorm.} \times M_2
\]

**UV** divergences arise in \( M_{4a}, M_{4b}, L_2, dm_2, B_2 \).

**IR** divergences arise in \( a_{4a}, a_{4b}, M_{4b}, M_{2*}, L_2, B_2 \).

Only \( M_2 \) is finite.
UV and IR separation by K-operation

Both $a_{4a}$ and $a_{4b}$ are IR divergent, but the sum $a_{4a} + a_{4b}$ is finite. Thanks to the Kinoshita-Lee-Nauenberg IR cancellation theorem.

Express the finite contribution in terms of the finite quantities:

$$a_{4a} + a_{4b} = \Delta M_{4a} + \Delta M_{4b} - \Delta L B_2,$$

where

$$\Delta M_{4a} \equiv M_{4a} - 2L_2^{UV} M_2,$$
$$\Delta M_{4b} \equiv M_{4b} - dm_2^{UV} M_{2*} - B_2^{UV} M_2 - L_2^R M_2 - dm_2^R M_{2*}$$

UV subtraction
IR subtraction

$$L_2 = L_2^{UV} + L_2^R, \quad B_2 = B_2^{UV} + B_2^R, \quad dm_2 = dm_2^{UV} + dm_2^R,$$

$$\Delta L B_2 \equiv L_2^R + B_2^R$$

UV terms are determined by K-operation.
IR terms are determined as residues of the UV terms.
The K-operation is a simple power counting rule of Feynman parameters.
Easy to implement it as a manipulation in a computer code.
Contribution from the 10th order Set V

Do the same separation for 389 Set V self-energy-like diagrams.
The integrands of $\Delta M_{X001} - \Delta M_{X389}$ are automatically generated.
The residual finite renormalization term is obtained as

$$A^{(10)}_1[\text{Set V}] = \Delta M_{10} + \Delta M_8 \left(-7 \Delta L_2 B\right) + \Delta M_6 \left\{ -5 \Delta L_4 B + 20 (\Delta L_2 B)^2 \right\}$$

$$+ \Delta M_4 \left\{ -3 \Delta L_6 B + 24 \Delta L_4 B \Delta L_2 B - 28 (\Delta L_2 B)^3 \right\}$$

$$+ \Delta M_4 \left( 2 \Delta d m_4 \Delta L_2^{*}\right)$$

$$+ M_2 \left\{ - \Delta L_8 B + 8 \Delta L_6 B \Delta L_2 B - 28 \Delta L_4 B \left( \Delta L_2 B \right)^2 + 4 (\Delta L_4 B)^2 + 14 (\Delta L_2 B)^4 \right\}$$

$$+ M_2 \Delta d m_6 \left( 2 \Delta L_2^{*}\right)$$

$$+ M_2 \Delta d m_4 \left( -16 \Delta L_2 B \Delta L_2^{*} - 2 \Delta d m_2^{*} \Delta L_2^{*} + \Delta L_4^{*}\right),$$

where

$$\Delta M_{10} = \sum_{G=X001}^{X389} \Delta M_G .$$

Each $\Delta M_G$ is to be numerically evaluated.
Automatic code generation for Set V

X253 represents 18 vertex diagrams
6354 vertex diagrams → 389 integrals

Diagram information
X253: “abccdedeba”

1. Amplitude $M(X253)$
2. UV subtraction terms
   $M(X253)^R = M(X253) - (23$ UV terms)$
3. IR subtraction terms
   $\Delta M(X253) = M(X253)^R - (91$ IR terms)$

When they are numerically integrated by VEAGS, quadruple precision of real numbers is used.

HOKUSAI-BigWaterfall 2017-, 2.5 Pflops
HOKUSAI-GreatWave 2015-, 1 Pflops
RICC 2009-2017, 96Tflops
RSCC 2004-2009, 12Tflops
RIKEN Wako
Tools used for automatic code generation

Gencode $N$ ....... our code generator

- Identify diagram information by Perl program
- Create building blocks for a diagram
  linear algebra, eg. inversion of a matrix, by Maple
- Create an bare amplitude
  $\gamma$-trace calculation by FORM
- Create UV and IR quantities
  symbolic manipulation by FORM
- Control the whole process by Perl program

All 389 integrands had been constructed by 2006.
Numerical integration

Very difficult numerical integration

- Round-off problem
  - IR cancellation causes the problem.

Need very fast quadruple precision arithmetics

A clue is the library of double-double and quad-double arithmetic.


- Sharp-peak problem
  - An integrand has very sharp peaks at the surface volume of a 13 dimensional hypercube.

Need robust algorithm of multi-dimensional integration

A clue is VEGAS algorithm.

Monte-Carlo integration, error decreases slowly by \( 1/\sqrt{N} \)

Huge computational resources are required.

G. P. Lepage 1978
HOKUSAI GW & BW at RIKEN

1 PFLOPS Fujitsu PRIMEHPC FX100 (34560 cores)  
April 2015-
Cutting edge supercomputer  
Compatibility with the K computer  
Availability for highly parallelized programs

2.58 PFLOPS IA Cluster of Xeon Gold 6148 (33600 cores)  
October 2017-
Raising HPC environment of RIKEN  
Popular architecture  
High versatility

Makiko Nio (RIKEN)
Cross-check for integrals of Set V

- Reshuffle integration variables of the 389 integrals.
- 2017 calculation is therefore independent from 2015 calculation.
- Numerical results with different mappings are in good agreement.

Integrals showing relatively large discrepancies:

<table>
<thead>
<tr>
<th>integral</th>
<th>2017 result</th>
<th>2015 result</th>
<th>difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>X100</td>
<td>$-15.232(17)$</td>
<td>$-15.292(33)$</td>
<td>0.060</td>
</tr>
<tr>
<td>X141</td>
<td>$-12.496(17)$</td>
<td>$-12.557(35)$</td>
<td>0.060</td>
</tr>
<tr>
<td>X113</td>
<td>$-4.443(17)$</td>
<td>$-4.385(32)$</td>
<td>$-0.058$</td>
</tr>
<tr>
<td>X325</td>
<td>$11.539(17)$</td>
<td>$11.596(34)$</td>
<td>$-0.056$</td>
</tr>
<tr>
<td>X146</td>
<td>$-2.246(17)$</td>
<td>$-2.299(34)$</td>
<td>0.053</td>
</tr>
<tr>
<td>X236</td>
<td>$2.107(21)$</td>
<td>$2.056(18)$</td>
<td>0.051</td>
</tr>
<tr>
<td>X153</td>
<td>$14.845(17)$</td>
<td>$14.894(34)$</td>
<td>$-0.048$</td>
</tr>
<tr>
<td>X251</td>
<td>$-1.343(20)$</td>
<td>$-1.391(08)$</td>
<td>0.047</td>
</tr>
<tr>
<td>X044</td>
<td>$4.365(16)$</td>
<td>$4.412(28)$</td>
<td>$-0.047$</td>
</tr>
<tr>
<td>X144</td>
<td>$23.677(17)$</td>
<td>$23.724(37)$</td>
<td>$-0.047$</td>
</tr>
<tr>
<td>X252</td>
<td>$-10.865(17)$</td>
<td>$-10.909(34)$</td>
<td>0.044</td>
</tr>
<tr>
<td>X256</td>
<td>$-13.996(17)$</td>
<td>$-14.041(34)$</td>
<td>0.044</td>
</tr>
</tbody>
</table>
Independent calculation of Set V

Our 2017 calculation is continued. S. Volkov announced his preliminary result at his ACAT2019 Day 2 talk.

Latest results of Set V

\[ A_1^{(10)} [\text{Set V}] = \begin{cases} 7.668 \text{ (159)} & \text{AKN2019} \\ 6.782 \text{ (113)} & \text{S. Volkov, ACAT 2019} \end{cases} \]

- Both rely on numerical means.
- Approaches to Feynman diagrams are different. So, independent.
- Difference $-0.89 \text{ (20)}$ is about $4.5 \sigma$.
- Not seriously affect the current precision of lepton $g - 2$.
- But becomes serious for the future electron experiments.
New value of $A_1^{(10)}$

New and massive evaluation of Set V leads to

the mass-independent 10th-order $A_1^{(10)}$

$$A_1^{(10)} = 6.675 (192) \rightarrow 6.737 (159) \text{ at present}$$

If Volkov’s new result for Set V is used,

the mass-independent 10th-order $A_1^{(10)}$

$$A_1^{(10)} = 5.851 (113)$$

### QED mass-dependent terms for $a_e$

**Input parameters:**

\[
\begin{align*}
m_e/m_\mu &= 0.483\,633\,170\ (11) \times 10^{-2} & \text{CODATA2014} \\
m_e/m_\tau &= 0.287\,585\ (19) \times 10^{-3} & \text{PDG2018}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2^{(4)}(m_e/m_\mu)$</td>
<td>$0.519,738,676\ (24) \times 10^{-6}$</td>
<td>Elend1966</td>
</tr>
<tr>
<td>$A_2^{(4)}(m_e/m_\tau)$</td>
<td>$0.183,790\ (25) \times 10^{-8}$</td>
<td>Elend1966</td>
</tr>
<tr>
<td>$A_3^{(4)}(m_e/m_\mu, m_e/m_\tau)$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$A_2^{(6)}(m_e/m_\mu)$</td>
<td>$-0.737,394,164\ (24) \times 10^{-5}$</td>
<td>Samuel&amp;Li1991, Li, Mendel&amp;Samuel1993, Laporta1993, Laporta&amp;Remiddi1993</td>
</tr>
<tr>
<td>$A_2^{(6)}(m_e/m_\tau)$</td>
<td>$-0.658,273\ (79) \times 10^{-7}$</td>
<td>Samuel&amp;Li1991, Li, Mendel&amp;Samuel1993, Laporta1993, Laporta&amp;Remiddi1993</td>
</tr>
<tr>
<td>$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)$</td>
<td>$0.1909\ (1) \times 10^{-12}$</td>
<td>Passera2006</td>
</tr>
<tr>
<td>$A_2^{(8)}(m_e/m_\mu)$</td>
<td>$0.916,197,070\ (37) \times 10^{-3}$</td>
<td>Kurz et al.2013, AHKN2012</td>
</tr>
<tr>
<td>$A_2^{(8)}(m_e/m_\tau)$</td>
<td>$0.742,92\ (12) \times 10^{-5}$</td>
<td>Kurz et al.2013, AHKN2012</td>
</tr>
<tr>
<td>$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$</td>
<td>$0.746,87\ (28) \times 10^{-6}$</td>
<td>Kurz et al.2013, AHKN2012</td>
</tr>
<tr>
<td>$A_2^{(10)}(m_e/m_\mu)$</td>
<td>$-0.003,82\ (39)$</td>
<td>AHKN2012, 2015</td>
</tr>
<tr>
<td>$A_2^{(10)}(m_e/m_\tau)$</td>
<td>$\mathcal{O}(10^{-5})$</td>
<td></td>
</tr>
<tr>
<td>$A_3^{(10)}(m_e/m_\mu, m_e/m_\tau)$</td>
<td>$\mathcal{O}(10^{-5})$</td>
<td></td>
</tr>
</tbody>
</table>
The fine-structure constant $\alpha$

To obtain the theory prediction of $a_e$, we need the non-QED value of the fine-structure constant $\alpha$.

Two values of $\alpha$ from $h/m(X)$ measurements of the atom interferometer:

$$\alpha^{-1}(\text{Rb}) = 137.035\,998\,995\,(85) \quad \text{R. Bouchendira et al. 2011}$$
$$\alpha^{-1}(\text{Cs}) = 137.035\,999\,046\,(27) \quad \text{R. H. Parker et al., 2018}$$

through the relation

$$\alpha^{-1}(X) = \left[ \frac{2R_{\infty}}{c} \times \frac{A_r(X)}{A_r(e)} \times \frac{h}{m(X)} \right]^{-1/2}.$$ 

$R_{\infty}$ ... the Rydberg constant
$A_r(X)$ ... relative atomic mass of a particle $X$
h ... Planck constant
c ... speed of light
$m(X)$ ... mass of an atom $X$
Electron $g-2$, theory v.s. experiment

Hadronic and weak contributions to $a_e$ are F. Jegerlehner, arXiv:1705.00263

$$a_e(\text{hadron}) = 1.693 \pm 0.12 \times 10^{-12}, \quad a_e(\text{weak}) = 0.030 \pm 0.023 \times 10^{-12}.$$  

With $\alpha(\text{Rb})$ or $\alpha(\text{Cs})$, the SM prediction of $a_e$ is

$$a_e(\text{theory} : \alpha(\text{Rb})) = 1.159652182.037 \pm 0.72 \times 10^{-12}$$
$$a_e(\text{theory} : \alpha(\text{Cs})) = 1.159652181.606 \pm 0.23 \times 10^{-12}$$

QED 10th, hadron, $\alpha(X)$

The Harvard measurement of $a_e$:


$$a_e(\text{expt.}) = 1.159652180.72 \pm 0.28 \times 10^{-12}$$

Difference between measurement and theory:

$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Rb})) = (-1.31 \pm 0.77) \times 10^{-12} \quad 1.7\sigma$$
$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Cs})) = (-0.88 \pm 0.36) \times 10^{-12} \quad 2.4\sigma$$
Electron $g-2$, theory v.s. experiment w/ Volkov’s result

Volkov’s new $A_1^{(10)}$[Set V] makes the theoretical prediction smaller by

$$0.886 \left( \frac{\alpha}{\pi} \right)^5 = 0.0599 \times 10^{-12}$$

and reduce the uncertainty due to the tenth-order QED

$$0.011 \times 10^{-12} \rightarrow 0.0076 \times 10^{-12}$$

Comparison of measurement and theory becomes

$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Rb})) = (-1.25 \pm 0.77) \times 10^{-12} \quad 1.6\sigma$$

$$a_e(\text{expt.}) - a_e(\text{theory} : \alpha(\text{Cs})) = (-0.82 \pm 0.36) \times 10^{-12} \quad 2.3\sigma$$

New expts on $a_e$ and $\alpha(\text{Cs})$ bring comparison to the level of $0.03 \times 10^{-12}$. Discrepancy in the tenth-order QED $A_1^{(10)}$ must be resolved.
Search of new physics through $\alpha$

One more value of $\alpha$ is obtained from the electron $g - 2$. Solve $\alpha(a_e)$ from $a_e(\text{expt.}) = a_e(\text{theory})$:

$$\alpha^{-1}(a_e) = 137.035\ 999\ 1496\ (13)(14)(330)$$

QED 10th, hadron, expt.

AKN2018, 2019

Difference between two other determinations of $\alpha$:

$$\alpha^{-1}(a_e) - \alpha^{-1}(\text{Rb}) = (0.155 \pm 0.091) \times 10^{-6}$$

1.7$\sigma$

$$\alpha^{-1}(a_e) - \alpha^{-1}(\text{Cs}) = (0.104 \pm 0.043) \times 10^{-6}$$

2.4$\sigma$

$$\alpha^{-1}(\text{Rb}) - \alpha^{-1}(\text{Cs}) = (-0.051 \pm 0.089) \times 10^{-6}$$

0.6$\sigma$

New physics or misinterpretation of the SM physics? Misinterpretation in experiment or theory?
α\( (a_e) \) and α\( (Rb) \) are used to determine exact values of some fundamental constants.

In the new SI, the Planck constant \( h \), the elementary charge \( e \), the Boltzmann constant \( k \), and the Avogadro number \( N_A \) become defined numbers like the speed of light \( c \):

\[
\begin{align*}
    h &= 6.626 \, 070 \, 15 \times 10^{-34} \text{ Js}, \\
    e &= 1.602 \, 176 \, 634 \times 10^{-19} \text{ C}, \\
    k &= 1.380 \, 649 \times 10^{-23} \text{ JK}^{-1}, \\
    N_A &= 6.022 \, 140 \, 76 \times 10^{23} \text{ mol}^{-1}.
\end{align*}
\]

Definition of kilogram is based on the Planck constant \( h \) after the new SI launches in 2019.

Good bye, the International Prototype Kilogram.

Mass-dependent terms for $a_\mu$

For higher-order terms, the electron-loop contributions are dominant.

<table>
<thead>
<tr>
<th>Term</th>
<th>Value (76)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(4)}<em>2 (m</em>\mu/m_e)$</td>
<td>1.094 258 3093 (76)</td>
<td>Elend1966</td>
</tr>
<tr>
<td>$A^{(4)}<em>2 (m</em>\mu/m_\tau)$</td>
<td>0.000 078 076 (11)</td>
<td>Elend1966</td>
</tr>
<tr>
<td>$A^{(4)}<em>3 (m</em>\mu/m_e, m_\mu/m_\tau)$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Term</th>
<th>Value (20)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(6)}<em>2 (m</em>\mu/m_\tau)$</td>
<td>0.000 360 671 (94)</td>
<td>Samuel&amp;Li1990,1992,1993 Laporta1993,Laporta&amp;Remiddi1993</td>
</tr>
<tr>
<td>$A^{(6)}<em>3 (m</em>\mu/m_e, m_\mu/m_\tau)$</td>
<td>0.000 527 738 (75)</td>
<td>Laporta1993,Laporta&amp;Remiddi1993</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Term</th>
<th>Value (60)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(8)}<em>2 (m</em>\mu/m_e)$</td>
<td>132.6852 (60)</td>
<td>AHKN2012, Kurz et al.2016a</td>
</tr>
<tr>
<td>$A^{(8)}<em>2 (m</em>\mu/m_\tau)$</td>
<td>0.042 4941 (53)</td>
<td>AHKN2012, Kurz et al.2016b</td>
</tr>
<tr>
<td>$A^{(8)}<em>3 (m</em>\mu/m_e, m_\mu/m_\tau)$</td>
<td>0.062 722 (10)</td>
<td>AHKN2012, Kurz et al.2016a</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Term</th>
<th>Value (86)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(10)}<em>2 (m</em>\mu/m_e)$</td>
<td>742.32 (86)</td>
<td>AHKN2012</td>
</tr>
<tr>
<td>$A^{(10)}<em>2 (m</em>\mu/m_\tau)$</td>
<td>$-0.0656$ (45)</td>
<td>AHKN2012</td>
</tr>
<tr>
<td>$A^{(10)}<em>3 (m</em>\mu/m_e, m_\mu/m_\tau)$</td>
<td>2.011 (10)</td>
<td>AHKN2012</td>
</tr>
</tbody>
</table>
The large enhancement factor comes from the light-by-light diagrams:

\[
\frac{2}{3} \pi^2 \ln(m_\mu/m_e) \sim 35
\]

A factor \( \pi \) comes from the integration over a Coulomb loop.

Insertion of the 2nd-order VP adds another log factor:

\[
\frac{2}{3} \ln(m_\mu/m_e) - \frac{5}{9} \sim 3
\]

T. Kinoshita started developing his numerical calculation method of QED to investigate this 6th-order light-by-light diagrams for \( a_\mu \).

QED 8th-order mass-dependent term $A_2^{(8)}$

Mass-dependence comes from diagrams w/ fermion loops:

- I(a), I(b), II(a), II(b), II(c)
  2nd and/or 4th-order VP functions
- I(c) Checked w/ the 1-dimensional integral of the VP function
- I(d) Checked w/ the Padé approximated VP function
- IV(b), IV(c), IV(d)
  Checked w/ the asymptotic expansion

Analytic calculations of $A_2^{(8)}$

Analytic calculation of $A_2$ and $A_3$ are a little easier than $A_1$. But still tough for the higher orders.

- Expansion parameters: $m_e/m_\mu = 1/206.7 \cdots$, $m_\mu/m_\tau = 1/16.8 \cdots$
- Different analytic structures: $m_{\text{loop}} \gg m_{\text{external}}$ or $m_{\text{loop}} \ll m_{\text{external}}$

Analytic expansions of the 8th order were obtained in 2014 and in 2016:

- $m_{\text{loop}} \ll m_{\text{external}}$ A. Kurz et al. PRD93, 053017(2016)
- $m_{\text{loop}} \gg m_{\text{external}}$ A. Kurz et al. PoS LL2014, 051(2014)

- Diagrams involving a light-by-light loop are very difficult to evaluate.
- The analytic expansion results are less accurate than the numerical results.
- Both are in good agreement.
QED 10th-order vertex diagrams

12,672 Feynman vertex diagrams divided into 32 subsets:
- 6,354 vertex diagrams w/o a fermion loop, Set V. difficult
- 6,318 diagrams w/ closed fermion loops, Set I-IV, IV. easier
QED 10th-order mass-dependent term $A_2^{(10)}$

All diagrams have been numerically calculated.

Many consistency checks:

- LO contribution can be estimated with help of renormalization group. **AHKN (2012)**
  
- Non-relativistic calculation is also useful for VI(k), the light-by-light diagram to which six photons are attached. **A. Kataev 1992, 1995, 2006**

- Padé approximated 4-loop vacuum-polarization function. **S. Karshenboim, 1993**

- Asymptotic expansion of the diagrams w/ 2nd- and/or 4th-order VP functions. **P. A. Baikov, A. Maier, and P. Marquard, 2013**

- S. Laporta, 1994
QED contributions to $a_\mu$, muon $g-2$

With $\alpha$(Cs) and $\alpha(a_e)$ the QED contribution to $a_\mu$ is

\[
\begin{align*}
    a_\mu(\text{QED : } \alpha(\text{Cs})) &= 1 165 847 189.31 \times 10^{-12} \\
    a_\mu(\text{QED : } \alpha(a_e)) &= 1 165 847 188.42 \times 10^{-12}
\end{align*}
\]

lepton-mass ratios, QED 8th, QED 10th, $\alpha$, combined

All are sufficiently accurate for the current and future measurements of $a_\mu$. Further numerical improvement on QED 8th and 10th is possible. Targets are diagrams involving a light-by-light scattering subdiagram. How about the 12th-order contribution?

The 6th-order light-by-light + three 2nd-order VP insertions:

\[
\left(\frac{\alpha}{\pi}\right)^6 \times \underbrace{10}_{\text{ways of insertion}} \times \underbrace{3^3}_{3 \text{ VP}} \times \underbrace{20}_{LbyL} \sim 0.8 \times 10^{-12}.
\]

The size of the whole 12th-order contribution might be $O(10^{-12})$.

Need a crude estimate of the 12th-order contribution.
QED $g-2$ up to the 8th-order contribution has been firmly established in last 3 years.

QED $g-2$ of the 10th order has been extensively calculated and checked.

QED $g-2$ is ready for the on-going new experiments of electron-positron $g-2$ and of muon $g-2$.

QED $g-2$ is served for the new SI. After the new SI launches, the fine-structure constant $\alpha$ is the unique source of uncertainties of other fundamental physical constants.

QED $g-2$ shows that we are able to compute many and complex Feynman diagrams using analytic/numerical methods with help of powerful computers.