Towards an efficient method to compute two-loop scalar amplitudes

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Abstract. We report on an ongoing work initiated by Prof. Shimizu, proposing a method to numerically compute two-loop scalar integrals as sums of two-dimensional integrals of generalised one-loop N-point functions analytically computed and integrated over some simple weight functions. The analytic computation of the generalised one-loop N-point functions in a systematic way motivates a novel approach sketched in this talk.

1. Introduction

After several years of run, the LHC delivered high quality data which push theorists to improve the accuracy of their predictions up to NNLO. A key ingredient in an automated evaluation of two-loop multileg processes is a fast and numerically stable evaluation of scalar Feynman integrals. The derivation of a fully analytic result remains beyond reach so far in the general mass case. On the opposite side, in particular for the calculation of two-loop three- and four-point functions in the general complex mass case relying on multidimensional numerical integration by means of sector decomposition [1–5] a reliable result has a high computing cost. Approaches based on Mellin-Barnes techniques [6–10] allow to perform part of the integrals analytically, yet, as far as we know, the number of integrals left over for numerical quadratures depends on the topologies considered and can remain rather costly. It would therefore be useful to perform part of the Feynman parameter integrations analytically in a systematic way to reduce the number of numerical quadratures.

2. Structure of a two-loop amplitude

Let us consider an arbitrary two-loop Feynman diagram with topology $\mathcal{T}$ involving N external legs with external momenta $\{p_i, i = 1, \cdots, N\}$ and I internal lines with internal masses $\{m_k^2, k = 1, \cdots, I\}$. To simplify we stick here to a scalar function. After the integration over the two loop momenta is performed, the integral representation of the diagram is given by:

$$ {}^{(2)}I_N^n\left(\{p_j\};\mathcal{T}\right) = \int_{(I\!R^+)^I} \left[\prod_{k=1}^{I} d\tau_k\right] \delta\left(1-\sum_{l=1}^{I}\tau_l\right) \left[\det(A)\right]^{I-\frac{3}{2}n} \left[\mathcal{F}(\{\tau_k\}) - i\,\lambda\right]^{n-I} \tag{1} $$

[4] Y. Shimizu passed away during the completion of this work.

with

$$\mathcal{F}(\{\tau_k\}) = \left\{ \sum_{i,j=1}^{2} \mathrm{Cof}[A]_{ij}\,(r_i \cdot r_j) \right\} - \det(A)\,\mathcal{C} \tag{2}$$

The matrix A, the momenta r_i and the scalar function $\mathcal{C}$ are defined by the form of the integrand before the integration over the loop momenta, namely:

$$[k_1 \quad k_2] \cdot A \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + 2\ [r_1 \quad r_2] \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \mathcal{C}$$

and $\mathrm{Cof}[A]$ is the matrix of cofactors of A. $\mathcal{F}$ is homogeneous of degree 3 in the τ_k's and it depends on the external momenta $\{p_j\}$, the internal masses $\{m_k^2\}$ and the topology $\mathcal{T}$ of the diagram. The parametric representation (1) is the actual starting point of this article.

Let us partition the set of Feynman parameter labels $\{1, \cdots, I\}$ into three subsets S_j and define three auxiliary parameters ρ_j, $j = 1, 2, 3$ accordingly as follows: i) S_1 contains the labels of the internal lines involving only k_1 not k_2, to S_1 is associated $\rho_1 \equiv \sum_{i \in S_1} \tau_i$; ii) S_2 contains the labels of internal lines involving only k_2 not k_1, to S_2 is associated $\rho_2 \equiv \sum_{i \in S_2} \tau_i$; iii) S_3 contains the labels of internal lines common to the two overlapping loops. *Each* of these lines involves the *same* combination[5] $k_1 + k_2$, so that the matrix element A_{12} weighting the scalar product $(k_1 \cdot k_2)$ is equal to the combination $\rho_3 \equiv \sum_{i \in S_3} \tau_i$. The ρ_j's thus fulfil the constrain

$$\rho_1 + \rho_2 + \rho_3 = \sum_{j=1}^{I} \tau_j = 1 \tag{3}$$

The elements of the matrix A read:

$$A_{12} = \rho_3, \quad A_{11} = \rho_1 + \rho_3, \quad \text{and} \quad A_{22} = \rho_2 + \rho_3 \tag{4}$$

Hence:

$$\det(A) = \rho_1\,\rho_2 + \rho_2\,\rho_3 + \rho_3\,\rho_1 \tag{5}$$

The determinant $\det(A)$ is clearly non negative. Let $|S_j|$ be the number of elements of S_j, with $|S_1| + |S_2| + |S_3| = I$. Let us introduce $|S_j|$ parameters u_{k_j} with $k_j \in S_j$ so as to reparametrise the τ_{k_j} summing up into ρ_j as follows:

$$\tau_{k_j} = \rho_j\,u_{k_j} \quad \text{with the constraint} \quad \sum_{k_j \in S_j} u_{k_j} = 1 \tag{6}$$

Accordingly the reparametrised integration measure takes the following factorised form:

$$\begin{aligned} &\left[\prod_{j=1}^{I} d\tau_j\right] \delta\left(1 - \sum_{i=1}^{I} \tau_i\right) \\ &= \prod_{k=1}^{3} \left\{ d\rho_k \rho_k^{|S_k|-1} \prod_{j_k \in S_k} \left[du_{j_k}\, \delta\left(1 - \sum_{l \in S_k} u_l\right)\right]\right\} \delta\left(1 - \sum_{i=1}^{3} \rho_i\right) \end{aligned} \tag{7}$$

[5] It could alternatively involve $k_1 - k_2$ *in every internal line common to the two overlapping loops*, depending on the convention adopted for the orientations of the loop momenta.

With this reparametrisation, the elements of the A matrix depend only on the parameters ρ_j and on none of the u_i's, so do Cof$[A]$ and $\det(A)$. In $\mathcal{F}$, the dependence in the u_i's enters through the factors $(r_i \cdot r_j)$, quadratically, and through the term $\mathcal{C}$, linearly. The term $\mathcal{F}$ may thus be seen as a polynomial of second degree in the u_i's and can thus be interpreted as building up the integrand of a "generalised" one-loop function represented as a Feynman integral over the u_i's. The integral representation of the two-loop diagram ${}^{(2)}I_N^n(\{p_j\};\mathcal{T})$ can thus be recast in the following form:

$$
{}^{(2)}I_N^n(\{p_j\};\mathcal{T}) = \int_{(I\!R^+)^3} \left[\prod_{k=1}^{3} d\rho_k \, \rho_k^{|S_k|-1} \right] \delta\left(1 - \sum_{l=1}^{3} \rho_l\right) \left[\rho_1\,\rho_2 + \rho_2\,\rho_3 + \rho_3\,\rho_1\right]^{I-\frac{3\,n}{2}} \; {}^{(1)}\widetilde{I}_{N'}^{n'} \tag{8}
$$

where we have introduced

$$
{}^{(1)}\widetilde{I}_{N'}^{n'} = \int_{(I\!R^+)^I} \prod_{k=1}^{3} \prod_{j \in S_k} du_j \, \delta\left(1 - \sum_{l \in S_k} u_j\right) \left[\overline{\mathcal{F}}(\{u_k\},\{\rho_l\}) - i\,\lambda\right]^{n-I} \tag{9}
$$

with $\overline{\mathcal{F}}(\{u_k\},\{\rho_l\}) = \mathcal{F}(\{\tau_i(\{u_k\},\{\rho_l\})\})$ and we have set $N' = I - 2$ and $n' = 2\,(n-2)$. The reparametrisation of ${}^{(2)}I_N^n(\{p_j\};\mathcal{T})$ according to eqs. (8), (9) has already been used in the literature [11–14] in order to perform the integration over all Feynman parameters fully numerically. We alternatively wish to advocate here the separate identification of ${}^{(1)}\widetilde{I}_{N'}^{n'}$ in eq. (9) with $n - I = -N' + n'/2$ as a N'-point function of "generalised one-loop type" in n' dimensions, and the possibility to compute ${}^{(1)}\widetilde{I}_{N'}^{n'}$ *analytically.*

The above qualificative "generalised one-loop type" refers to two kinds of generalisations.
1) After integrating over three of the u_i's in order to eliminate the $\delta(1-\sum_{l\in S_k} u_j)$-constraints, the effective kinematics of the "generalised" one-loop N'-point function in n' dimensions is encoded in a $(I-3)\times(I-3)$ matrix $G = G(\{p_j\},\{\rho_l\})$, a column $(I-3)$-vector $V = V(\{p_j\},\{\rho_l\})$ and a scalar function $C = (\{p_j\},\{\rho_l\})$, all of which functions of the external momenta $\{p_j\}$ and of the integration variables $\{\rho_k\}$ seen as external parameters. Let us note that this effective kinematics of the "generalised" one-loop function depends on the ρ_j seen as "external" parameters beside the external momenta p_k's, and that it may span a larger parameter space than the one involved in standard one-loop N'-point functions involved in collider processes at one loop.
2) Unlike for the standard one-loop function, the integration domain of the parameters u_k's is not the usual $(I-3)$-simplex defined by $\Sigma_{(I-3)} = \{u_k \geq 0, k = 1,\cdots, I-3 | \sum_{k=1}^{I-3} u_k = 1\}$ but instead the polysimplicial set[6] $\Sigma_{(|S_1|-1)} \times \Sigma_{(|S_2|-1)} \times \Sigma_{(|S_3|-1)}$; The quantity $\overline{\mathcal{F}}$ formally reads:

$$
\overline{\mathcal{F}} = U^T \cdot G \cdot U - 2\,V^T \cdot U - C \tag{10}
$$

where U is the column $(I-3)$-vector gathering the yet unintegrated $(I-3)$ variables u_k parametrising the polysimplicial integration domain $\Sigma_{(|S_1|-1)} \times \Sigma_{(|S_2|-1)} \times \Sigma_{(|S_3|-1)}$.

3. "Generalised" one-loop functions

Although long-tested standard techniques developed for the genuine one-loop case [15,16] might be customised to treat the new ones at hand, the two issues mentioned in the previous section motivate the development of a novel approach which tackles both these issues in a systematic

[6] The polysimplicial set depends on the topology $\mathcal{T}$ of the two-loop diagram considered. It is understood that, in case some of the $|S_j|$ equals 1, the corresponding trivial set factor $\Sigma_{(|S_j|-1)}$ shall be omitted.

and straightforward way while computing the"generalised 1-loop type functions". This novel method is described in details in ref. [17] for the real mass case and has been extended for complex masses in ref. [18] and for infrared divergent cases in ref. [19].
The key point of this novel method is an extensive use of a "Stokes-type" identity:

$$\frac{1}{D^{\alpha+1}(\{u_k\})} = \frac{1}{2\,\alpha\,\Delta_d}\left[\frac{d-2\,\alpha}{D^{\alpha}(\{u_k\})} - \nabla_u^T \cdot \left(\frac{U - G^{-1}.V}{D^{\alpha}(\{u_k\})}\right)\right] \tag{11}$$

where $D(\{u_k\})$ is a second order polynomial in the d variables u_k which has the structure defined in eq. (10) and the quantity Δ_d is given by:

$$\Delta_d = V^T.G^{-1}.V + C$$

The relation (11) is not very useful unless the power α is such that $d - 2\,\alpha = 0$, in this case only the boundary term remains which enables to perform one integration trivially. In general, the power with which the polynomial $D(\{u_k\})$ appears in a one-loop N-point function is not equal to $d/2$. So we need a formula to shift the power of D given by:

$$\int_0^\infty \frac{d\xi}{(D+\xi^\nu)^\mu} = \frac{1}{\nu}\,B\left(\frac{1}{\nu},\mu-\frac{1}{\nu}\right)\frac{1}{D^{\mu-1/\nu}} \tag{12}$$

where $B(x, y)$ is the Euler beta function, provided that $1/\nu$, μ and $\mu - 1/\nu$ are non-negative integers.
As a proof of concept, the three- and four-point one-loop function have been recalculated yet with systematic analytic continuation to arbitrary kinematics i.e. not only those restricted to one-loop processes. As an example, let us sketch the computation of a one-loop four-point function, more details are given in refs. [17–19]. After the integration over the loop momentum, the one-loop four-point function can be written as:

$$^{(1)}I_4^4 = \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^{1-u_1-u_2} du_3 \,\frac{1}{(D(U) - i\,\lambda)^2} \tag{13}$$

with

$$D(U) = U^{\,T}\cdot G\cdot U - 2\,V^T\cdot U - C, \quad U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The only assumption made here is that the imaginary part of the denominator keeps a constant sign when the u_i's span the simplex. The 3×3 matrix G is a Gram matrix[7], V is a 3-vector and C is a scalar; both V and C depend on the internal masses m_i.
After having used three time the "Stokes-type" identity eq. (11), we get rid of the integration over the u_i's and we end up with a sum of integrals over the first octant of $\mathbb{R}^3$ weighted by some coefficients constructed from the Gram matrix, the vector V as well as the reduced Gram matrices and the reduced V vectors:

$$^{(1)}I_4^4 = \sum_{i\in S_4}\;\sum_{j\in S_4\setminus\{i\}}\;\sum_{k\in S_4\setminus\{i,j\}} \frac{\bar{b}_i}{\det(G)}\,\frac{\bar{b}_j^{\{i\}}}{\det(G^{\{i\}})}\,\frac{\bar{b}_k^{\{i,j\}}}{\det(G^{\{i,j\}})} \tag{14}$$

$$\times\, L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \tag{15}$$

[7] Depending on the parametrisation, it exists several Gram matrices but they all have the same determinant.

with

$$
\begin{aligned}
&L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \\
&\quad = \kappa \int_0^{+\infty} \frac{d\xi}{(\xi^2 - \Delta_3 - i\lambda)} \int_0^{+\infty} \frac{d\rho}{(\rho^2 + \xi^2 - \Delta_2^{\{i\}} - i\lambda)} && (16) \\
&\quad \times \int_0^{+\infty} \frac{d\sigma}{(\sigma^2 + \rho^2 + \xi^2 - \Delta_1^{\{i,j\}} - i\lambda)\,(\sigma^2 + \rho^2 + \xi^2 + \tilde{D}_{ijk} - i\lambda)^{1/2}} && (17)
\end{aligned}
$$

where:

$$\Delta_3 = V^T \cdot G^{-1} \cdot V + C \quad \text{and} \quad \bar{b}_i = \left(G^{-1} \cdot V\right)_i \tag{18}$$

The quantities $\Delta_2^{\{i\}}$ (resp. $\Delta_1^{\{i,j\}}$) and the coefficients $\bar{b}_j^{\{i\}}$ (resp. $\bar{b}_k^{\{i,j\}}$) obey to formulae of the same type as eq. (18) but which now involve the reduced Gram matrices $G^{\{i\}}$ and the reduced vectors $V^{\{i\}}$ (resp. the reduced Gram matrices $G^{\{i,j\}}$ and the reduced vectors $V^{\{i,j\}}$). These reduced matrices (resp. vectors) are constructed from the original Gram matrix (resp. vector V) by removing the line(s) and the column(s) (resp. the line(s)) whose label(s) belong(s) to the set defined in the exponent. The integration variables ξ, ρ and σ are the three extra variables introduced to adjust the power of the denominator. One can argue that we have progressed next to nothing but the integrals defined in eq. (17) are simpler concerning the analyticity than the orignal integral over the simplex. The Δ's and the $\tilde{D}$'s are complex numbers in the case of complex masses and the result of the integrations will depend on the sign of the imaginary parts of these latter quantities. There are eight cases to distinguish but they share a common structure. Let us give two examples.

1) case $\mathrm{Im}(\Delta_3) < 0$**,** $\mathrm{Im}(\Delta_2^i) > 0$**,** $\mathrm{Im}(\Delta_1^{ij}) > 0$**,** $\mathrm{Im}(\tilde{D}_{ijk}) < 0$

$$
\begin{aligned}
&L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \\
&= -\Bigg\{ \int_{\widehat{(0,1)}^+} \frac{du}{u^2\, P_{ijk}\, Q_i - R_{ij}\, T} \\
&\qquad \left[-\ln\left(u^2\,(P_{ijk} + R_{ij} + Q_i) + T\right) + \ln\left(T\, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}\, \frac{(R_{ij} + Q_i)}{Q_i}\right)\right] \\
&\quad + \int_0^1 \frac{du}{u^2\, P_{ijk}\, Q_i - R_{ij}\, T} \\
&\qquad \Bigg[\ln\left(u^2\, P_{ijk} + (R_{ij} + Q_i + T)\right) - \ln\left(\frac{(R_{ij} + Q_i)}{Q_i}\,(Q_i + T)\right) \\
&\qquad + \ln\left(u^2\, Q_i + T\right) - \ln\left(T\, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}\right) \\
&\qquad + \eta\left(\frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T)\right) - \eta\left(T\, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{(R_{ij} + Q_i)}{Q_i}\right)\Bigg] \\
&\quad - \int_{\Gamma^+} \frac{du}{u^2\, P_{ijk}\, Q_i - R_{ij}\, T}\, \eta\left(T\, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}\, \frac{R_{ij}}{Q_i}, \frac{R_{ij} + Q_i}{R_{ij}}\right)\Bigg\}
\end{aligned} \tag{19}
$$

where the quantities T, Q_i, R_{ij} and P_{ijk} are linear combinations of Δ_3, $\Delta_2^{\{i\}}$, $\Delta_1^{\{i,j\}}$ and $\tilde{D}_{ijk}$.

2) case $\mathrm{Im}(\Delta_3) > 0$**,** $\mathrm{Im}(\Delta_2^i) > 0$**,** $\mathrm{Im}(\Delta_1^{ij}) > 0$**,** $\mathrm{Im}(\tilde{D}_{ijk}) < 0$

$$
\begin{aligned}
& L_4^4(\Delta_3, \Delta_2^{\{i\}}, \Delta_1^{\{i,j\}}, \tilde{D}_{ijk}) \\
& \quad = - \int_0^1 \frac{du}{u^2\, P_{ijk}\, Q_i - R_{ij}\, T} \\
& \qquad \left[\ln\left(u^2\, P_{ijk} + (R_{ij} + Q_i \;+ T) \right) \; - \; \ln\left(\frac{(R_{ij} + Q_i)}{Q_i} \, (Q_i \;+ T) \right) \right. \\
& \qquad - \ln\left(u^2\, (P_{ijk} + R_{ij} + Q_i) + T \right) \; + \; \ln\left(T\, \frac{(P_{ijk} + R_{ij})}{P_{ijk}} \, \frac{(R_{ij} + Q_i)}{Q_i} \right) \\
& \qquad + \ln\left(u^2\, Q_i + T \right) \qquad\qquad\quad - \; \ln\left(\frac{T}{P_{ijk}} \, (P_{ijk} + R_{ij}) \right) \\
& \qquad \left. + \eta\left(\frac{(R_{ij} + Q_i)}{Q_i}, (Q_i + T) \right) - \eta\left(T\, \frac{(P_{ijk} + R_{ij})}{P_{ijk}}, \; \frac{(R_{ij} + Q_i)}{Q_i} \right) \right]
\end{aligned}
\tag{20}
$$

The structure of eq. (19) and eq. (20) are similar. Note that in the former equation, the contour of integration of the first term is no more along the real axis but has to be deformed in order to prevent of being crossed by the cut of the logarithm $\ln\left(u^2\,(P_{ijk} + R_{ij} + Q_i) + T\right)$. Using Cauchy's theorem, the contour deformation can be written as a part along the imaginary positive axis plus another part along the real axis between 1 and $+\infty$.

4. Conclusion

We sketched a novel method to compute the scalar massive two-loop N-point functions using analytically computed one-loop building blocks. A long way shall still be scouted out to extend it to full-fledged two-loop tensor integrals appearing in general gauge theories.

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