# FROM GRAVITON TO CHAOS

Pavel Stránský

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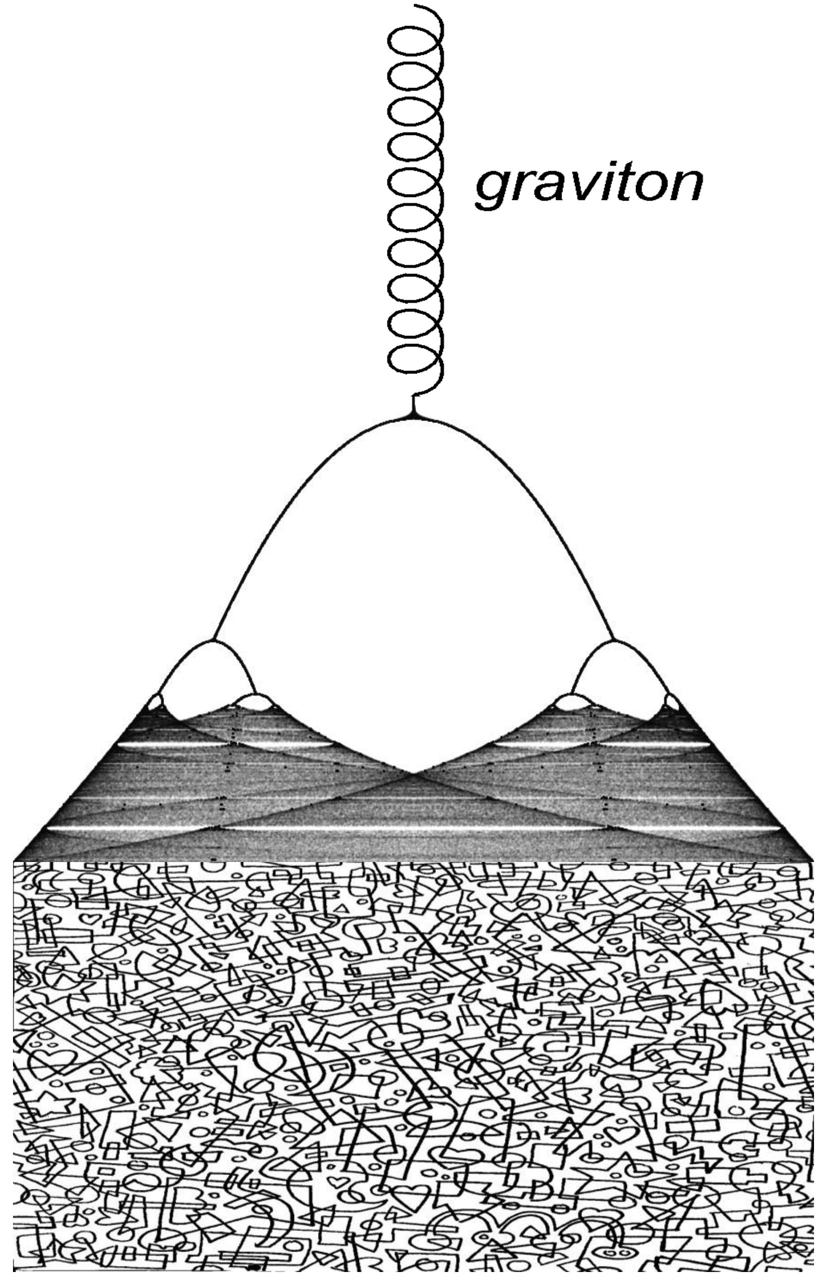
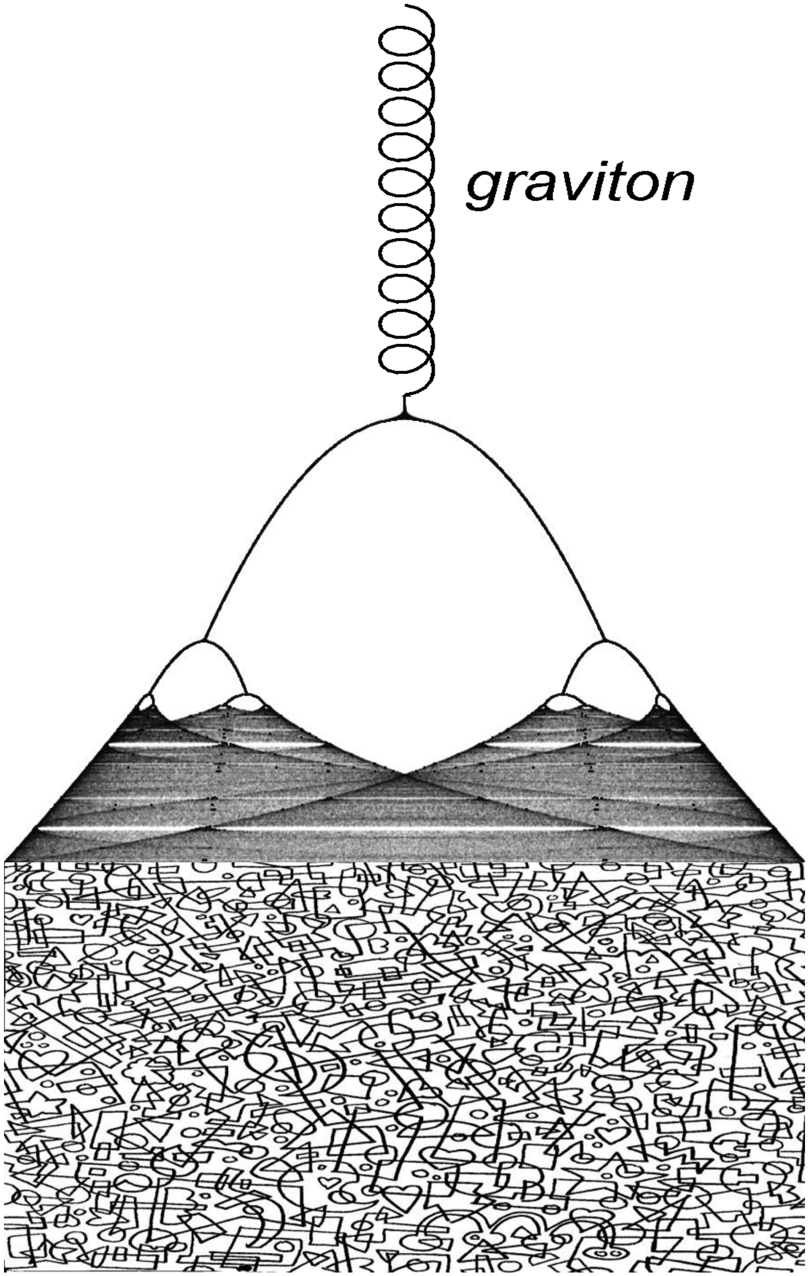



Off-site meeting, IP NP, Little Rock

6th April 2018

... to be gradually
converted into


FROM
CHAOS
TO
KVASNIČÁK



# Outline

Summary

0. Logistic map
1. Celestial mechanics
2. Meteorology
3. Double pendulum
4. Fractals
5. Chaos in curved spaces
6. The invariant set postulate

What will not be mentioned:

- heteroclinic tangles
- quantum chaos
- cellular automata
- Benford law
- time series and $1/f$
- algorithmic complexity

# Summary

## Edward Lorenz (1960)

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

## Robert May (1976)

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realised that simple nonlinear systems do not necessarily possess simple dynamical properties.

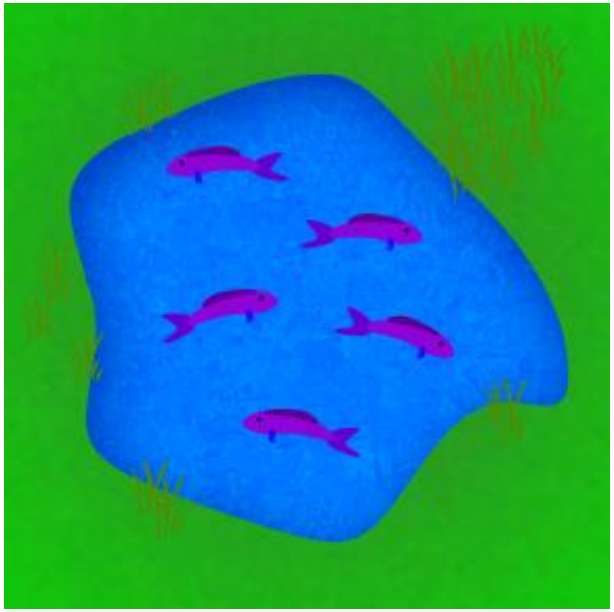
# 0. Logistic map

$$x_{n+1} = r x_n \underline{\qquad\qquad}$$

population (at time *n*+1)

growth rate

dying due to overpopulation

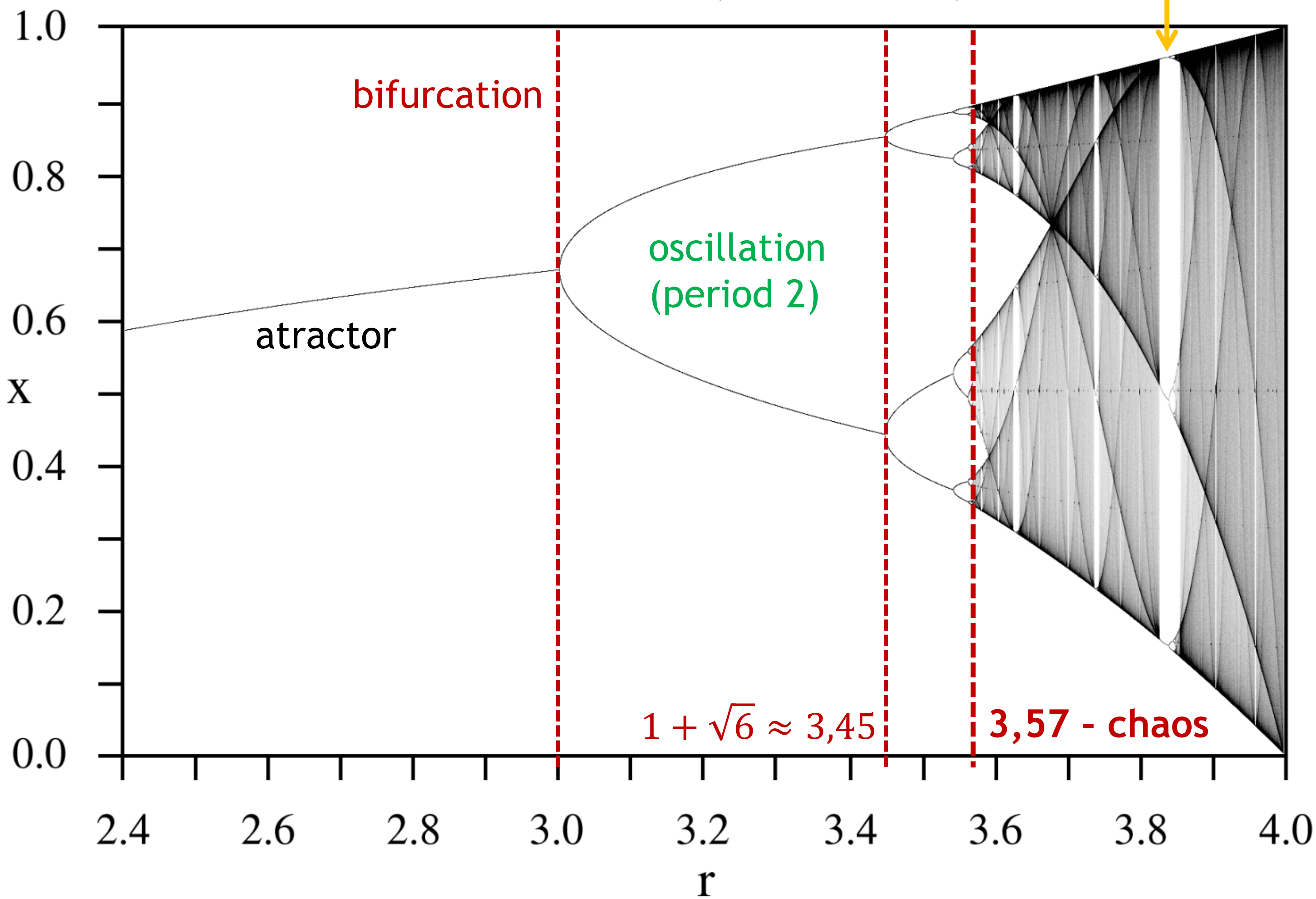


P.F. Verhulst, *Recherches mathématiques sur la loi d'accroissement de la population*, Nouv. mém. de l'Academie Royale des Sci. et Belles-Lettres de Bruxelles **18**, 1 (1845)

# Logistic map

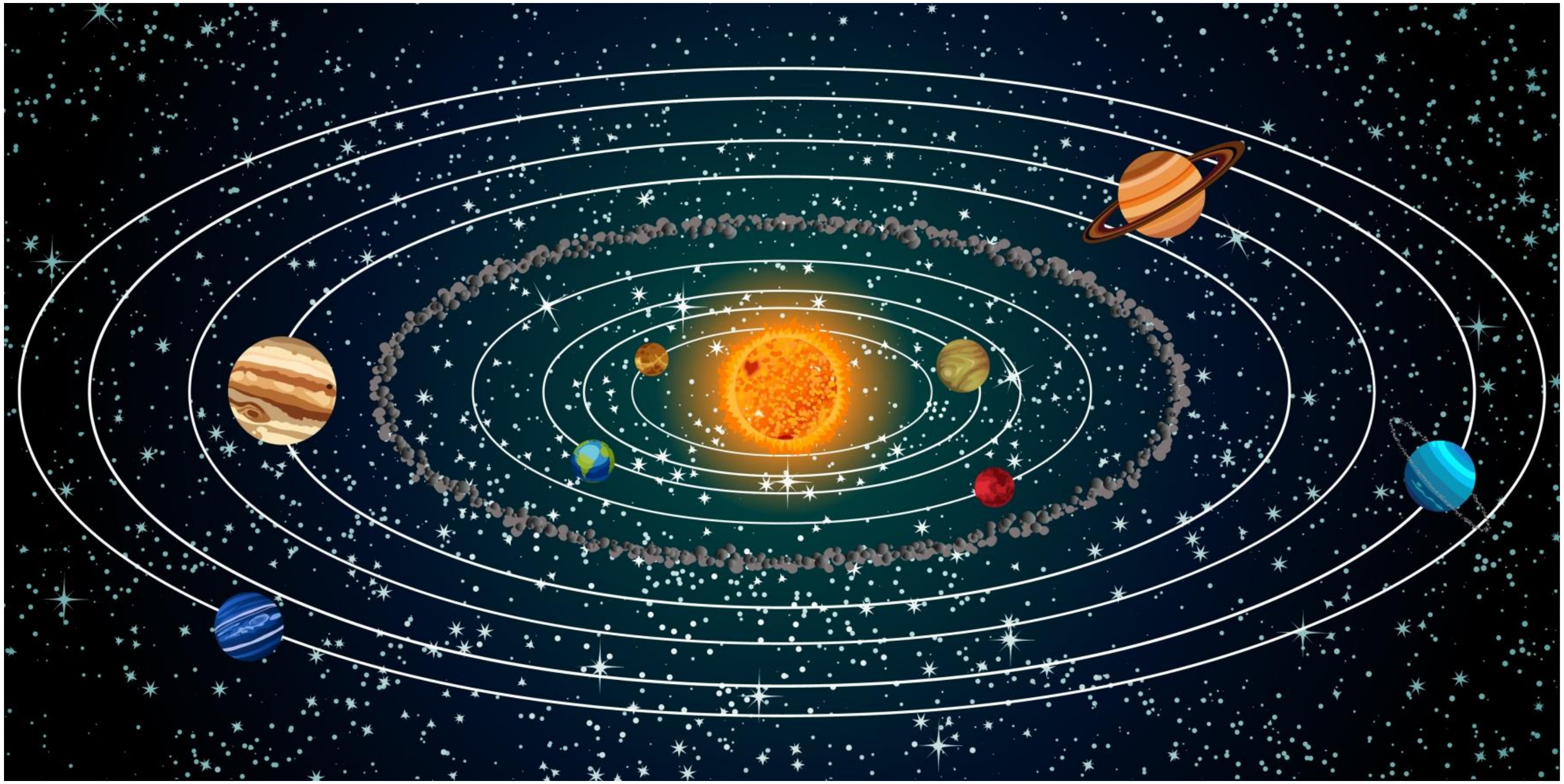
T.Y. Li, J.A. Yorke, *Period three implies chaos*, Amer. Math. Monthly 82, **985,** 1975

$$x_{n+1} = r x_n (1 - x_n)$$



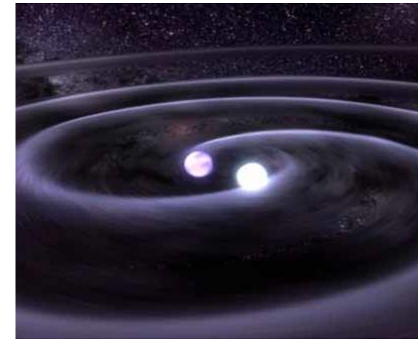

https://www.wolframalpha.com/input/?i=logistic+system

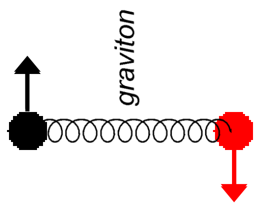
# 1. Celestial mechanics



# Two-body system

(bodies attracted to each other by the gravitation force)





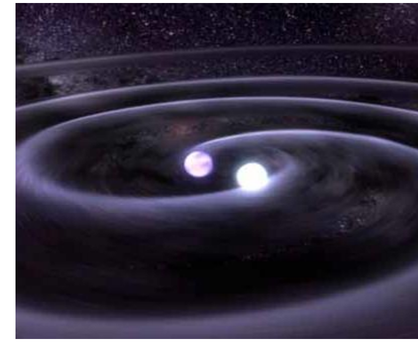

(masses of the bodies are equal)

Simplification:

- the bodies are negligibly small
- the bodies have no internal structure

# Two-body system

(bodies attracted to each other by the gravitation force)



(masses of the bodies are equal)

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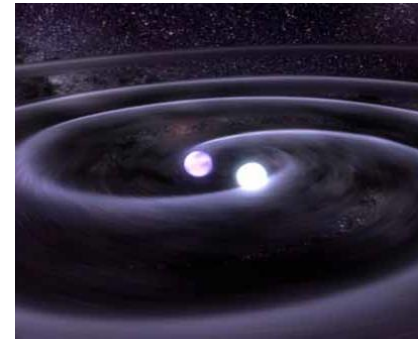
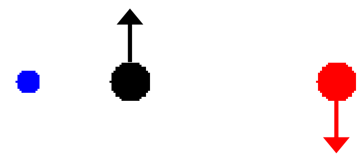
**periodic elliptic** motion (1609 - Johannes Kepler and his laws)

# Two-body system + Third body

+



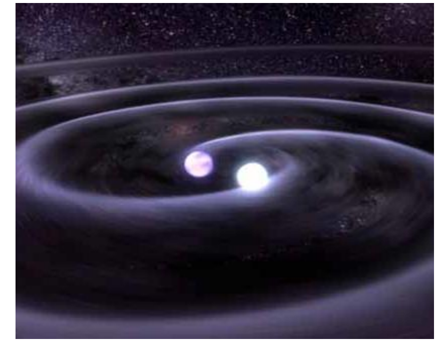


$M = M = 5\, m$

(Simplification: motion restricted to a plane)

# Two-body system + Third body

+



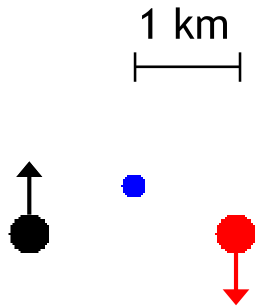
$$M = M = 5\,m$$

(Simplification: motion restricted to a plane)

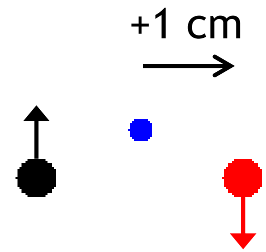
**periodic** stable motion

# Three-body system

## - unstable motion




blue satellite slightly shifted



# Three-body system

- unstable motion

# Poincaré’s Story: The planetary many-body problem

**1887**

- At the occasion of the 60th birthday of Sweedish and Norwegian king Oscar II (to be celebrated in 1889), Sweedish mathematician Gösta Mittag-Leffler announces a scientific competition with the aim of finding a general solution of the many-body celestial system
- Prize for the winner: *gold medal and 2500 golden crowns*

## An unsolved problem for the competition:

Consider a system of arbitrarily many constituents that attracts each other according to the Newton’s law of gravitation. Assuming that the constituents never collide, find coordinates of any of them in the form of a *well-behaved function of time.*

# Poincaré’s Story: The planetary many-body problem

## 1887

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Henri Poincaré
(1854-1912)

## 1888

- **Henri Poincaré** applies his work called *The three-body problem and the equations of dynamics*

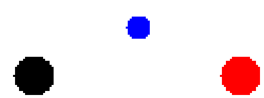
He establishes the first of the three pillars of the modern physics:

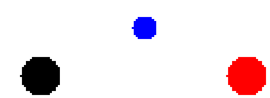
- Chaos theory
- Quantum mechanics
- Theory of relativity

in the solutions of equations of motion, introducing topology to celestial mechanics, and cracking the lid of the Pandora’s box of chaos.

# Three-body system

## - unstable motion



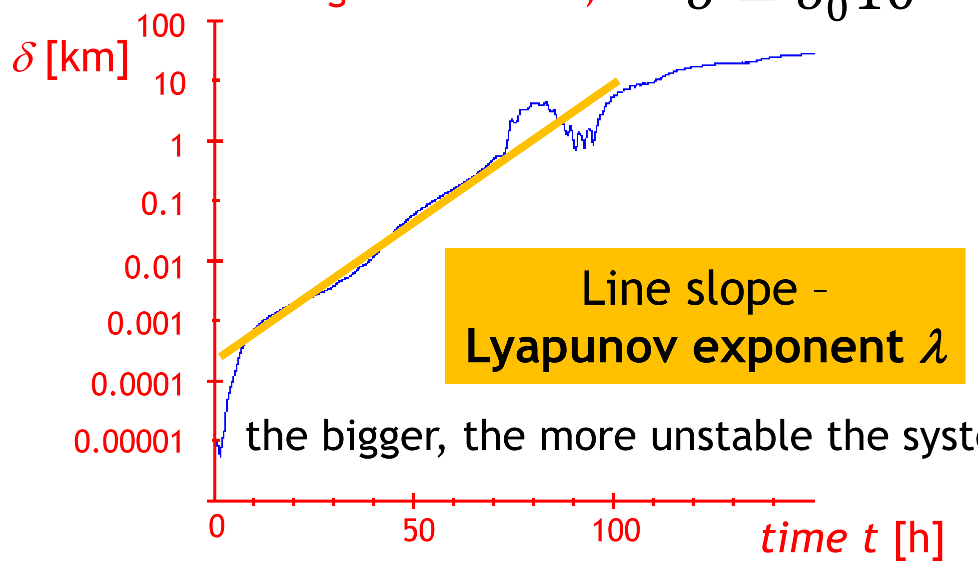


*distance between the blue satellites*

(left simulation – right simulation)

$$\delta = \delta_0 10^{\lambda t}$$

Deviation grows exponentially




Line slope –

**Lyapunov exponent $\lambda$**

the bigger, the more unstable the system

(How many times can you fold a piece of paper?)

**7 times**

Britney Gallivan *85

**12 times**

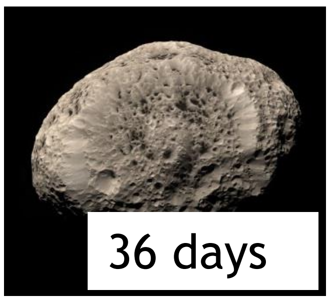
# Lyapunov time

$$\tau = \frac{1}{\lambda}$$

- **estimates** for how long one can predict the future of a system

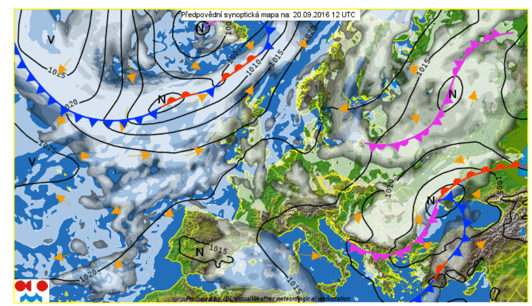
## Examples

Hyperion (one of the Saturn's moons):



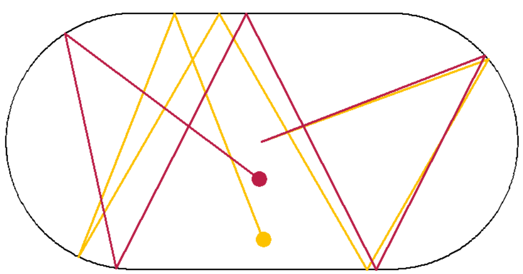

- Rotation axis changes chaotically in time
- Consequence of the resonance with another Saturn's moon Titan

Weather forecast: A couple of hours to days

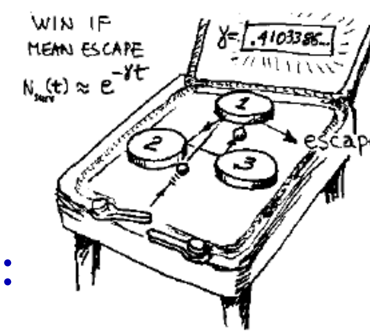


Stadium billiard:

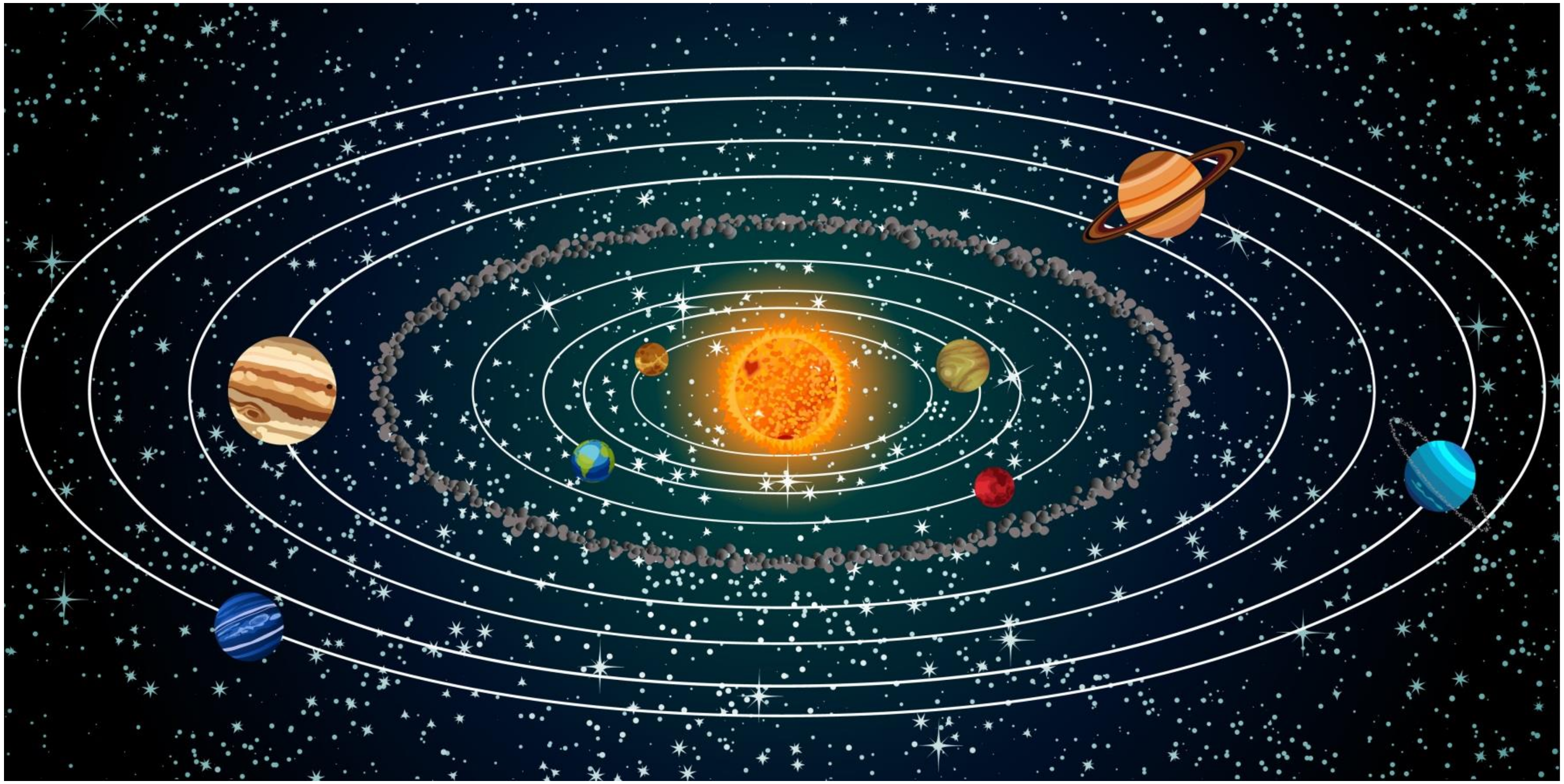
Few seconds (a couple fo bounces)



Pinball game:

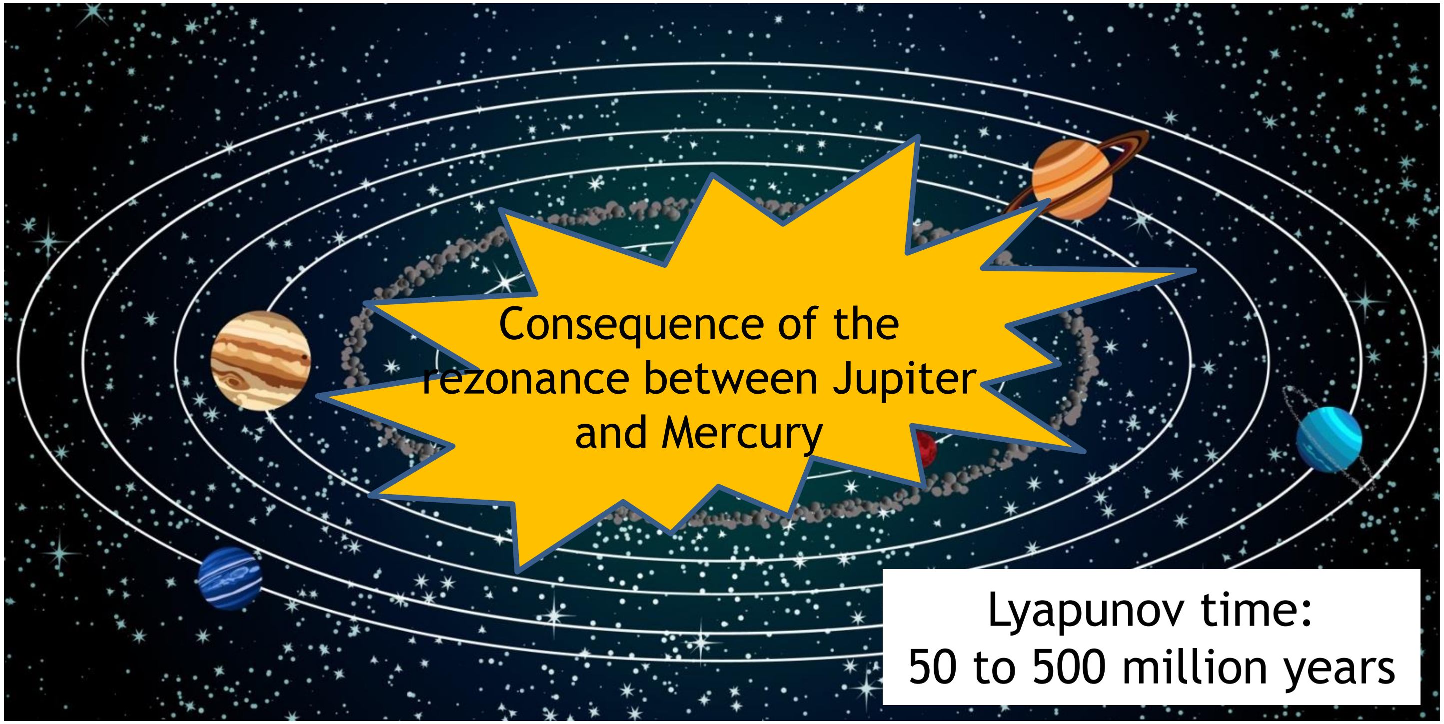


# ???



Is the Solar system stable?

# NO!




J. Laskar a M. Gastineau (2009):

- Calculate very precisely the future of the solar system, starting from 2501 initial conditions differing only in the Mars position (shifted by 0.38mm in each case)
- Obtain **20 collision solutions (i.e. about 1%)** of various types**:**
  - Mercury hits Venus
  - Mercury falls into the Sun
  - Mercury deviates Mars onto a collision trajectory with the Earth


Wayne B. Hayes, *Is the outer Solar System chaotic?* Nature Physics **3,** 689 (2007)
J. Laskar, M. Gastineau, *Existence of collisional trajectories of Mercury, Mars and Venus with the Earth,* Nature **459**, 817 (2009)

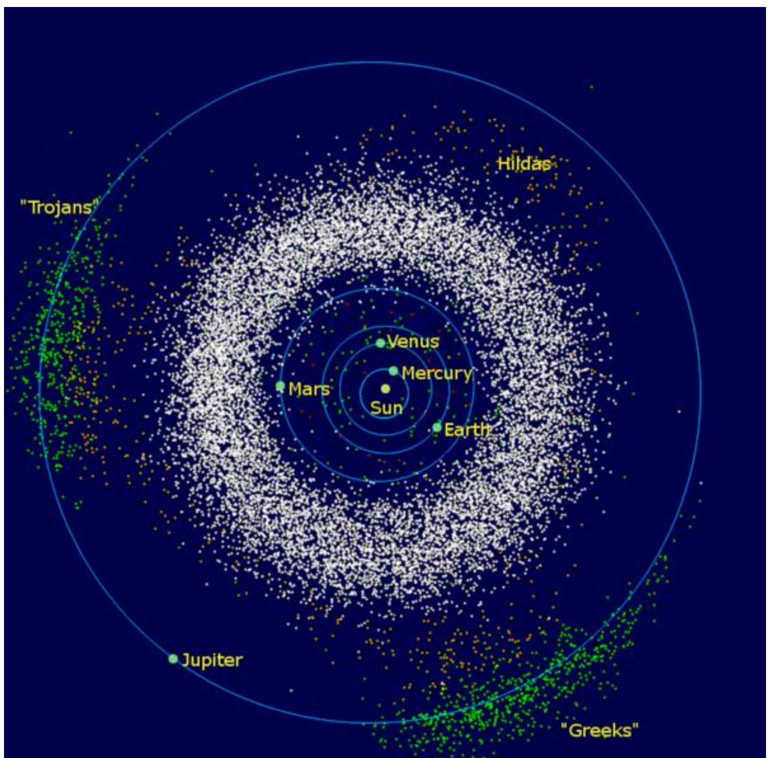
# KAM theorem

## (Andrej Kolmogorov, Vladimir Arnol’d, Jürgen Moser, 1960)

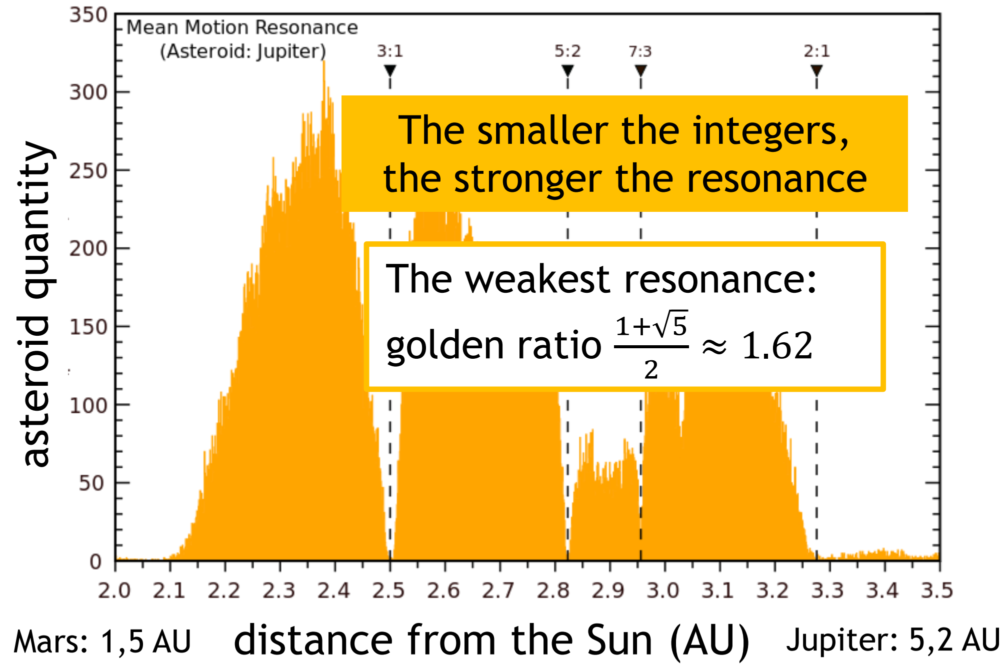
Chaotic behaviour is caused by **resonances** – transfer of energy between the components (degrees of freedom) of the system

### Gaps in the Main asteroid belt

(D. Kirkwood 1874)




- caused by resonances of the asteroids’ orbits with Jupiter



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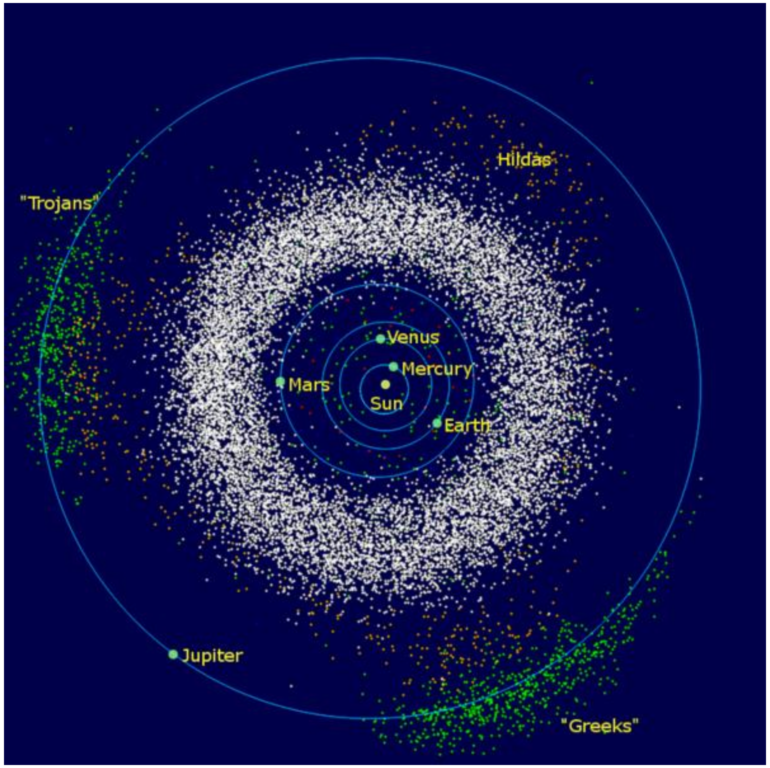
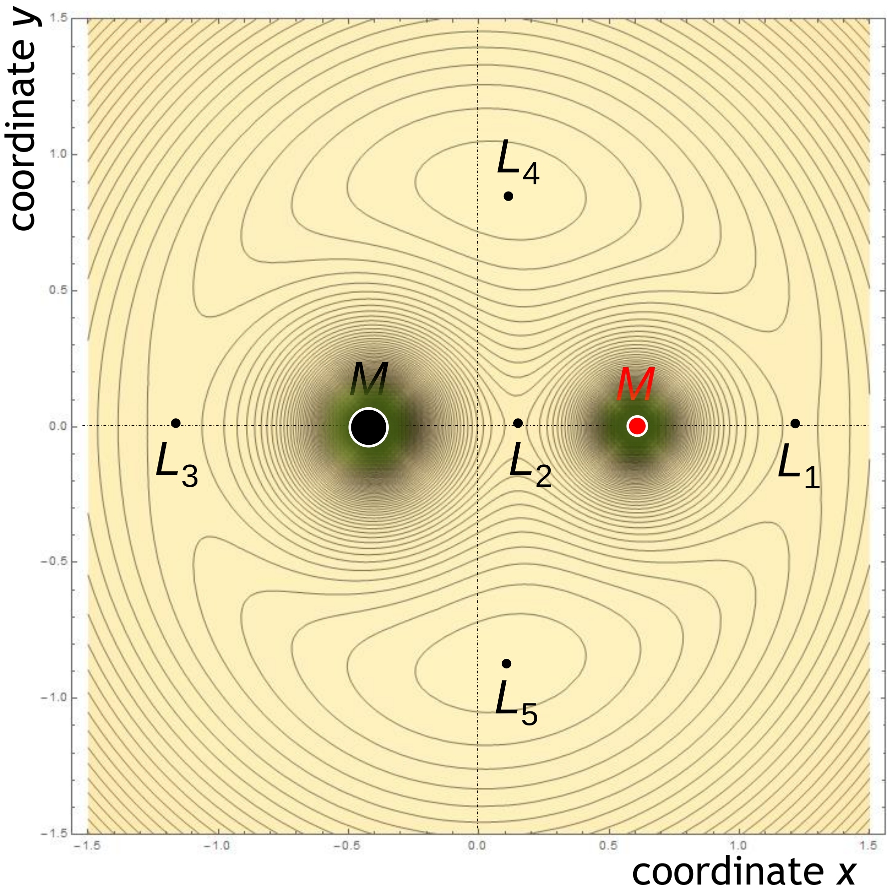



# Reduced three-body problem

- *M* > *M*, third body with negligible mass m ≈ 0
- The motion of all the three bodies restricted to a plane
- Solved in the system connected with *M*, *M*




## Poincaré section

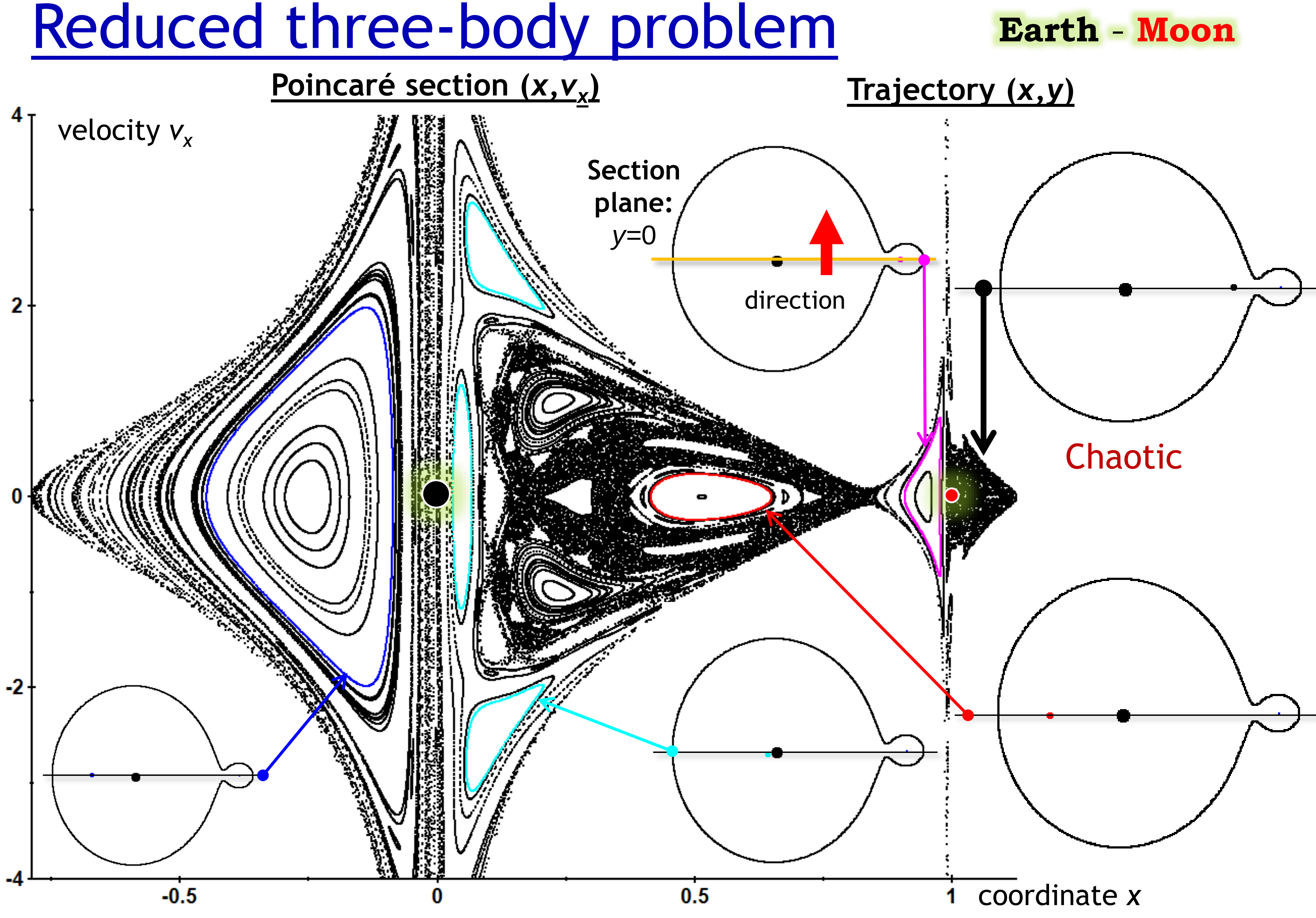
- „stroboscopic map“ – trajectory observed only at specific times (for example when *y*=0)
- A plot of points (coordinate, velocity)
- In a system with just two degrees of freedom, **each point of the section belongs to only one trajectory**

Stable (quasiperiodic) trajectories and **unstable** (chaotic) trajectories can be distinguished with the naked eye.

$L_1, \ldots, L_5$ – Lagrange points (equilibrium; centrifugal force cancels out gravitational force)

N.B. This is what Poincaré considered in his essay.

# Reduced three-body problem



# Hamiltonian systems

$$H = H(p_1, \ldots p_f, q_1, \ldots, q_f)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

**State of the system:**
a point in the $2f$ dimensional phase space

**Conservative system:**

$$H(p_1, \ldots p_f, q_1, \ldots, q_f) = E$$

(Trajectories restricted to $2f-1$ dimensional hyperspace)

**Integrals of motion:**

- connected to additional symmetries of the system

$$I_i = I_i(p_1, \ldots p_f, q_1, \ldots, q_f)$$

$$\{I_i, I_j\}_{\text{Poisson}} = 0$$

**Integrable system:**

- Canonical transformation to action-angle variables

$$\tilde{H} = \tilde{H}(J_1, \ldots, J_f)$$

$$\dot{J}_i = 0 \qquad \dot{\phi}_i = \frac{\partial \tilde{H}}{\partial J_i} \equiv \omega_i(J_1, .$$

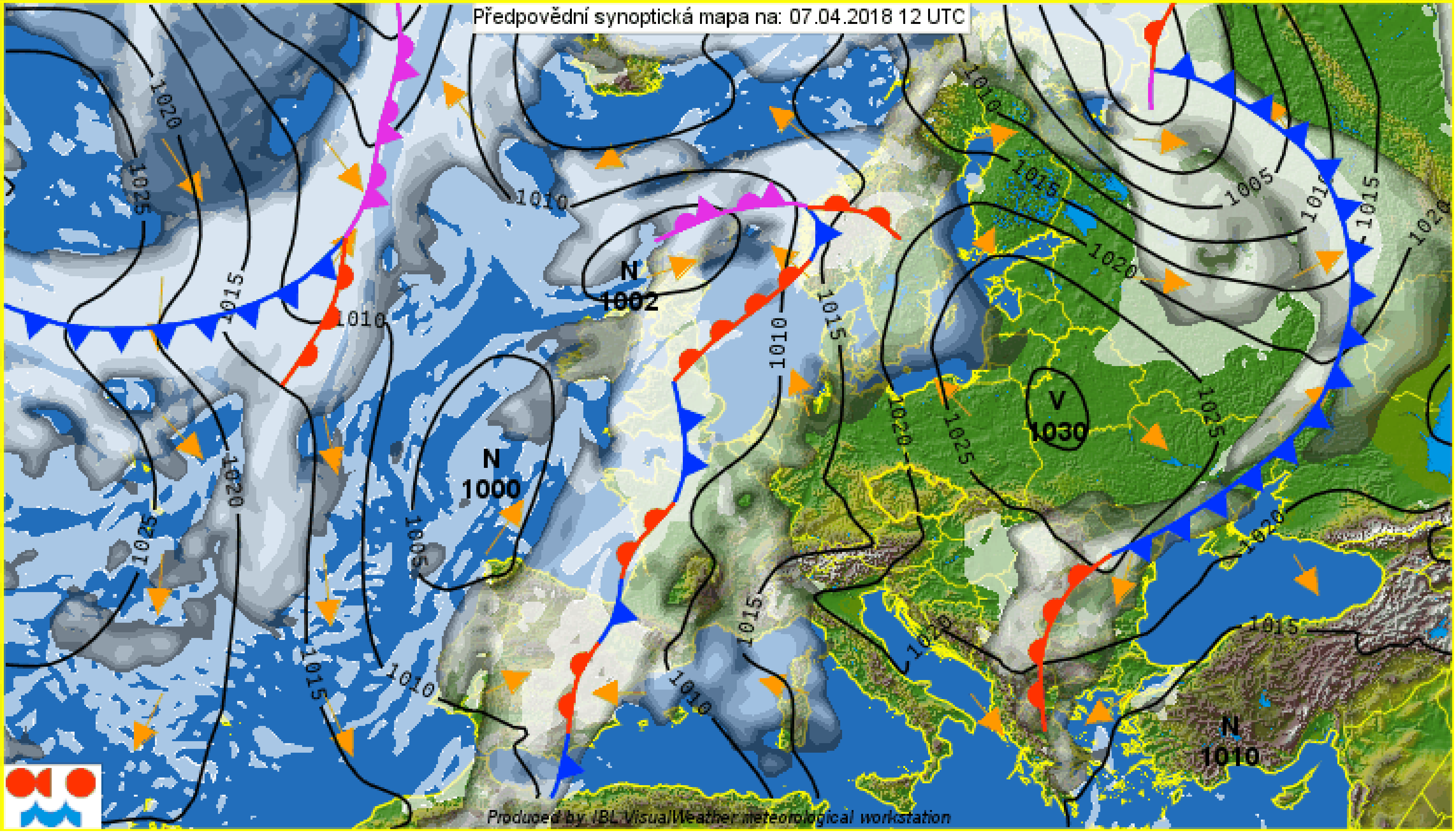
Number o inde ent int i

Number of degrees of freedom $f$

**Nonintegrability: prerequisite for chaos**

Quasiperiodic **(regular)** motion on a torus

# 2. Meteorology

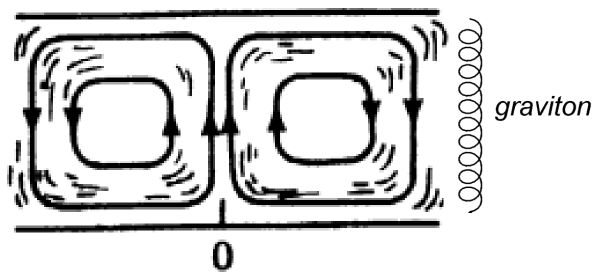


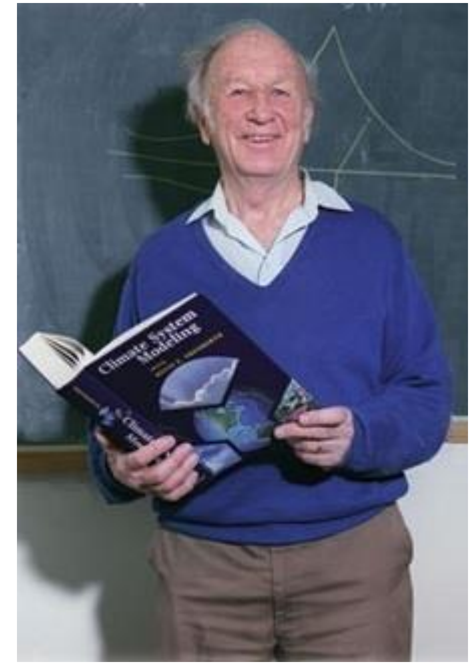
# Lorenz system

- simple model for atmospheric convection

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\tau - z) - y$$
$$\dot{z} = xy - \beta z$$

Bénard cell



3 variables (not spatial coordinates!)

$\boldsymbol{x}$: convection intensity

$\boldsymbol{y}$: temperature difference between the ascending and descending current

$\boldsymbol{z}$: distortion of vertical temperature profile from linearity

3 parameters

$\boldsymbol{\sigma}$: Prandtl number

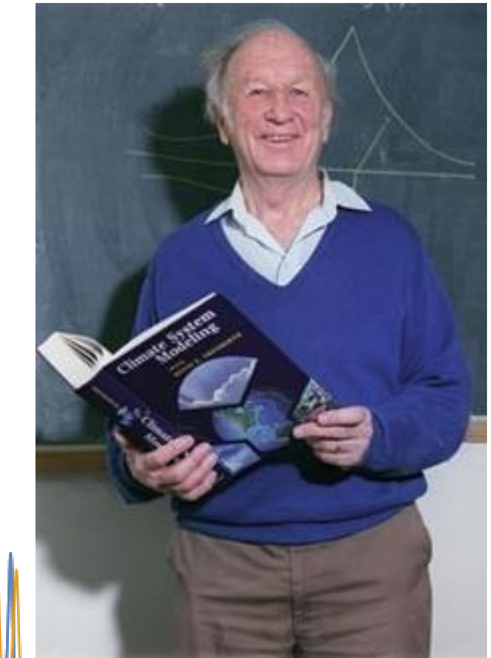
$\boldsymbol{\tau}$: Rayleigh number

$\boldsymbol{\beta}$: physical proportion

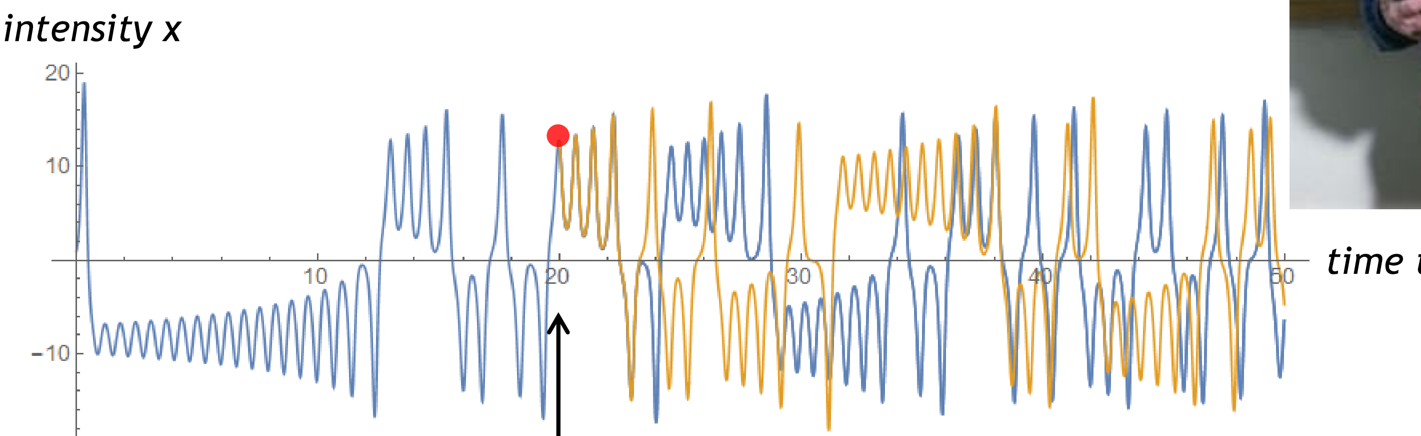
Lorenz's choice $\sigma = 10, \tau = 28, \beta = \frac{8}{3}$

Edward N. Lorenz, *Deterministic nonperiodic flow*, J. Atmos. Sci. **20**, 130 (1963)

# Lorenz’s story



Lorenz was computing the “weather forecast” using his model.
The computer precision was 6 digits ($x$=14,7139 m/s), but the terminal output was rounded to 3 digits ($x$=14,7 m/s).




In the evening Lorenz wrote down a partial result. The following day he resumed the calculation using rounded value ($I$=14,7 m/s).
After a few time steps he gets **qualitatively** different weather.

1963: One flap of the **sea gull** wings may affect the weather far away.

1972: Does the flap of a Butterfly’s wings in Brasil set off a tornado in Texas?

# Lorenz's story

**The Butterfly Effect**

metaphor for the physical chaos

*„The butterfly, with its seeming frailty and lack of power, is a natural choice for a symbol of the small that can produce the great.“*

- sensitivity to initial conditions
- sensitivity to tiny perturbations

In the evening Lorenz wro
day he resumed the calcu ).
After a few time steps he

Solution in the form of the **strange attractor** (fractal dimension *d*=2,04)

(it resembles the wings of a butterfly)

1963: One flap of the **sea gull** wings may affect the

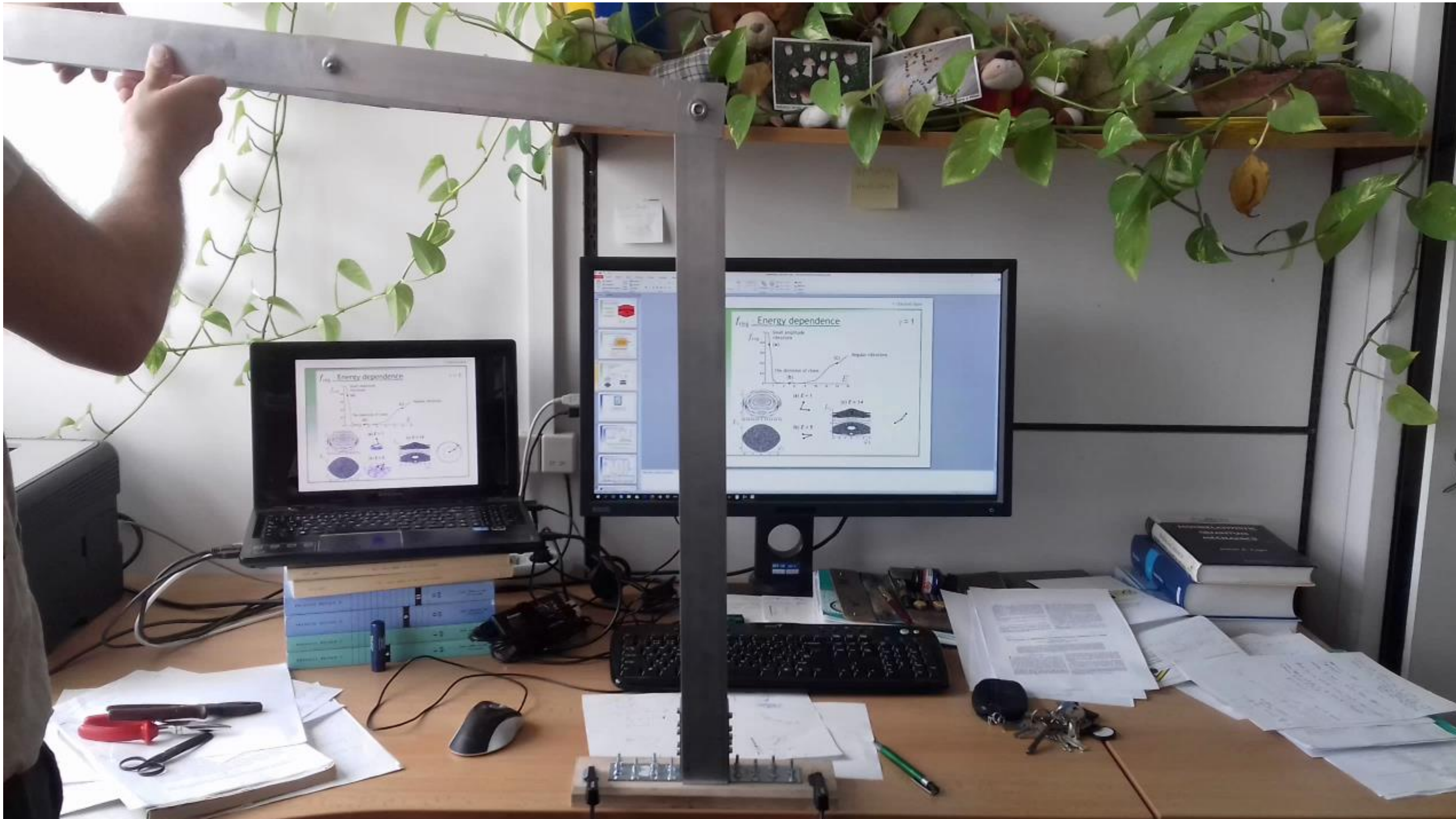
1972: Does the flap of a Butterfly's wings in Brasil

# 3. Double pendulum

# When you see it, you must have it.

**Double pendulum construction.**

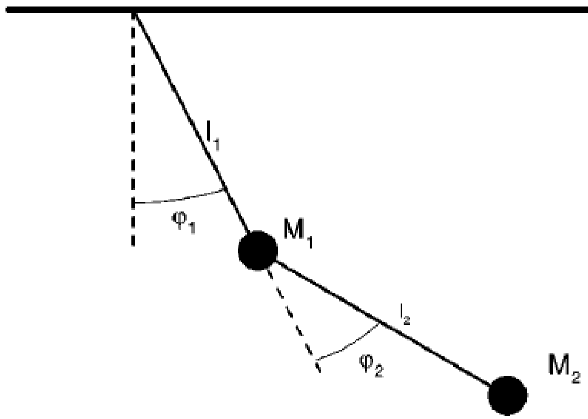
UNDER CONSTRUCTION




*graviton*


graviton
When the construction has been successful.

# Double pendulum Hamiltonian




$$\mu = \frac{M_2}{M_1}$$

$$l = \frac{l_2}{l_1}$$

$$\gamma = g M_1 l_1$$

3 fundamental parameters

- in the following considered only the case $m = l = 1$ (equal masses and lengths) and $\gamma = 0$ (no gravity) or $\gamma = 1$

Angular momenta
(associated with the angles $\varphi$)

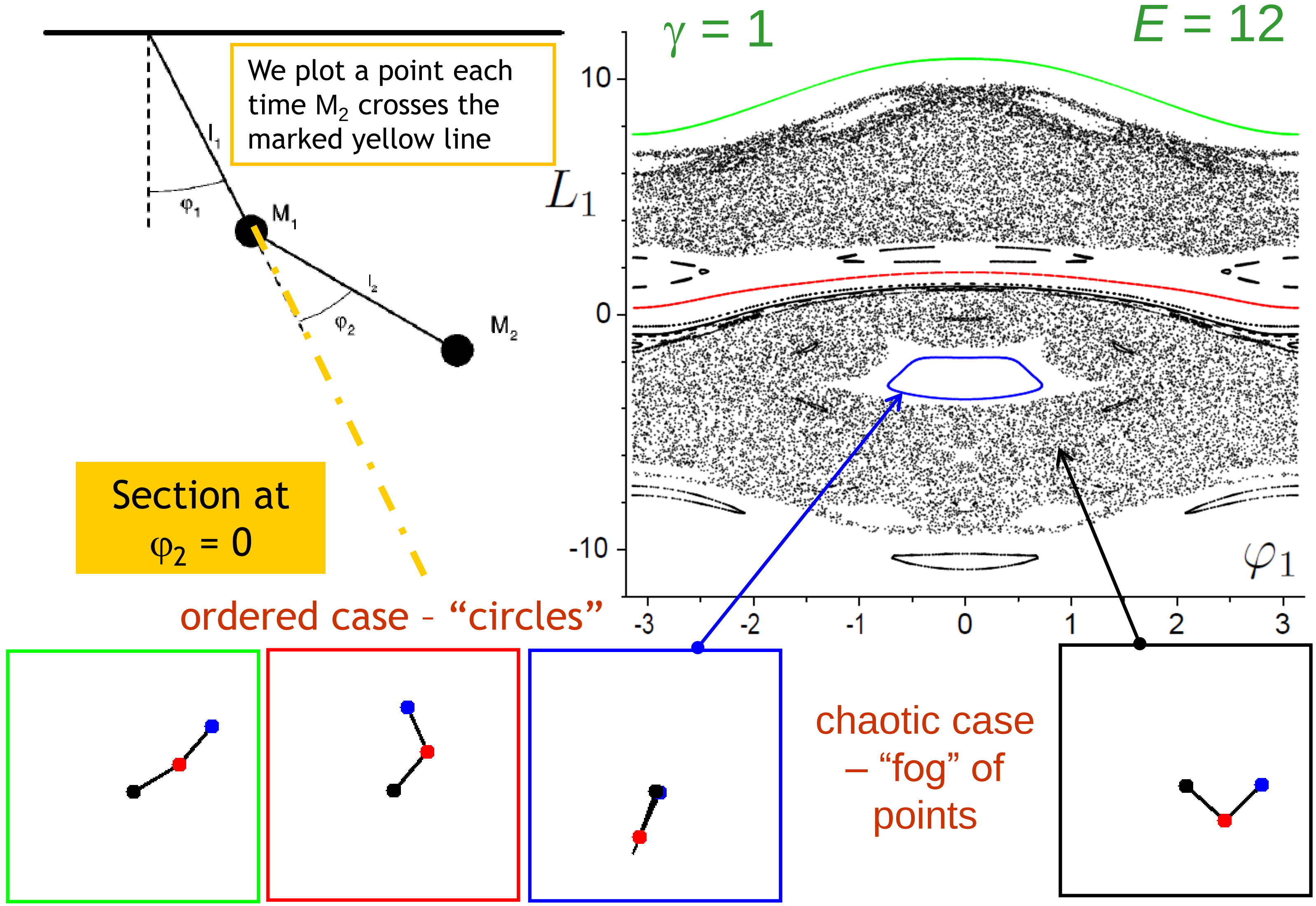
$$T = \frac{1}{2}\frac{1}{1+\mu\sin^2\varphi_2}\left[L_1^2 - \frac{l+\cos\varphi_2}{l}L_1L_2 + \frac{1+\mu+2\mu l\cos\varphi_2+\mu l^2}{\mu l^2}L_2^2\right]$$

$$V = \gamma\left[(1+\mu)(1-\cos\varphi_1) + \mu l\,(1-\cos(\varphi_1+\varphi_2))\right]$$

Integrable for $\gamma$ = 0 (no gravity)
$L_1$ is then the additional integral of motion

L. Perotti, Phys. Rev. E **34**, 066218 (2004)

# Poincaré sections




Various initial conditions at energy $E = 12$

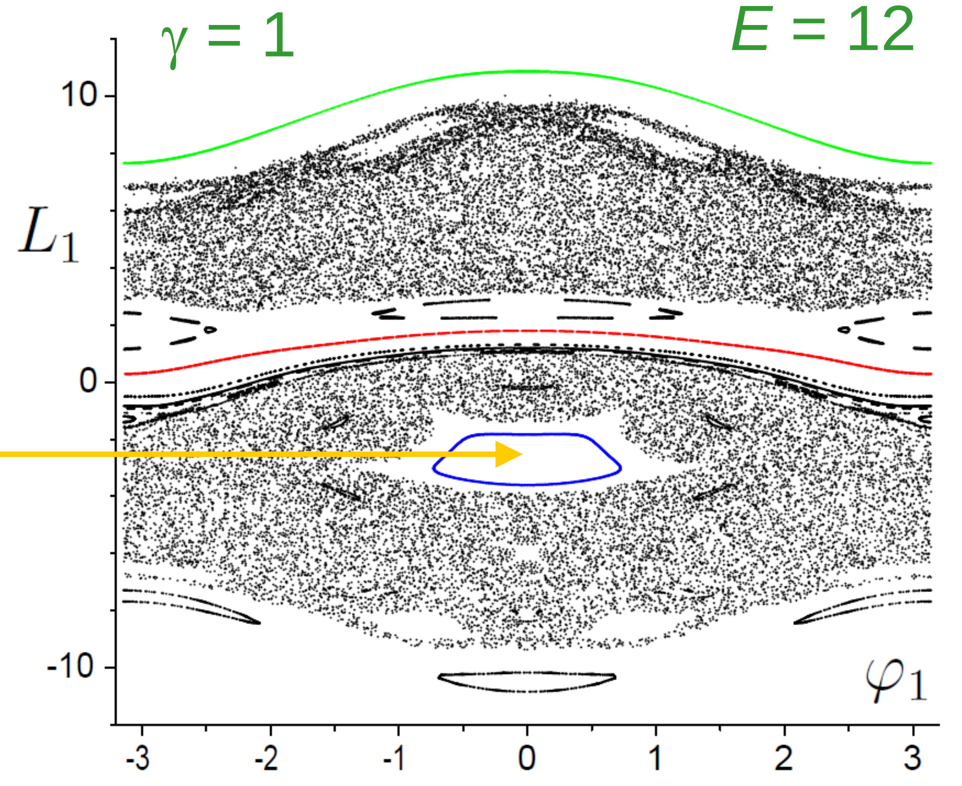
# Fraction of regularity

Measure of classical chaos

Surface of the section covered with regular trajectories

$$f_{\mathrm{reg}} = \frac{S_{\mathrm{reg}}}{S_{\mathrm{tot}}}$$

Total kinematically accessible surface of the section



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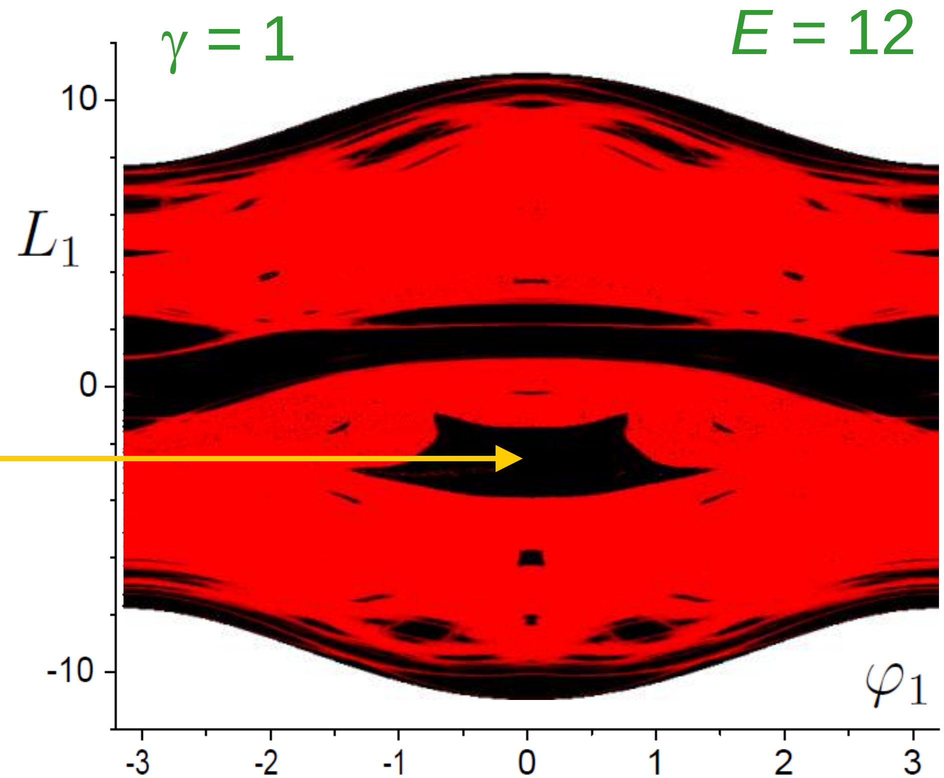


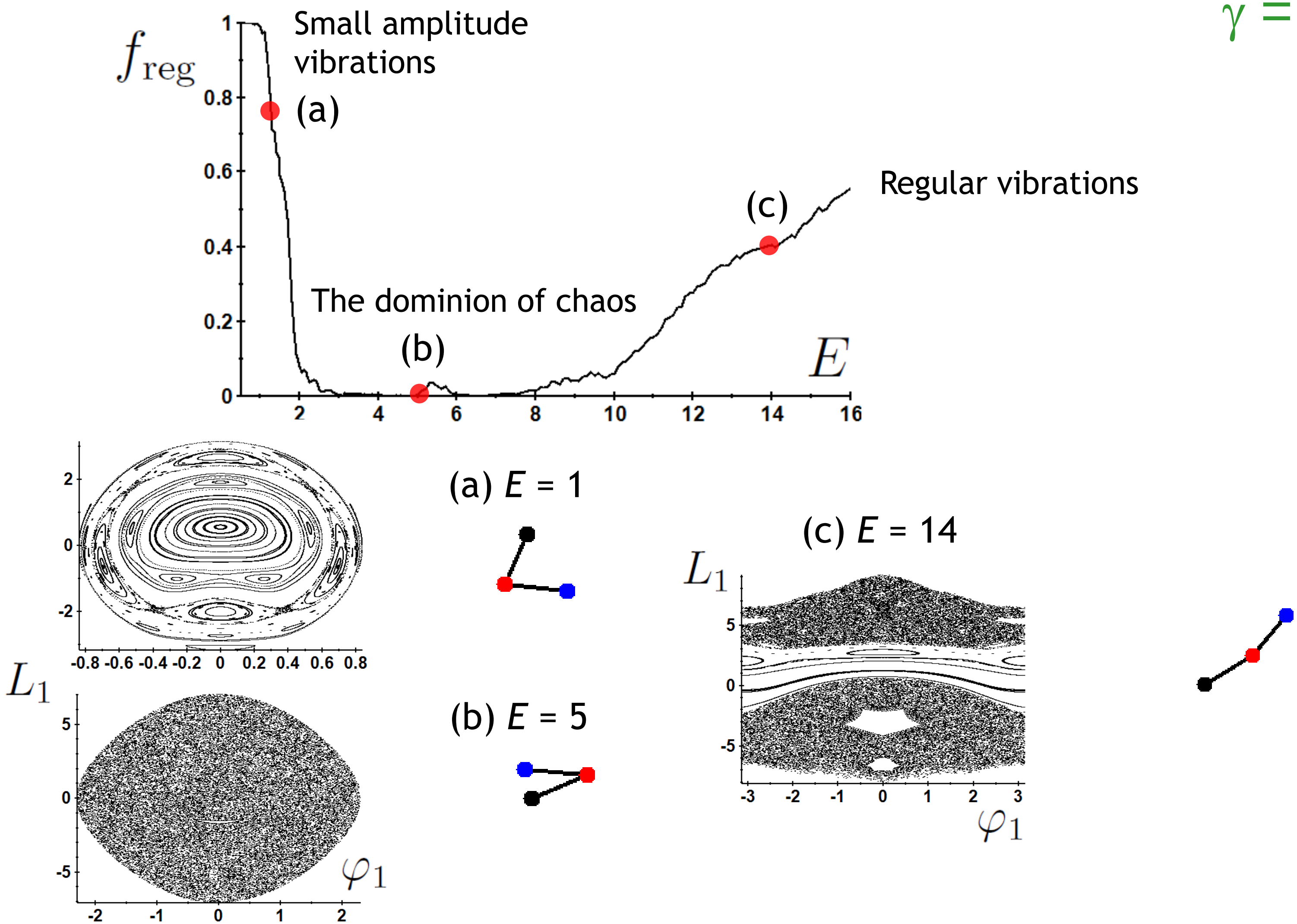

REGULAR area

CHAOTIC area

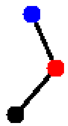
$f_{\mathrm{reg}}$=0.29

# $f_{reg}$ depends on energy!

$\gamma = 1$

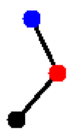


# Quasiperiodic X unstable trajectories



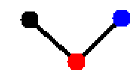
1. Lyapunov exponent

$$\lambda \equiv \max_{\delta(0)} \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta(t)|}{|\delta(0)|}$$



$\delta(0)$ $\delta(t) \sim t^{\alpha}$

Regular: at most polynomial divergence



$\delta(0)$

Chaotic: exponential divergence

$\delta(t) \sim \mathrm{e}^{t}$

Divergence of two neighboring trajectories

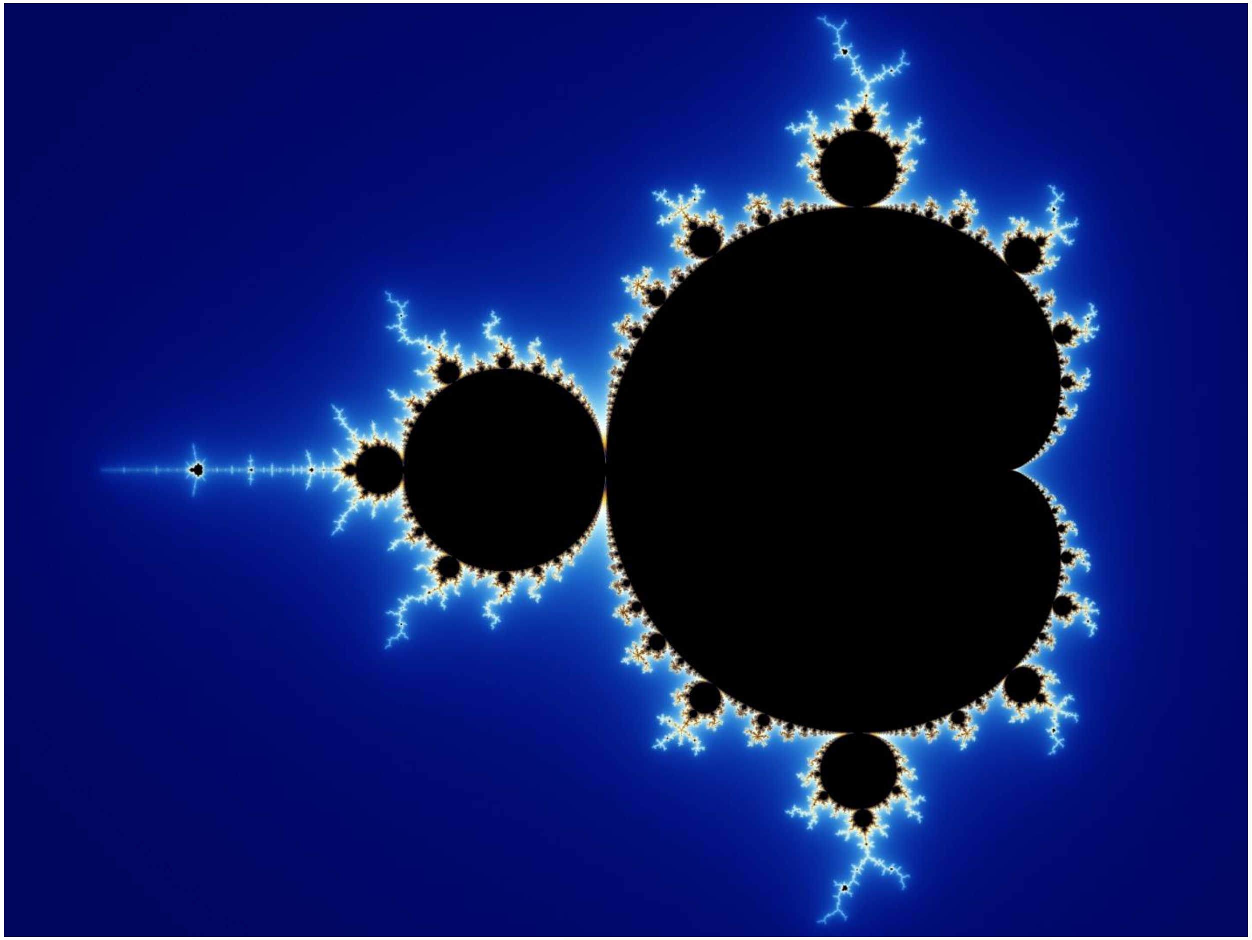
**2. SALI (Smaller Alignment Index)**

$$\mathrm{SALI}(t) = \min\left\{\left|\frac{\delta_1(t)}{|\delta_1(t)|} + \frac{\delta_2(t)}{|\delta_2(t)|}\right|, \left|\frac{\delta_1(t)}{|\delta_1(t)|} - \frac{\delta_2(t)}{|\delta_2(t)|}\right|\right\} \in \left[0, \sqrt{2}\right]$$

- two divergencies
- fast convergence towards zero for chaotic trajectories

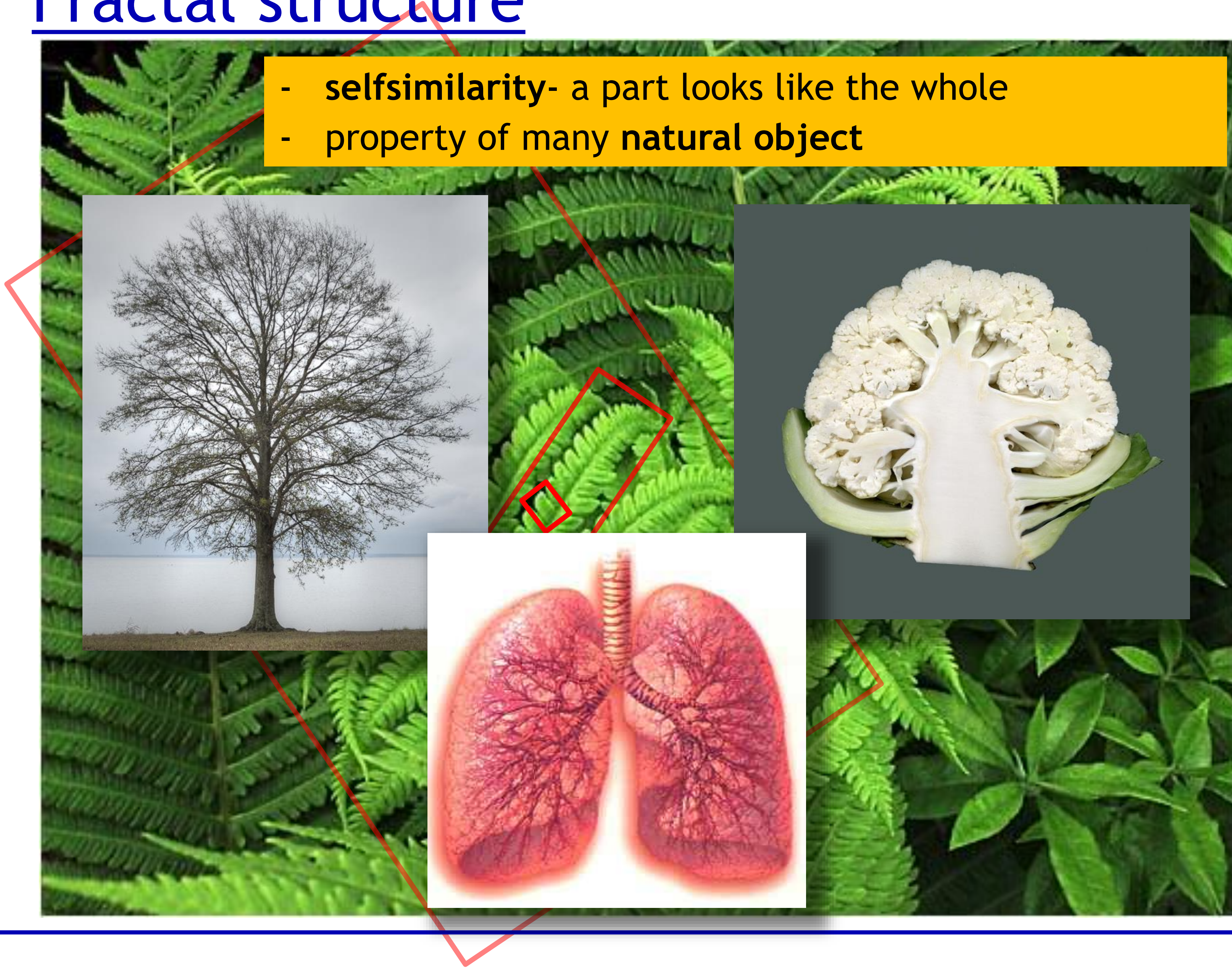
Ch. Skokos, J. Phys. A: Math. Gen **34**, 10029 (2001); 37 (2004), 6269

# Fractals



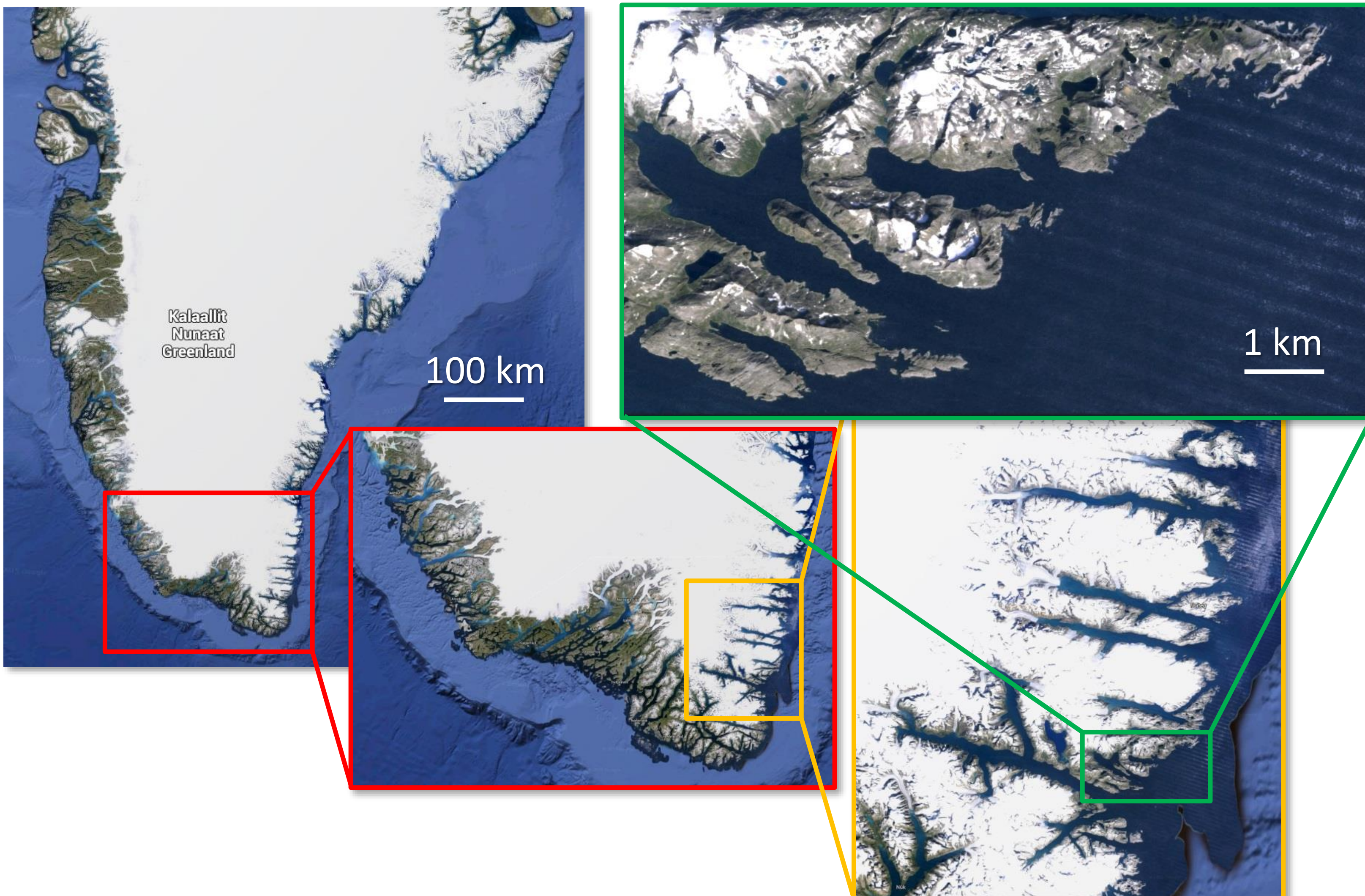
# Fractal structure

- **selfsimilarity-** a part looks like the whole
- property of many **natural object**



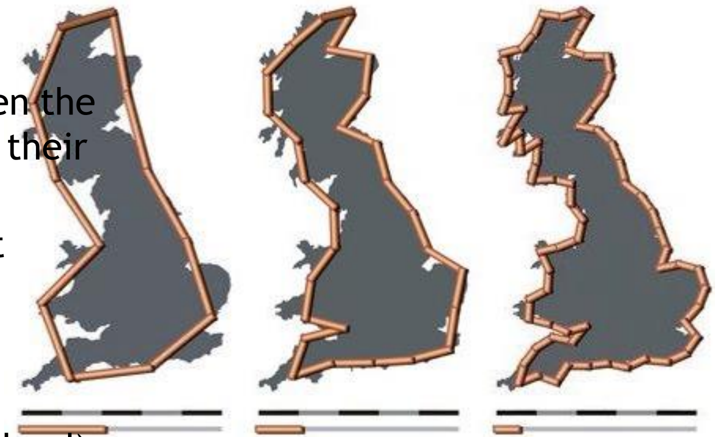
# Fractal structure

Length of the sea coastline



# Fractal (fractional) dimension

## Length of the sea coastline (Great Britain)

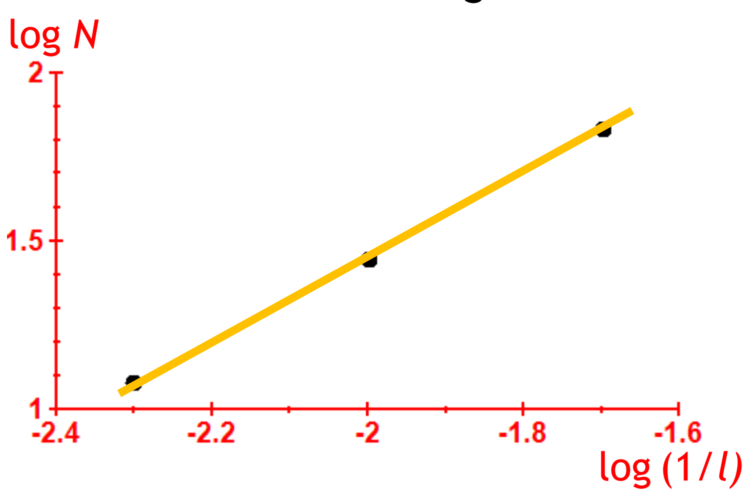


1950 – Lewis F. Richardson studies the correlation between the tendency of countries to declare a war and the length of their common border

He finds out that the border lengths taken from different sources vary extremely. Today's values for the GB:

- Ordance Survey: 17 820 km
- Coastal Guide Europe: 18 838 km
- CIA World Factbook: 12 429 km (includes Northern Ireland)

| | | | |
|---|---|---|---|
| Measure length *l*: | 200 km | 100 km | 50 km |
| Number of mesurements *N*: | 12 | 28 | 68 |
| Length of the coastline: | 2400 km | 2800 km | 3400 km |




$$N = \frac{1}{l^d}$$

$$\log N = d \log \frac{1}{l}$$

line slope: **fractal dimension *d*** ≈ 1,25

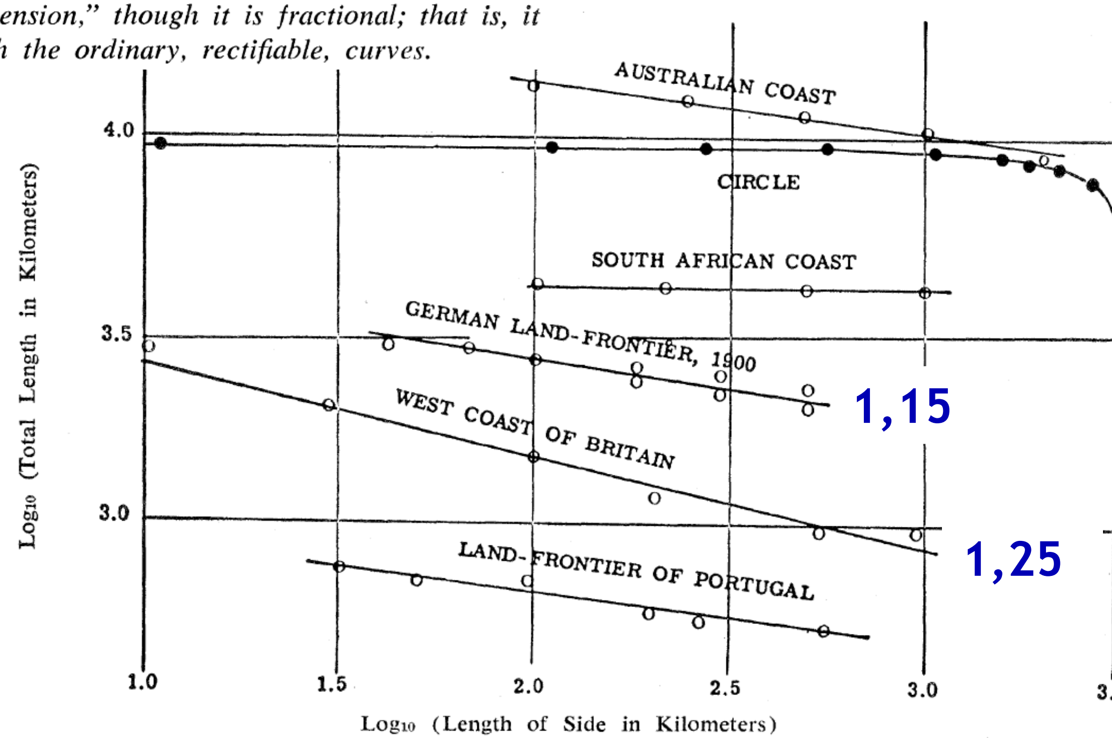
Benoît Mandelbrot, *How long is the coast of Britain? Statistical self-similarity and fractional dimension*, Science **156,** 636 (1967)

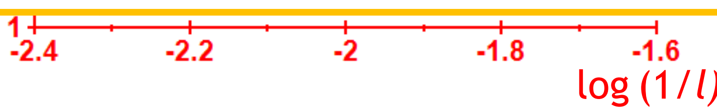
# Fractal (fractional) dimension

## Length of the sea coastline (Great Britain)

1950 – Lewis F. Richardson studies the correlation between the

Abstract. *Geographical curves are so involved in their detail that their lengths are often infinite or, rather, undefinable. However, many are statistically "self-similar," meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity* D *that has many properties of a "dimension," though it is fractional; that is, it exceeds the value unity associated with the ordinary, rectifiable, curves.*

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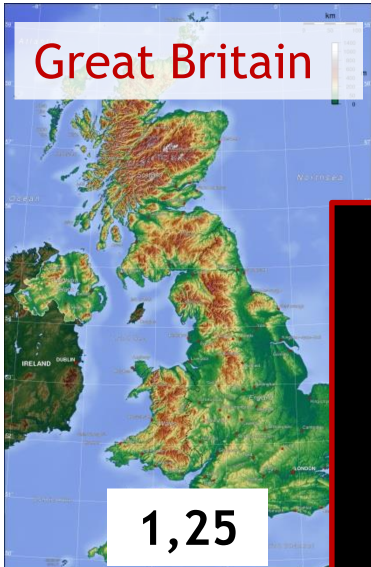
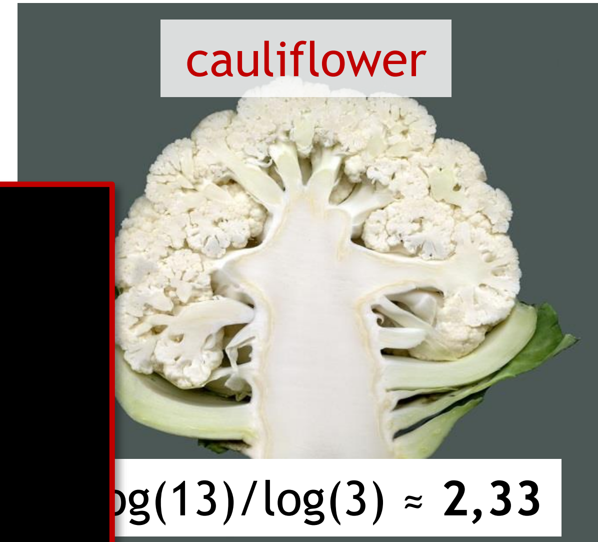
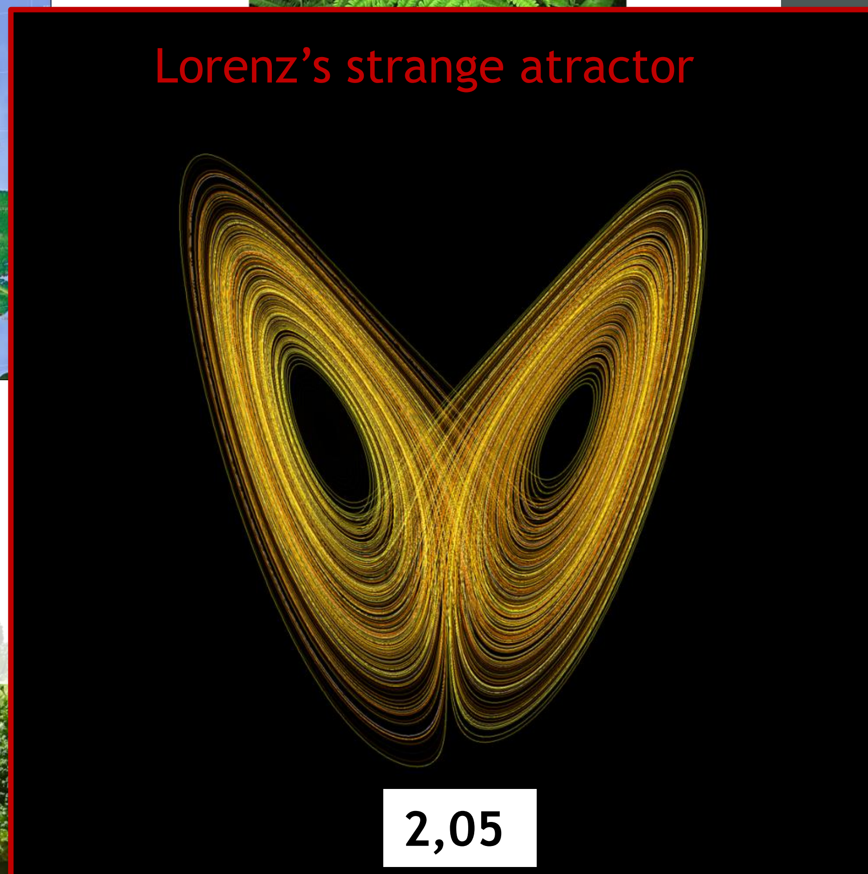
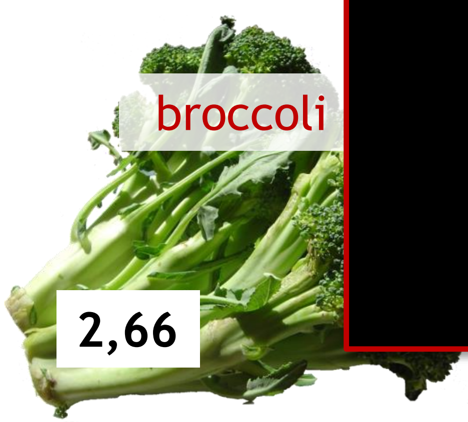
# Fractal dimension - examples

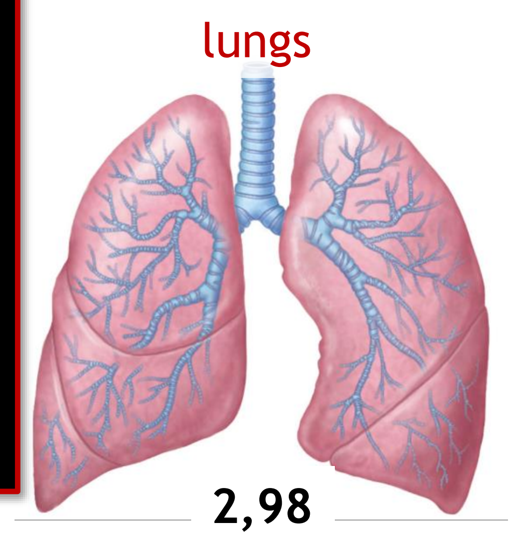






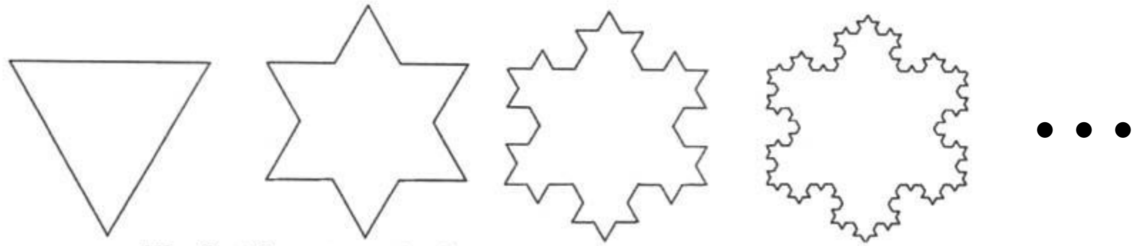
# Artificial fractals

## Koch curve:

(Helge von Koch, 1904)

- fractal dimension

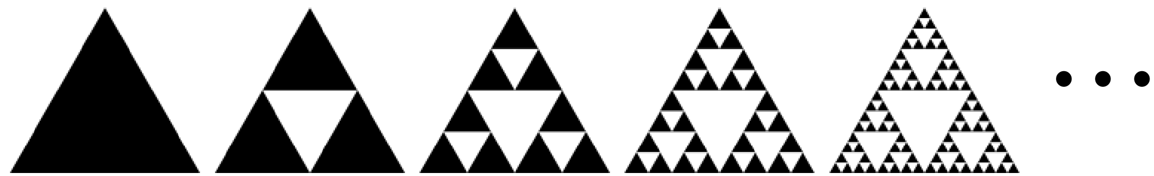
$$d = \frac{\log 4}{\log 3} \approx 1.26$$



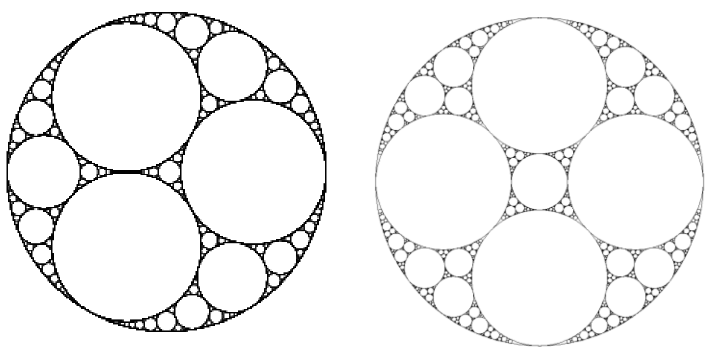
## Sierpińsky triangle:

(Wacław Sierpińsky, 1915)

$$d = \frac{\log 3}{\log 2} \approx 1.59$$



## Apollonian circles:



-fractal dimension
(depends on the type)

$$d \approx 1.3$$

... and more and more

# Mandelbrot set

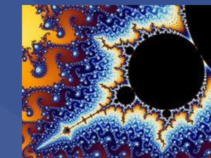
A set of all complex numbers c,
for which the series

$$z_{n+1} = z_n^2 + c$$

is bounded

YouTube animation: https://www.youtube.com/watch?v=PD2XgQOyCCk

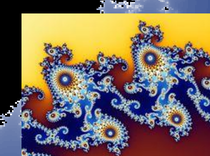
# Mandelbrot set




- 1978 – defined by Robert W. Brooks and Peter Matelski, giving the first sketch of its shape
- fractal dimension of the border ***d*=2**

## Benoît Mandelbrot

- 1975 introduces the notion **fractal**
- 1980 uses the computer to draw the Mandelbrot set for the first time

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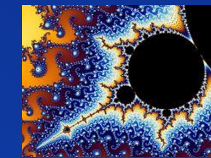

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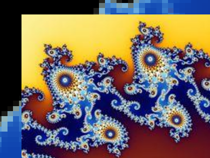
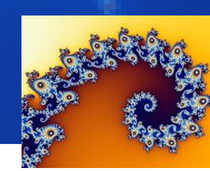
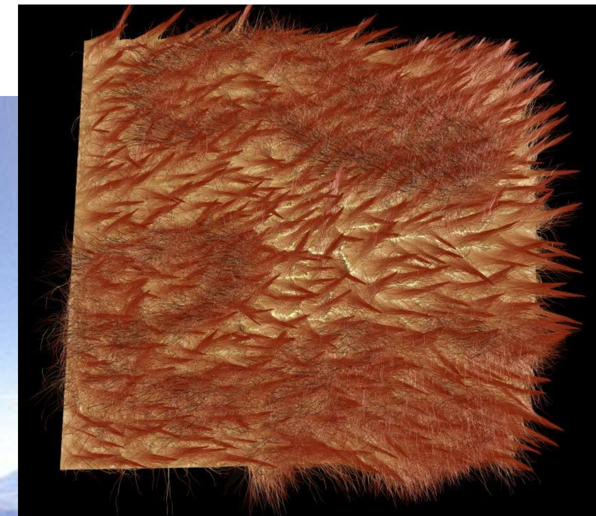


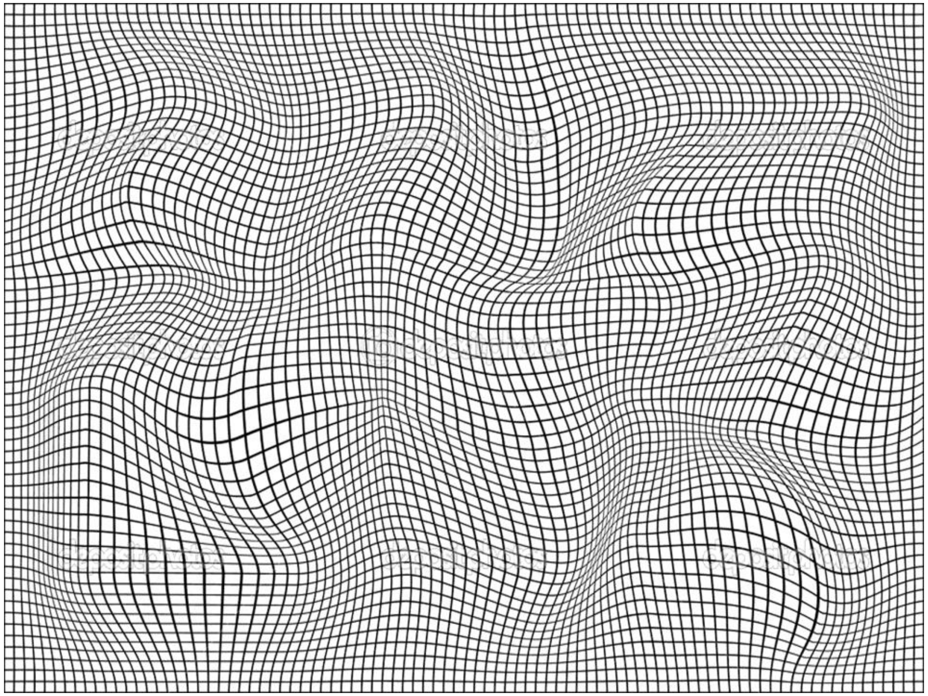


# Application of the fractals – computer graphics

- generating of structures with given fractal dimension
- computer games, movies (*Star Trek II: The Wrath of Khan* - 1982)



# 4. Chaos in curved spaces



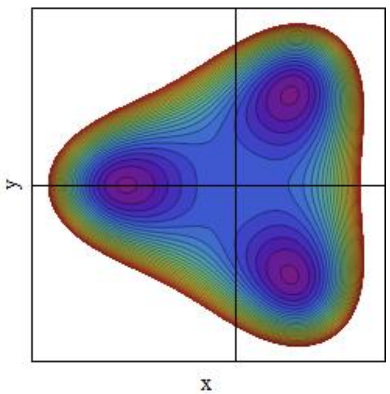
# Geometrical embedding

Hamiltonian in the flat Eucleidian space with a potential:

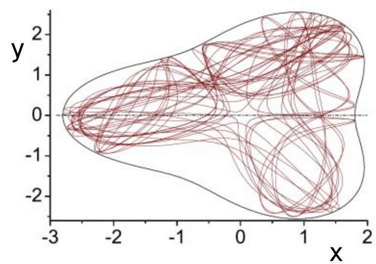
Hamiltonian of a free particle in a curved space:

$$H = \frac{1}{2M}\boldsymbol{p}^2 + V(\boldsymbol{x}) \quad \longrightarrow \quad H' = \frac{1}{2M} g_{ij}(\boldsymbol{x}) p^i p^j$$

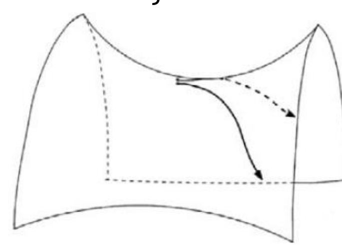
A suitable metric $g_{ij}$




Potential

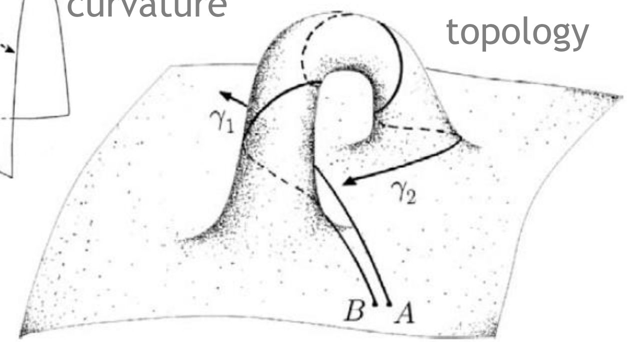



Trajectory



curvature

Geodesic




topology

Bridge:

- The equations of motion (Hamilton, Newton) correspond with the geodesic equation

Why embedding:

- Riemannian geometry brings in the notion of curvature that could help clarify the sources of instability, and in the same time quantify the amount of chaos in non-ergodic systems

L. Casetti, M. Pettini, E.D.G. Cohen, Phys. Rep. **337**, 237 (2000)

# Geodesics & Maps

- Generalization of a straight line
- Describe a ”free motion” in a curved space
- “Shortest path” between two points


Visualisation of a curved space - mapping onto the flat space

# Flat space
(dynamics)

**Potential energy** $V$
Time $t$
Forces $\partial V$
Curvature of the potential $\partial^2 V, (\partial V)^2$

**Trajectories**
Hamiltonian equations of motion

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}$$

Tangent dynamics equation

$$\frac{\mathrm{d}^2\delta^i}{\mathrm{d}t^2} + \left(\frac{\partial^2 V}{\partial x^i \partial x^j}\right)\delta^j = 0$$

# Curved space
(geometry)

**Metric** $g_{ij}$
Arc-length $s$
Christoffel's symbols $\Gamma^i_{jk}$
Riemannian tensor
Ricci tensor $R^i_{jkl}, R_{jl} = R^i_{jil}, R = g^{jl}R_{jl}$
Scalar curvature

**Geodesics**
Geodesic equation

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0$$

Equation of the geodesic deviation (Jacobi equation)

$$\frac{\mathrm{D}^2\delta^i}{\mathrm{d}s^2} + R^i_{jkl}\frac{\mathrm{d}x^j}{\mathrm{d}s}\delta^k\frac{\mathrm{d}x^l}{\mathrm{d}s} = 0$$

$$\tilde{x}^i(t) = x^i(t) + \delta^i(t)$$

Lyapunov exponent $\lambda = \max_{\boldsymbol{\delta}(\mathbf{0})} \lim_{t\to\infty} \frac{1}{t}\ln\frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|}$

# Examples of embedding

## 1. Jacobi metric

$$g_{ij} = 2\left[E - V(\boldsymbol{x})\right]\delta_{ij}$$

- conformal
- length element $\mathrm{d}s^2 = 4\left[E - V(\boldsymbol{x})\right]\mathrm{d}t^2$
- nonzero scalar curvature $R^{(N=2)} = \dfrac{(\nabla V)^2}{(E-V)^3} + \dfrac{\Delta V}{(E-V)^2}$ (záporná pouze pro $\Delta V < 0$)

## 2. Eisenhart metric

- manifold dimension extended by two

$$\mathrm{M} \times \mathbb{R}^2 \quad \left(x^0 = t, x^1, \ldots, x^N, x^{N+1}\right)$$

- length element = time element $\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j - 2V(\boldsymbol{x})\mathrm{d}x^0\mathrm{d}x^0 + 2\mathrm{d}x^0\mathrm{d}x^{N+1} = \mathrm{d}t^2$
- only one nonvanishing Christoffel symbol $\Gamma^i_{00} = \dfrac{\partial V}{\partial x^i}$
- vanishing scalar curvature $R = 0$

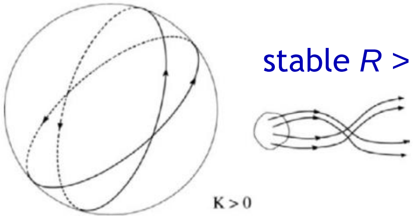
L. Casetti, M. Pettini, E.D.G. Cohen, Phys. Rep. **337**, 237 (2000)
M. Pettini, *Geometry and Topology in Hamiltonian Dynamics and Statistical Mechanics* (Springer New York, 2007)

# Curvature and instability

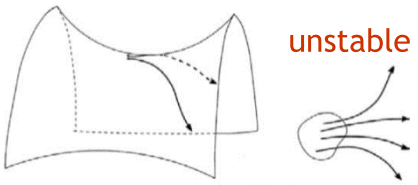
- *R* = *const* (izotropní varieta) — Equation of the geodetic deviation: $$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}s^2}+\frac{1}{2}R\xi^i=0$$

Equation of motion

- harmonic oscillator with frequency $\omega=\sqrt{R/2}$




- exponential growth with Lyapunov exponent $l=\sqrt{-R/2}$




- *R* < 0 — Unstable motion with estimated Lyapunov exponent $l\geq\sqrt{-\max R/2}$

- dim *f* = 2 $$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}s^2}+\frac{1}{2}R(s)\xi^i=0 \qquad R(s)=R_0+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega s)+b_n\sin(n\omega s)\right]$$

Equation of motion of a harmonic oscillator with its length (stiffness) modulated in time
Unstable if the frequency $\omega_0=\sqrt{R_0/2}$ in resonance with any of the frequency of the Fourier expansion, even if *R*(s) > 0 on the whole manifold:

**Parametric instability** – *R* is not sufficient to determine chaotic motion

# Curvature and instability

Besides **solving** the equation for the geodesic deviation, can one deduce something about the instability only from the curvature?

$$g_{ij}^E = \frac{E}{|E - V(\boldsymbol{x})|}\delta_{ij}$$

## 3. Israeli metric

Using the Israeli metric and connection form, the equation of the geodesic deviation is expressed as

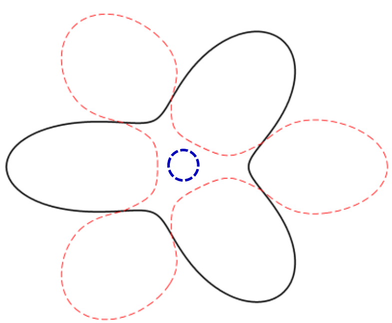
$$\frac{\mathrm{D}^2\boldsymbol{\delta}}{\mathrm{d}t^2} = -\mathcal{V}\mathcal{P}\boldsymbol{\delta}$$

$$\mathcal{P}^{ij} = \delta^{ij} - \frac{v^i v^j}{\boldsymbol{v}^2}$$ - projector into a direction orthogonal to the velocity

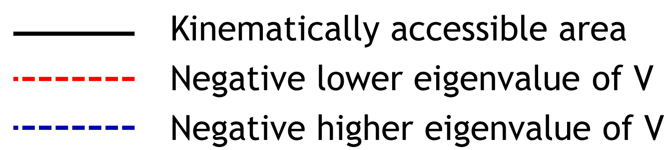
$$\mathcal{V}_{ij} = \frac{3}{M^2\boldsymbol{v}^2}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j} + \frac{1}{M}\frac{\partial^2 V}{\partial x^i \partial x^j}$$

Stability matrix

**Conjecture:** A negative eigenvalue of the *Stability matrix* $\mathcal{V}$ inside the kinematically accessible area induces instability of the motion.



Example of **unstable** configuration




L. Horwitz et al., Phys. Rev. Lett. **98**, 234301 (2007)

# Properties of the stability matrix

$$\mathcal{S}_{ij} = \frac{1}{M}\left[\frac{3}{2\,|K(\boldsymbol{x})|}\partial_i V \partial_j V + \partial^2_{ij} V\right]$$

1. When $|K(\boldsymbol{x})|$ is big enough, $\mathcal{S}$ becomes the Hessian matrix for the tangent dynamics
2. Eigenvalues can only decrease within the kinematically accessible domain

The size of the negative eigenvalue region can only **grow with energy**, or remain the same

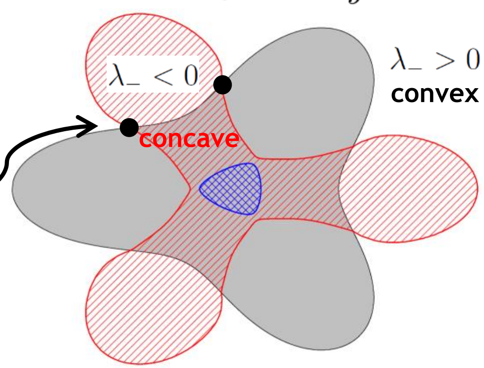
*f* = 2

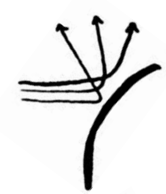
3. The lower eigenvalue $\lambda_-$ is continuous on the boundary of the accessible domain
4. The lower eigenvalue $\lambda_-$ is zero on the boundary when

$$(\partial_x V)^2\,\partial^2_{yy} V + (\partial_y V)^2\,\partial^2_{xx} V - 2\,(\partial_x V)\,(\partial_y V)\,\partial^2_{xy} V = 0$$

$$\det\begin{pmatrix} \partial^2_{xx}V & \partial^2_{xy}V & \partial_x V \\ \partial^2_{xy}V & \partial^2_{yy}V & \partial_y V \\ \partial_x V & \partial_y V & 0 \end{pmatrix} = 0$$

condition for inflexion points of the curve $V(x,y) = 0$



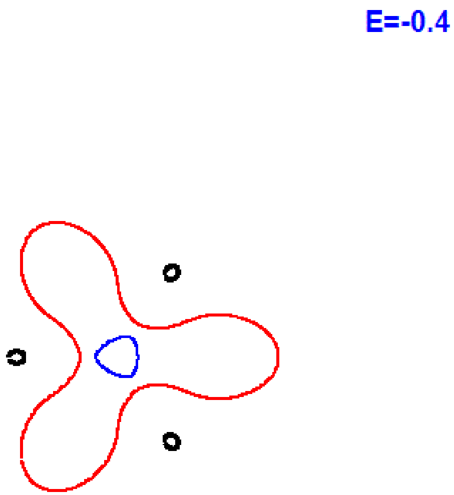
**concave** potential surface - dispersing

**convex** potential surface - focusing

The curvature-based criterion for the onset of chaos can be **partly** translated into the language of the shape of the equipotential contours.

# Instability threshold

## Scenario A - Penetration




$$V = A\left(x^2+y^2\right) + Bx\left(x^2-3y^2\right) + C\left(x^2+y^2\right)^2$$

$$A \approx -0.588 \quad B \approx 0.809 \quad C = 1$$

- region of negative $\lambda_-$, which exists outside the accessible region, starts **overlapping** with it at some energy $E$

- equipotential contours undergoes the **convex-concave transition**

# Instability threshold

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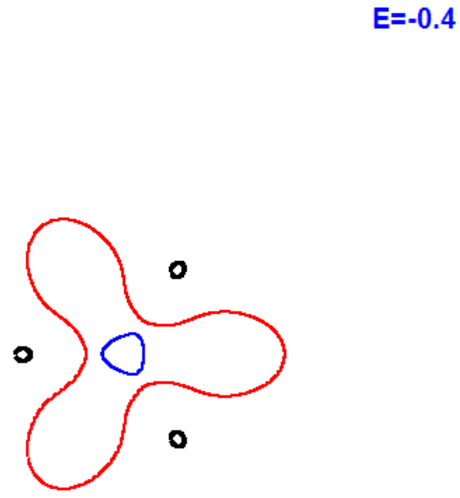


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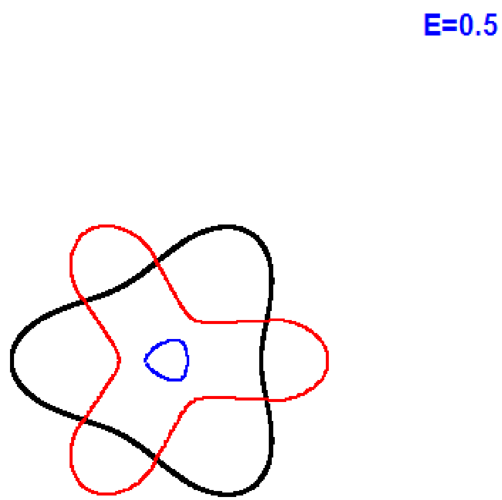


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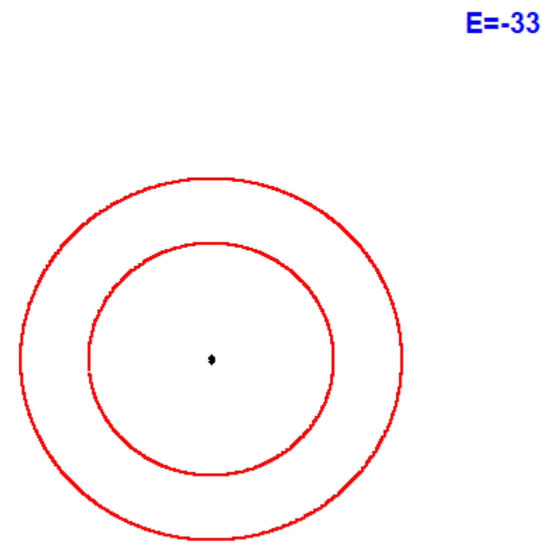
## Scenario B - Creation




$$V = 5\rho + \rho^2 + 4\rho^3 + \rho^4$$

$$\rho = r - r_0 \quad r_0 \approx 2.973$$

- region of negative $\lambda_-$ eventually **appears** somewhere inside the accessible region at some energy $E$

- all the equipotential contours convex

- necessary condition $\partial^2_{xx}V + \partial^2_{yy}V < 0$

# Závěr

Edward Lorenz (1960): Přítomnost jasně udává budoucnost,
ale přibližná přítomnost neudává budoucnost ani přibližně.

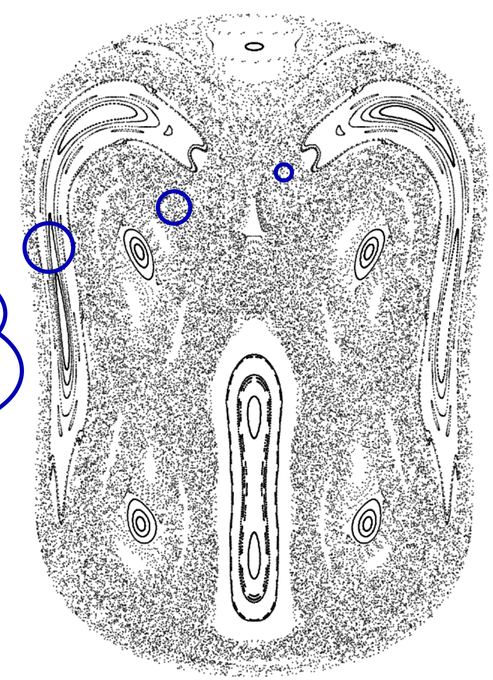
Klasická fyzika je deterministická, ale jelikož je nemožné mít k dispozici absolutně přesné polohy a hybnosti všech těles a absolutně přesnou výpočetní sílu, budoucnost nelze předpovědět. Předpověditelnost je omezena Ljapunovovým časem.

## Deterministický chaos

Na co se nedostalo:

- heteroklinická změť
- chaos v kvantové fyzice
- komplexní systémy
- celulární automaty
- Benfordův zákon
- časové řady a 1/f šum
- algoritmická komplexita

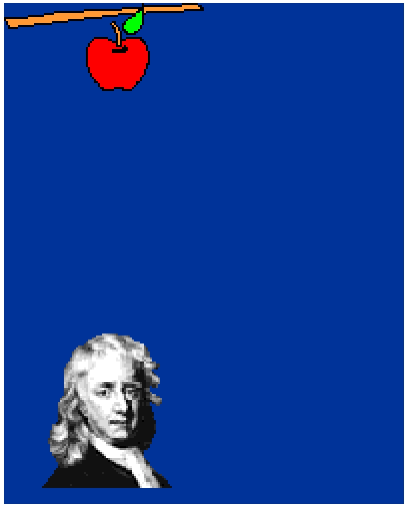
DÍKY ZA POZORNOST



# Fyzika 1. druhu - **kódování**

Pozorováním světa a prováděním experimentů získáváme jednoduchá pravidla, kterými se svět řídí

- **(přírodní) zákony**
- **rovnice**



Newton (1680)

# Fyzika 2. druhu - **dekódování**

Zabýváme se detailně důsledky pravidel a zákonů

- Co se stane, když zákony **upravíme** nebo pozměníme?
- Jaká jsou všechna **možná řešení** rovnic (tedy i ta, která bezprostředně nepozorujeme)?

$$F_g = G\frac{m_1 m_2}{r^2} \Rightarrow \text{???}$$