Outline lecture 2

1. Where does the mixing matrix U appear
2. neutrino oscillations in vaccuum
   - in quantum mechanics for 2 generations
   - (how to get the same eqn in QFT)
3. 3 generations
   - why is 2 generations often an acceptable approximation ?

Where do mixing matrices appear?

Only one mass eigenstate basis for $\{e_R^\alpha\}$, $\{\nu_R^I\} = (\nu_R^2, \nu_R^1)$....so sit there (means $U_{R\nu}$ unphysical). What to do for ℓ^a?? Take mass basis of charged leptons :

$$\ell_L^e \equiv \begin{pmatrix} U_{ei}\nu_L^i \\ e_L \end{pmatrix} \;,\; \ell_L^\mu \equiv \begin{pmatrix} U_{\mu j}\nu_L^j \\ \mu_L \end{pmatrix} \;,\ell_L^\tau \equiv \begin{pmatrix} U_{\tau k}\nu_L^k \\ \tau_L \end{pmatrix}$$

and Lagrangian becomes

$$i\,(U_{ej}^*\overline{\nu_L^j}\;\;\overline{e_L})\,\gamma^\mu \mathbf{D}_\mu \begin{pmatrix} U_{ek}\nu_L^k \\ e_L \end{pmatrix} + i\,(U_{\mu j}^*\overline{\nu_L^j}\;\;\overline{\mu_L})\,\gamma^\mu \mathbf{D}_\mu \begin{pmatrix} U_{\mu k}\nu_L^k \\ \mu_L \end{pmatrix} + ..$$

3×3 mixing matrix $U_{\alpha,i}$ appears at $W^\pm$ vertices (like CKM)

$$\rightarrow \quad -i\,\frac{gU_{ej}^*}{\sqrt{2}}\overline{\nu_L^j}\gamma^\mu W_\mu^+ e_L + \ldots$$

but flavour-diagonal Z vertex :

$$\propto \sum_\alpha -i\,\frac{g}{2}U_{\alpha j}^*\overline{\nu_L^j}\gamma^\mu Z_\mu^+ U_{\alpha k}\nu_L^k = \delta_{jk}\frac{g}{2}\overline{\nu_L^j}\gamma^\mu Z_\mu^+\nu_L^k$$

Oscillations

a relativistic muon decays at the top of the atmosphere, produces a ν.
Cannot reconstruct $(E_\nu, \vec{k}_\nu)$ well enough to identify if ν is ν_3 or ν_2...
The ν travels to the SK detector, where it produces another μ

$\Rightarrow$ must sum in *amplitude* possibility to travel as ν_2 or ν_3
$\Leftrightarrow$ neutrino propagation is a quantum process
NB : in these notes $\Delta^2 = m_j^2 - m_i^2$, contrary to $\Delta = (m_j^2 - m_i^2)/4E$ of board-lectures.

neutrinos "oscillate"(QM version : easy to rederive)

A relativistic neutrino, with momentum $\vec{k}$, is produced in muon decay at $t = 0$ (at Tokai/edge atmosphere). Describe as a quantum mechanical state :

$$|\nu(t=0)\rangle = |\nu_\mu\rangle$$

It travels a distance L in time t to the detector (SuperK)

$$|\nu(t)\rangle$$

where it produces an μ in CC scattering. With what probability ?

$$\mathcal{P}_{\mu\to\mu}(t) = |\langle\nu_\mu|\nu(t)\rangle|^2 \quad = \quad ?$$

1. Suppose massive neutrinos (two generations for simplicity). Flavour and mass eigenstates related by : $\nu_\alpha = U_{\alpha i}\nu_i$

$$\begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix}.$$

2. Suppose time evolution in the mass basis described by

$$i\frac{d}{dt}\begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} = \begin{bmatrix} E_2 & 0 \\ 0 & E_3 \end{bmatrix} \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} \qquad , \quad E_i^2 = k^2 + m_i^2$$

3. If produce relativistic ν_μ at $t = 0$, then at t later :

$$|\nu(t)\rangle = \sum_j U_{\mu j}|\nu_j(t)\rangle = \sum_j U_{\mu j} e^{-iE_j t}|\nu_j\rangle$$

Amplitude for neutrino to produce charged lepton α in CC scattering in detector after t :

$$|\langle \nu_\alpha | \nu(t) \rangle| = \left| \sum_j U_{\mu j} e^{-iE_j t} U^*_{\alpha j} \right|$$

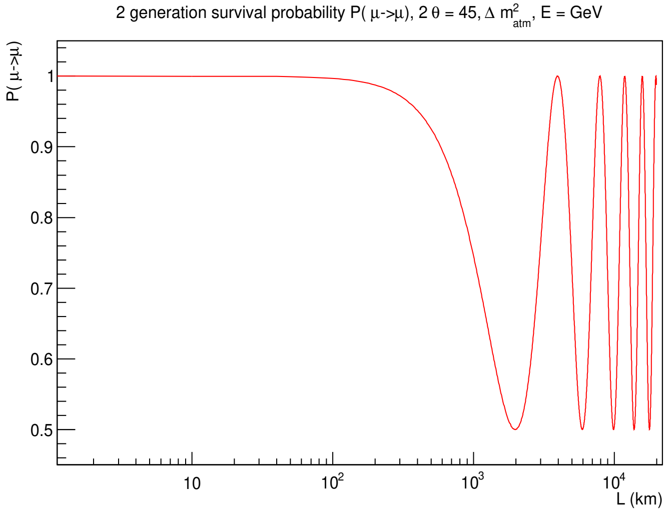
So in 2 generation case, using $t = L$, $E_3 - E_2 \simeq \frac{m_3^2 - m_2^2}{2E} \equiv \frac{\Delta^2_{32}}{2E}$:

$$\mathcal{P}_{\mu\to\tau}(t) = \left| \sin\theta \cos\theta \left(e^{i\Delta^2_{32} L/4E} - e^{-i\Delta^2_{32} L/4E} \right) \right|^2$$

$$= \sin^2(2\theta) \sin^2 \left(L \frac{\Delta^2_{32}}{4E} \right)$$

$$\mathcal{P}_{\mu\to\mu}(t) = 1 - \sin^2(2\theta) \sin^2 \left(L \frac{\Delta^2}{4E} \right) = 1 - \sin^2(2\theta) \sin^2 \left(1.27 \frac{L}{km} \frac{\Delta^2}{\mathrm{eV}^2} \frac{\mathrm{GeV}}{4E} \right)$$

$E = \nu$ energy, L source-detector distance, $\Delta^2_{32} \sim 10^{-3}\mathrm{eV}^2$
$E \sim 10$ GeV for atmospheric νs ; $L : 20km \to 10000km$



$$\mathcal{P}_{\mu\to\mu}(L) = 1 - \sin^2(2\theta)\sin^2\left(1.27\frac{L}{km}\frac{\Delta^2}{\mathrm{eV}^2}\frac{\mathrm{GeV}}{4E}\right)$$

$$\begin{aligned}\Delta^2_{32} &= 2.5\times 10^{-3}\mathrm{eV}^2\\ E &\sim 0.6\mathrm{GeV(T2K)}\\ &\sim \mathrm{MeV(reactors)}\\ &\sim 10\mathrm{GeV(atmospher}\end{aligned}$$

decoherence of neutrinos for large $L/E \gg 1/\Delta^2$

- at production, 2 superposed wavepackets of masses m_2,m_3.
- group velocity of packets

$$v_i = \frac{\partial E}{\partial p} = \frac{p}{E} \simeq 1 - \frac{m_i^2}{2E^2}$$

- after distance L, packets have separated by

$$(v_2 - v_3)L \simeq \frac{\Delta_{23}^2}{E^2} L \simeq \frac{L}{\ell_{osc}} \frac{1}{E}$$

- no interference if larger than size of packets $\sim 1/(\delta E)$ where packet energy uncertain by δE. so no oscillations once

$$\frac{L}{\ell_{osc}} \gtrsim \frac{E}{\delta E}$$

can make similar estimate doing sum over paths, phases should sum coherently

Oscillations in QFT (skeletal version...probably with typos, sorry)

Question = neutrinos are relativistic ; should we not do oscillations in QFT ?

Here show that QFT is equivalent to "the Schrodinger eqn" (which I will want to use for oscillations in matter later...)
In second quantised field theory, the eqns of motion for the number operator $\hat{n}$ are (Heisenberg rep, t-dep ops)

$$\frac{d}{dt}\hat{n} = +i[\hat{H}, \hat{n}]$$

where the Hamiltonian $\hat{H}$ can be taken as free $= \hat{H}_0 \sim \sum \omega \hat{n}_\omega$.

Oscillations in QFT

1 : Work in second quantised formalism for neutrino field :

$$\hat{\psi}_j(x) = \sum_{s=+,-} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \left(e^{-ip\cdot x} \hat{a}_j^s(\vec{p}) u^s(p) + e^{ip\cdot x} \hat{b}_j^{s\,\dagger}(\vec{p}) v^s(p) \right)$$

where s is helicity, j is generation, $\hat{a}^\dagger$ creates particles, et $\hat{b}^\dagger$ creates anti-particules. *Define $\hat{a}$ for energy= mass eigenstates. But formalism is covariant...*

2 : We want to know time/space evolution of a beam neutrinos (no anti), of + helicity and momentum $\vec{p}$. Define number operator :

$$\hat{n}_{jk}^{sr}(\vec{p}) = \hat{a}_j^{+\dagger}(\vec{p}) \hat{a}_k^{+}(\vec{p})$$

covariant in flavour space (indices jk),*and now drop spin indices for simplicity of notation*.

3 : The eqns of motion for the number operator $\hat{n}$ are

$$i\frac{d}{dt}\hat{n} = -[\hat{H}_0, \hat{n}]$$

with free Hamiltonian in mass eigenstate basis is :

$$H_0 = \sum_n \int \frac{d^3k}{(2\pi)^3}\,\omega_{nn}(|\vec{k}|)\Big(\hat{n}^{++}nn(\vec{k}) + \hat{n}^{--}_{nn}(\vec{k})\Big)\ ,\ \omega_{nn} = \sqrt{\vec{k}^2 + m_n^2}$$

4 : Want to Calculate $i\frac{d}{dt}\hat{n}_{jk}(\vec{p}) = -[\hat{H}_0, \hat{n}_{jk}(\vec{p})]$

using comm. relations $[a_j^\dagger(\vec{k}), a_k(\vec{q})] = (2\pi)^3\delta^3(\vec{k}-\vec{q})2\omega\delta_{jk}$.

$$\begin{aligned} = & -\int \frac{d^3k}{(2\pi)^3}\Big(\omega_2(\vec{k})\hat{a}_2^\dagger(\vec{k})\hat{a}_2(\vec{k}) + \omega_1(\vec{k})\hat{a}_1^\dagger(\vec{k})\hat{a}_1(\vec{k})\Big)\hat{a}_j^\dagger(\vec{p})\hat{a}_k(\vec{p}) \\ & -\hat{a}_j^\dagger(\vec{p})\hat{a}_k(\vec{p})\Big(\omega_2(\vec{k})\hat{a}_2^\dagger(\vec{k})\hat{a}_2(\vec{k}) + \omega_1(\vec{k})\hat{a}_1^\dagger(\vec{k})\hat{a}_1(\vec{k})\Big) \end{aligned}$$

$$= -\begin{bmatrix} 0 & (\omega_1-\omega_2)\hat{a}_1^\dagger(\vec{p})\hat{a}_2(\vec{p}) \\ (\omega_2-\omega_1)\hat{a}_2^\dagger(\vec{p})\hat{a}_1(\vec{p}) & 0 \end{bmatrix} \qquad (1)$$

5 : To obtain an observable, and relate to the quantum mechanical calculation, take the expectation value of the operator equation in a state corresponding to abeam of +ve helicity neutrinos of momentum $\vec{p}$. For simplicity, suppose only one neutrino in the beam. Then

$$\langle \hat{n}_{jj}(\vec{p}) \rangle \equiv [f_{jj}](\vec{p})$$

is the density matrix for the 2-state neutrino system.

6 : The QM density matrix for the state $|\nu(t)\rangle = s|\nu_1(t)\rangle + c|\nu_2(t)\rangle$is

$$|\nu(t)\rangle\langle\nu(t)| = \left[\begin{array}{cc} s^2|\nu_1(t)\rangle\langle\nu_1(t)| & sc|\nu_1(t)\rangle\langle\nu_2(t)| \\ sc|\nu_2(t)\rangle\langle\nu_1(t)| & c^2|\nu_2(t)\rangle\langle\nu_2(t)| \end{array} \right]$$

so the eqn of motion is

$$i\partial_t[f] = \left[\begin{array}{cc} \omega_1 & 0 \\ 0 & \omega_2 \end{array} \right][f] - [f]\left[\begin{array}{cc} \omega_1 & 0 \\ 0 & \omega_2 \end{array} \right]$$

which gives same eqn as the expectation value of eqn (**??**).

$\Rightarrow$ so the simple quantum mechanical formulae are ok !

But aren't there three generations?

$$\mathcal{P}_{\alpha\beta}(L) = |U_{\alpha 1} U^*_{\beta 1} + U_{\alpha 2} U^*_{\beta 2} e^{-i\Delta_{21} L/2E} + U_{\alpha 3} U^*_{\beta 3} e^{-i\Delta_{31} L/2E}|^2$$

Reminder : U

two mass differences :

$$|\Delta m^2_{3j}| = |m^2_3 - m^2_j| \simeq 2.5\times 10^{-3}\,\mathrm{eV}^2 \;\gg\; \Delta m^2_{21} \simeq 7.5\pm\times 10^{-5}\,\mathrm{eV}^2$$

$\Rightarrow$ mixing matrix $e_\alpha \to \nu_i$ (2 Majorana phases in P) :

$$U_{\alpha i} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix}}_{\text{atm. + LBL disa.}} \underbrace{\begin{bmatrix} c_{13} & & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix}}_{\text{reac.disa. + LBL app.}} \underbrace{\begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{sol + reac.disa.}} P$$

$$= \begin{bmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{bmatrix} P$$

$$\theta_{23} \simeq \pi/4 \pm \pi/40 \quad \theta_{12} \simeq \pi/6 \quad \theta_{13} \simeq 8^o$$

(global fits of www.nu-fit.org)

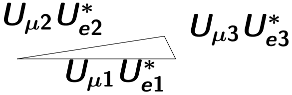
The drunken Unitarity triangle

Not hear much about "leptonic unitarity triangle"
1.not measure elements at tree in CC
2. Also, it drinks.
Amplitude to oscillate from flavour α to β over distance L :

$$\mathcal{A}_{\alpha\beta}(L) = U_{\alpha 1} U^*_{\beta 1} + U_{\alpha 2} U^*_{\beta 2} e^{-i(m_2^2 - m_1^2)L/(2E)} + U_{\alpha 3} U^*_{\beta 3} e^{-i(m_3^2 - m_1^2)L/(2E)}$$

at $L = 0$ unitarity : $\Rightarrow \mathcal{A}_{\alpha\beta} = 1$ for $\alpha = \beta$
$\mathcal{A}_{\alpha\beta} = 0$ for $\alpha \neq \beta$

$\Leftrightarrow$ unitarity triangle(in complex plane)



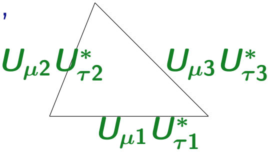
At $L = t \neq 0$, two of the vectors rotate in the complex plane,
with frequencies $(m_j^2 - m_1^2)/2E$
oscillations $\leftrightarrow$ time-dependent non-unitarity

About two- flavour analyses : atm/LBL ν_μ disappearance

Amplitude to oscillate from flavour μ to τ over distance L :

$$\mathcal{A}_{\mu\tau}(L) = U_{\mu 1}U^*_{\tau 1} + U_{\mu 2}U^*_{\tau 2}e^{-i(m_2^2 - m_1^2)L/(2E)} + U_{\mu 3}U^*_{\tau 3}e^{-i(m_3^2 - m_1^2)L/(2E)}$$

At $L \sim E/(m_3^2 - m_1^2)$, vector "3" rotates, frequency $(m_j^2 - m_1^2)/2E$



$\Rightarrow$ "Atmospheric" neutrinos, also LBL (ν_μ disappearance via Δm^2_{31} oscillations) :

$$\mathcal{A}_{\mu\tau}(L) \simeq U_{\mu 1}U^*_{\tau 1} + U_{\mu 2}U^*_{\tau 2} + U_{\mu 3}U^*_{\tau 3}e^{-i(m_3^2 - m_1^2)L/(2E)}$$

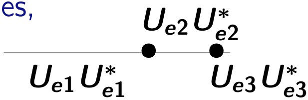
$U_{\mu 3}U^*_{\tau 3}$ oscillates on timescale $t = L \sim (m_3^2 - m_1^2)/E$
$U_{\mu 2}U^*_{\tau 2} \sim$ stationary, measure θ_{23}

About two- flavour analyses : solar and Kamland

Amplitude to oscillate from flavour e to e over distance L :

$$\mathcal{A}_{ee}(L) = U_{e1}U^*_{e1} + U_{e2}U^*_{e2}e^{-i(m_2^2-m_1^2)L/(2E)} + U_{e3}U^*_{e3}e^{-i(m_3^2-m_1^2)L/(2E)}$$

At $L \sim 2E/(m_2^2 - m_1^2)$, vector 2 rotates,
frequency $(m_2^2 - m_1^2)/2E$
vec. 3 spins rapidly



$\Rightarrow$ "Solar" + "KamLAND" (reactor $\overline{\nu_e}$ for $L \sim 100$ km) neutrinos
$\Leftrightarrow$ ν_e disappearance over long baselines $L \sim (m_2^2 - m_1^2)/2E$
two-ν approx works because θ_{13} is small ($U_{e3} = sin\theta_{13}$) :

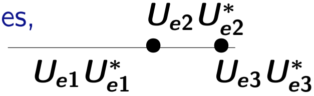
$$\mathcal{A}_{ee} \simeq |U_{e1}|^2 + |U_{e2}|^2 e^{-i(m_2^2-m_1^2)\tau/(2E)}$$

measure θ_{12}

Amplitude to oscillate from flavour e to e over distance L :

$$\mathcal{A}_{ee}(L) = U_{e1}U_{e1}^* + U_{e2}U_{e2}^* e^{-i(m_2^2-m_1^2)L/(2E)} + U_{e3}U_{e3}^* e^{-i(m_3^2-m_1^2)L/(2E)}$$

At short enough L, only third vector rotates, frequency $(m_3^2 - m_1^2)/2E$



$\Rightarrow$ reactor θ_{13} by $\overline{\nu_e}$ disappearance ; select short baseline such that only $|U_{e3}(t)|^2$ moves

$$\begin{aligned} \mathcal{A}_{ee} &\simeq (|U_{e1}|^2 + |U_{e2}|^2) + |U_{e3}|^2 e^{-i(m_3^2-m_1^2)L/(2E)} \\ &= c_{13}^2(c_{12}^2 + s_{12}^2) + s_{13}^2 e^{-i(m_3^2-m_1^2)L/(2E)} \end{aligned}$$