Differential distributions at N^{3}LL+NNLO with `RadISH+NNLOJet`

Pier Monni
CERN

Mainly based on
"Fiducial distributions in Higgs and DY production at N^{3}LL+NNLO"
arXiv: 1805.05916, with
W. Bizon, X. Chen, Gehrmann-De Ridder, Gehrmann, Glover, A. Huss,
E. Re, L. Rottoli, and P. Torrielli

EW subgroup meeting - LAL Orsay 23 May 2018

Outline

- Predicting the small-p_T regime:
  - methods/tools - differences pertinent to this discussion
- Review of resummation in `RadISH` - relevant aspects
  - Theory and generation of radiation
  - Matching to fixed order and estimate of uncertainties
  - Treatment of Landau singularity
- Predictions for fiducial distributions at N^3LL+NNLO
  - resummation vs. fixed-order, uncertainties, and theory vs. data

Small-p_T spectrum

Resummations

- Collins, Soper, Sterman (CSS)
- Resummation in impact parameter space
- Codes: e.g.
  - DYRES (NNLL)
  - ResBos (NNLL)

[see talk by C.-P. Yuan]

- SCET
- Resummation in impact parameter/momentum space
- Codes: e.g.
  - CuTe (approx. N^3LL)

[see talks by V. Vaidya and F. Tackmann]

- Branching algorithm
- Resummation in momentum space
- Codes: e.g.
  - RadISH (N^3LL)

[this talk]

(N)LOPS

(LL most times for p_T, but fully exclusive)

Powheg+Py/Hw

Pythia

MC@NLO+Py/Hw

Sherpa

Geneva

- Logarithmic order *

LL NLL NNLL

$$\ln \Sigma(p_t) \equiv \ln \left(\int_0^{p_t} dp_t' \frac{d\Sigma}{dp_t'} \right) = \sum_n \left(\alpha_s^n \ln^{n+1}(M/p_t) + \alpha_s^n \ln^n(M/p_t) + \alpha_s^n \ln^{n-1}(M/p_t) + \dots \right)$$

- Resummations require matching to fixed-order; e.g. NLO (DYRES, CuTe, ResBos) or NNLO (RadISH+NNLOJet)

* In the case of p_T, a power-suppressed component is also present as $p_T \to 0$

All-order predictions: (practical) differences

- Regardless of the deep theoretical differences between the formulations (not relevant for this meeting), all above approaches can reach the same logarithmic accuracy.

- However, other differences persist in the default choices of the various approaches
  - *Resummation scheme* : starting from NNLL onwards, one has some freedom in deciding *how much* of the subleading corrections (i.e. beyond the nominal logarithmic accuracy) are either kept in the Sudakov exponent or are *expanded out* (not at fixed order!). Analogous differences arise from the scale setting procedure, e.g. at the differential/cumulative level (typically SCET), or from approximating the measurement function in a factorisation theorem,…
    - Each of the above codes uses its own scheme, hence inevitably leading to numerical (always logarithmically subleading) differences. The higher the order, the smaller the difference

  - *Scales/uncertainties* : resummation uncertainties (due to missing higher-order corrections) are estimated in different ways: e.g. resummation, renormalisation, factorisation scales (RadISH, DYRES), additional resummation scales (typical in SCET, sometimes used in CSS too)

  - *Turning off resummation effects at large p_T* : theoretically ambiguous, several choices are adopted; profile scales (e.g. SCET), modified b-space logarithms (e.g. DYRES), constraint on phase space of the radiation (e.g. RadISH). These effects are always regular (non-logarithmic) in pT, differences reduce with higher orders.

  - *Treatment of the Landau pole* : cutoff, b* prescription, NP models (e.g. gaussian smearing); differences are suppressed by powers of the c.o.m. energy

RadISH: a brief theory overview

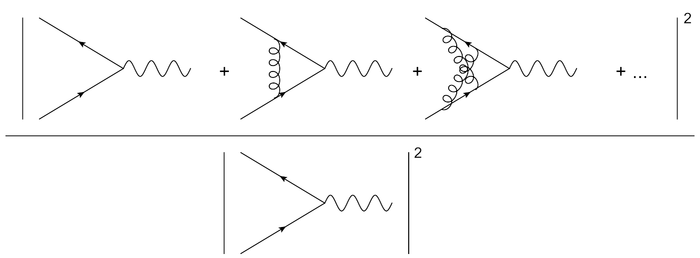
- The formulation is based on the concept of rIRC safety, that allows one to parametrise the all-order squared amplitudes in terms of lower-order building blocks, and to identify the precise phase space regions that contribute at a given logarithmic order

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

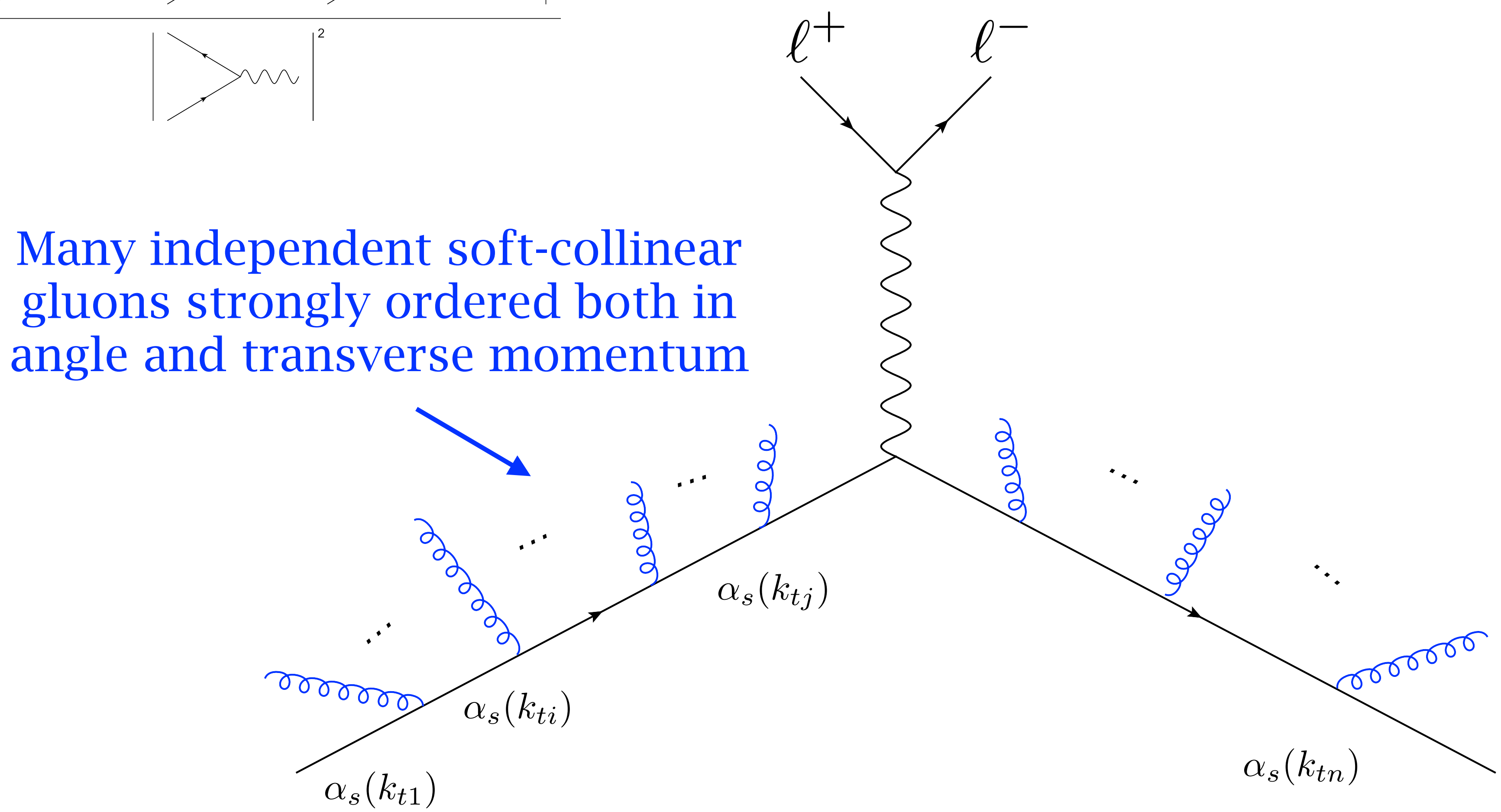
All-order form factor

$$V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$$

$\mathcal{V}(\Phi_B) =$



e.g. LL



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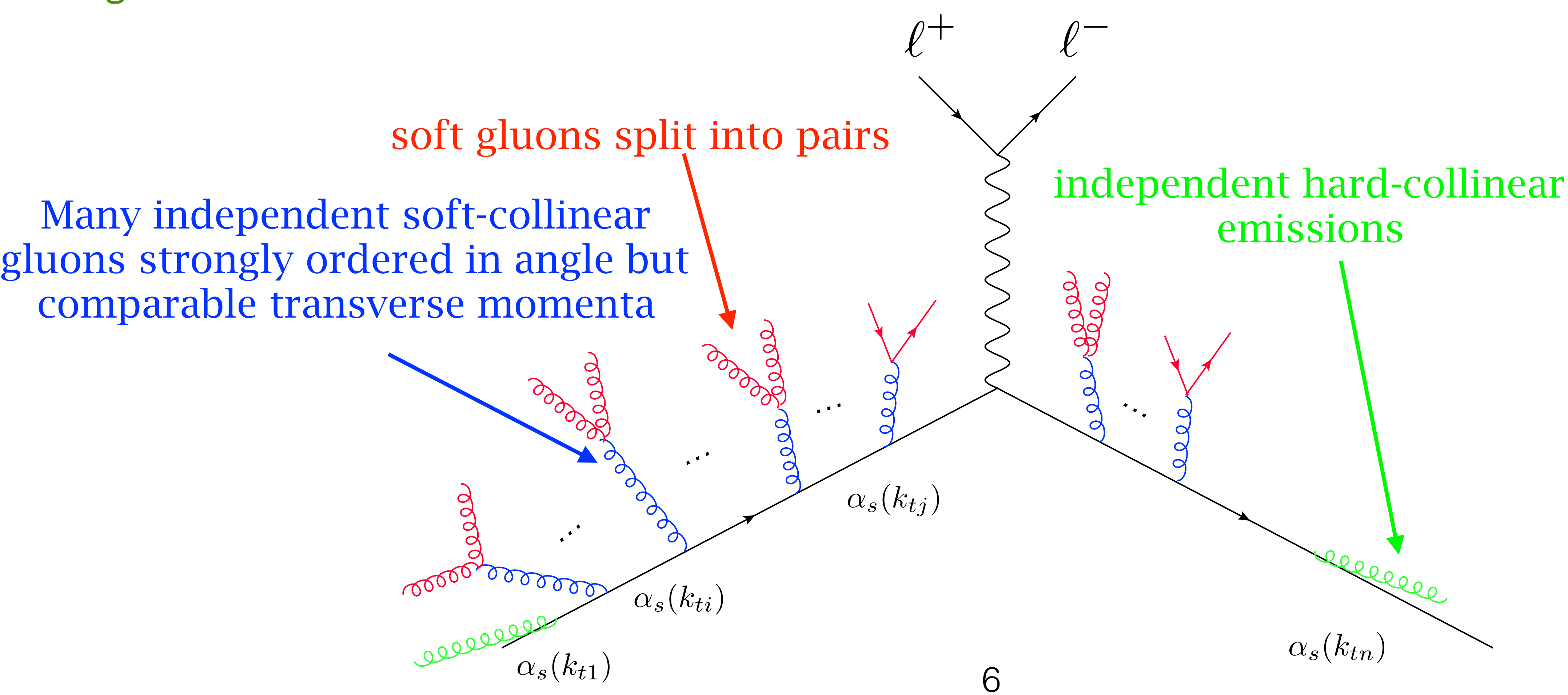
e.g. NLL



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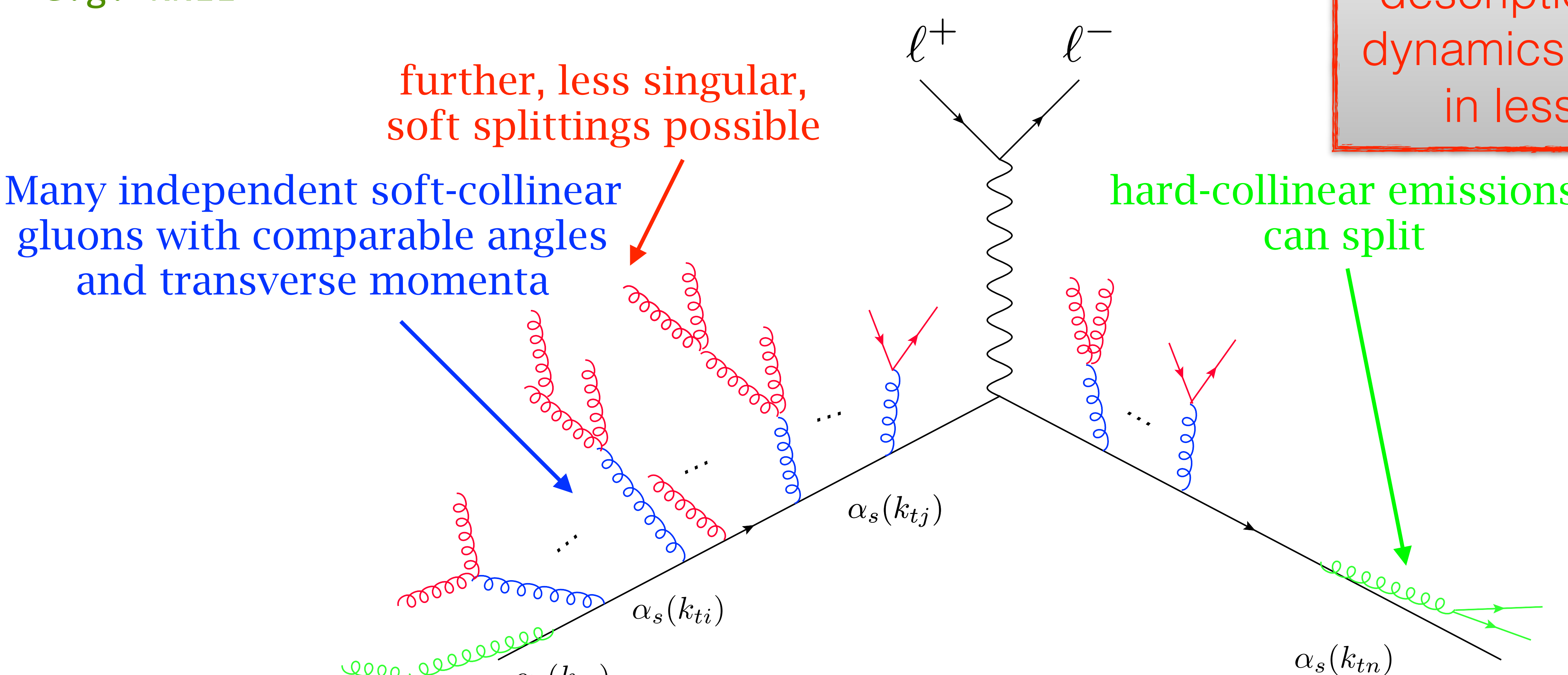
$$V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$$

Higher logarithmic order implies a more accurate description of the radiation dynamics and its kinematics in less singular limits

e.g. NNLL



RadISH: a brief theory overview

- Given that the observable is only sensitive to the total pt of the radiation, one can organise the radiation into clusters of at least one emission (defined at the squared-amplitude level - no jets involved) and integrate over the emissions within each cluster (including corresponding real-virtual corrections) *at fixed k_t and rapidity (up to four partons at N[3]LL)*. *Analytical* cancellation of corresponding IRC poles

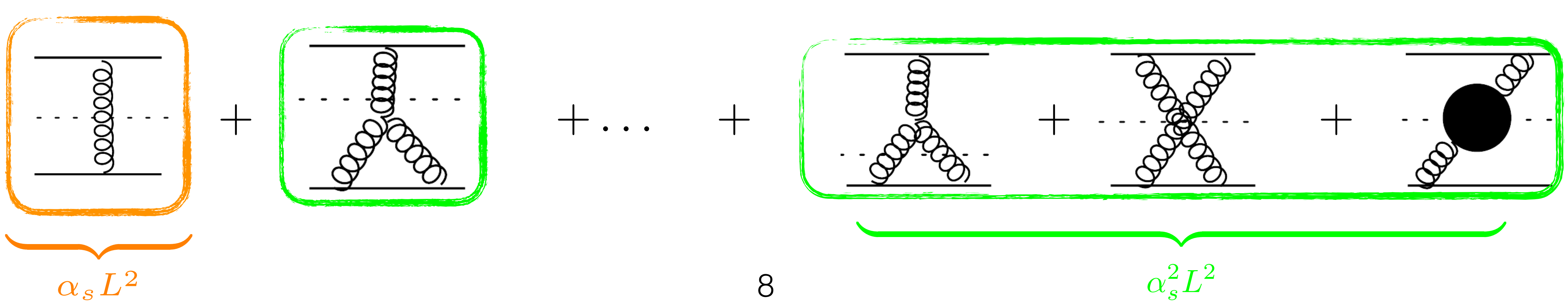
$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

$$\sum_{n=0}^{\infty} |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \longrightarrow |M_B(\tilde{p}_1, \tilde{p}_2)|^2$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left(|M(k_i)|^2 + \int [dk_a][dk_b] |\tilde{M}(k_a, k_b)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} - \vec{k}_{ti}) \delta(Y_{ab} - Y_i) \right.\right.$$

$$\left.\left. + \int [dk_a][dk_b][dk_c] |\tilde{M}(k_a, k_b, k_c)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} + \vec{k}_{tc} - \vec{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \right) \right\}$$

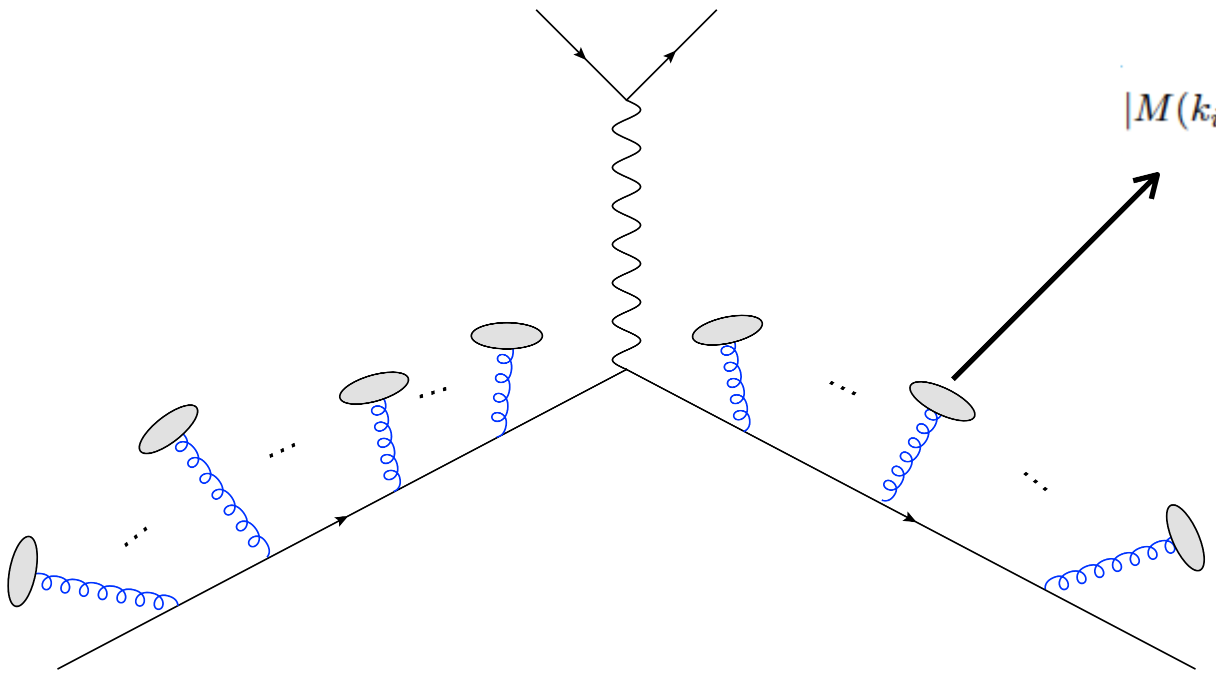
e.g. NLL



RadISH: a brief theory overview

- The events are then obtained by generating a ISR shower of such clusters of emissions

  ➡ e.g. gluon emissions off quark legs



$$|M(k_i)|^2 + \int [dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2 \delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)$$
$$+ \int [dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2 \delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i) + \dots$$

The integrated emission probabilities up to N³LL can be expressed in terms of known anomalous dimensions/ coefficient functions:

[Catani, Grazzini ’11][Catani et al. ‘12]
[Gehrmann, Luebbert, Yang ‘14]
[De Florian, Grazzini ’01][Davies, Stirling ‘84]
[Li, Zhu ’16][Vladimirov ’16]

- When a cluster gets unresolved, one needs to cancel the IRC singularities against the ones in the virtual corrections to the quark form factor $\mathcal{V}(\Phi_B)$. This is done by means of a phase space slicing:
  - clusters with total transverse momentum smaller than $Q_0 = \epsilon k_{t1}$ (*unresolved*) are handled analytically and combined with the form factor : this defines a no-emission probability (Sudakov radiator)

    (Transverse momentum of hardest cluster → k_{t1})
  - *resolved* clusters are generated numerically with a Monte Carlo algorithm, similar in spirit to a (*semi-inclusive*) parton shower. The limit $\epsilon \to 0$ is numerically stable (cutoff at $\epsilon \sim e^{-20}$)

- In the resolved radiation one has $k_{ti} \sim k_{t1}$, which ultimately sets the boundary with the NP regime

Implementation: recoil of leptonic system

- The generation starts with fully differential Born (i.e. p p -> Z) events, which are *then* showered if they pass the phase-space cuts

$$d\Sigma = d\sigma_{\text{Born}} \otimes d\sigma_{\text{Radiation}}$$

No kinematic cross talk (i.e. lepton momenta are left untouched by the shower)

$$d\Sigma = d\sigma_{\text{Born}} \times d\sigma_{\text{Radiation}}$$

- Formally speaking, the resummation is valid in a Born-like kinematics. Therefore we decide to preserve the Born event, while modifying the transverse momentum of the Z boson.

- This violation of momentum conservation is regular in p_T, and hence ambiguous in the resummed event. For this reason it is avoided in the following (technically recovered in matching to fixed-order at a later stage)

    - If necessary it could be easily implemented (e.g. through a simple boost). We consider including this feature in future work

- As a consequence, the action of fiducial cuts as well as the definition of dynamical scales will differ from the fixed-order counterpart unless p_T -> 0 (more on this later)

- RadISH generates N^3LL events for the resummed cross section, and for its fixed-order expansion (used in matching). The matching to fixed-order is performed at the *histogram level*

Matching to fixed order

- Away from the $p_T \to 0$ limit, regular terms due to momentum conservation as well as exact matrix elements become relevant. These are restored at fixed-order in perturbation theory via a two-step matching procedure. The NNLO fixed-order is provided by `NNLOJet`
[Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, A. Huss, T.A. Morgan ’16]
- Firstly, we want to ensure that the resummation does not affect the high-p_T tail of the spectrum
  - modify the available rapidity range for each cluster so that the corresponding phase space closes up at large p_T. This is realised by means of the following replacement

$$|\eta_i| \lesssim \ln\frac{Q}{k_{t1}} \to \frac{1}{p}\ln\left(\frac{Q^p}{k_{t1}^p}+1\right) \qquad \longrightarrow \qquad \frac{d\Sigma}{dp_t} \sim \frac{1}{p_t^{1+p}}\,, \quad \text{for } p_t \gg Q$$

This induces (only) regular terms in the p_T spectrum

  - p is a free parameter in the calculation
- combine the resulting *histograms* for the resummed cumulative distribution and its fixed-order expansion with the fixed-order counterpart at (ideally) N^3LO, i.e.

$$\Sigma_{\rm FO}(p_t) = \sigma^{\rm N^3LO} - \Sigma^{\tt NNLOJet}(p_T > p_t) \qquad \Sigma_{\rm MAT}(p_t) = \frac{\Sigma_{\rm RES}(p_t)}{\mathcal{L}(\mu_F)}\left[\mathcal{L}(\mu_F)\frac{\Sigma_{\rm FO}(p_t)}{\Sigma_{\rm EXP}(p_t)}\right]_{\rm EXPANDED}$$

Asymptotic value of $\Sigma_{\rm RES}(p_t)$

We set the N^3LO correction to zero in the DY case (currently unknown). This ambiguity is subleading (N^4LL) in the differential distribution

Choice of matching scheme is also ambiguous. Difference between *multiplicative* and *additive* solution found to be small

$$\Sigma_{\rm MAT}(p_t) = \Sigma_{\rm RES}(p_t) + \Sigma_{\rm FO}(p_t) - \Sigma_{\rm EXP}(p_t)$$

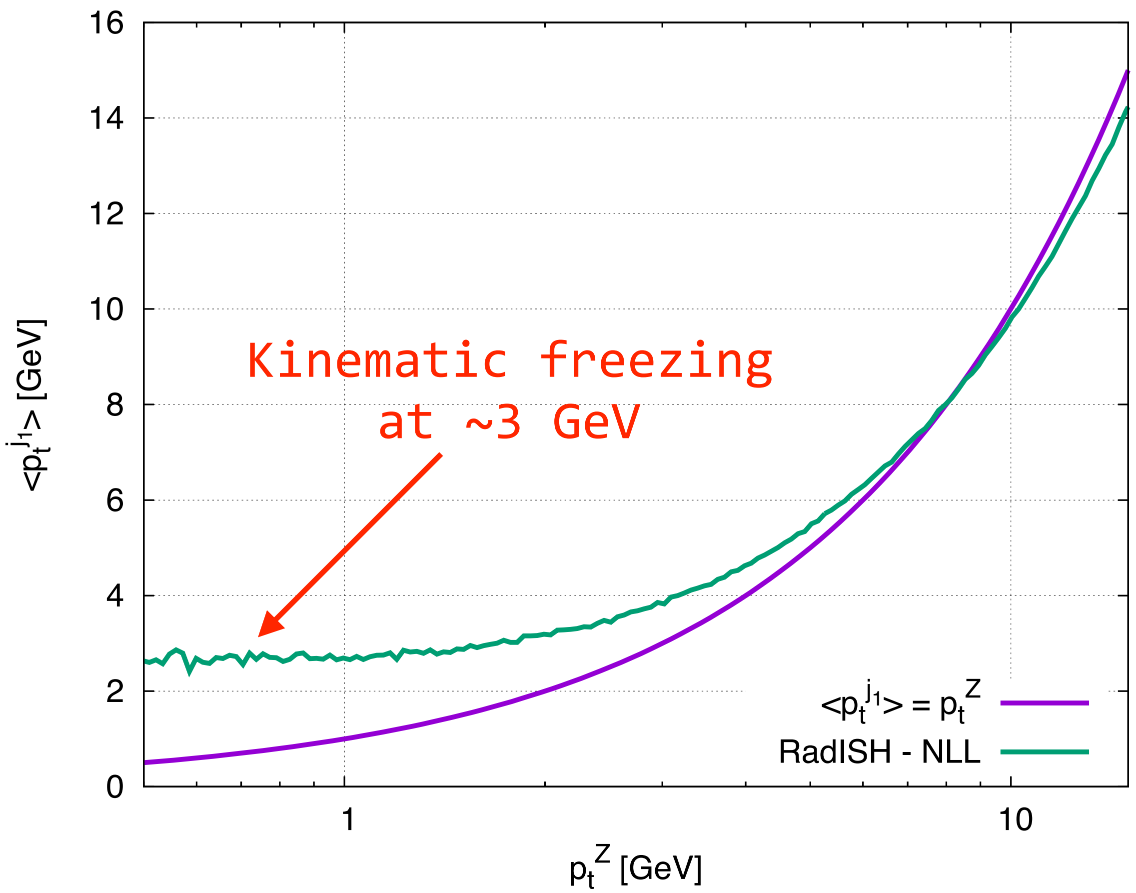
Small-p_T limit and Landau pole

- The strong coupling in the Sudakov radiator and in the emission probabilities features a Landau singularity at

$$\alpha_s(\mu_R)\beta_0 \ln \frac{Q}{k_{t1}} = \frac{1}{2} \longrightarrow k_{t1} \simeq 0.09\,\mathrm{GeV}\ (\text{for } \mu_\mathrm{R} = \mathrm{Q} = \mathrm{M_Z})$$

- We cut off the region below this scale by setting the emission probability to zero; parton densities are also frozen. This is the only actual cutoff in the calculation.

- In practice, this cutoff is (almost) *never* an issue :
  - the small-pT limit is dominated by events with $k_{ti} \sim k_{t1} \gg p_t$ (power-like spectrum)



➡ e.g. in RadISH, at LL

$$\frac{d^2\Sigma(p_t)}{dp_t d\Phi_B} \simeq 4p_t\sigma^{(0)}(\Phi_B)\int_{\Lambda_\mathrm{QCD}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2p_t\sigma^{(0)}(\Phi_B)\left(\frac{\Lambda^2_\mathrm{QCD}}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}}$$

NB: this fact does not imply that there are no NP corrections at scales of order ~ GeV, i.e. from intrinsic k_t, not estimated here

Computational setup

- The results below are obtained with the following fiducial cuts, according to the 8 TeV ATLAS measurement

$$p_t^{\ell^\pm} > 20 \text{ GeV}, \qquad |\eta^{\ell^\pm}| < 2.4, \qquad |Y_{\ell\ell}| < 2.4, \qquad 46 \text{ GeV} < M_{\ell\ell} < 150 \text{ GeV}$$

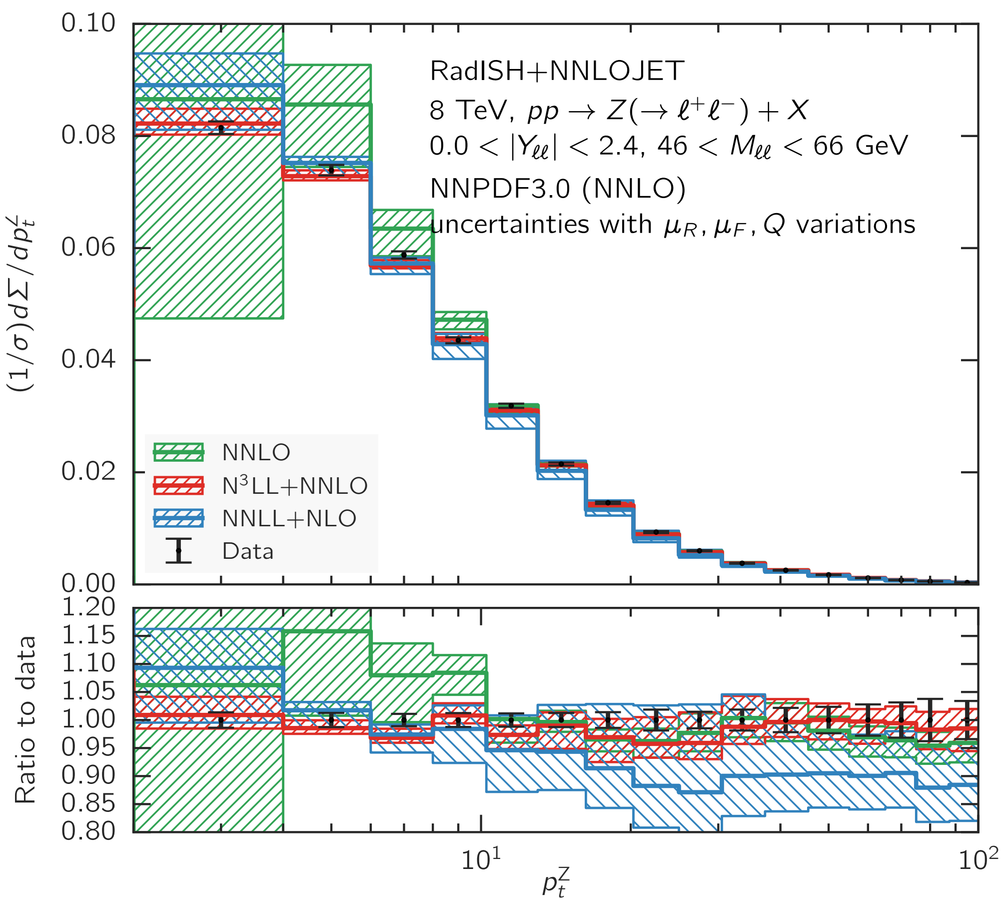
- The PDF set is NNPDF3.0 and the central scales are

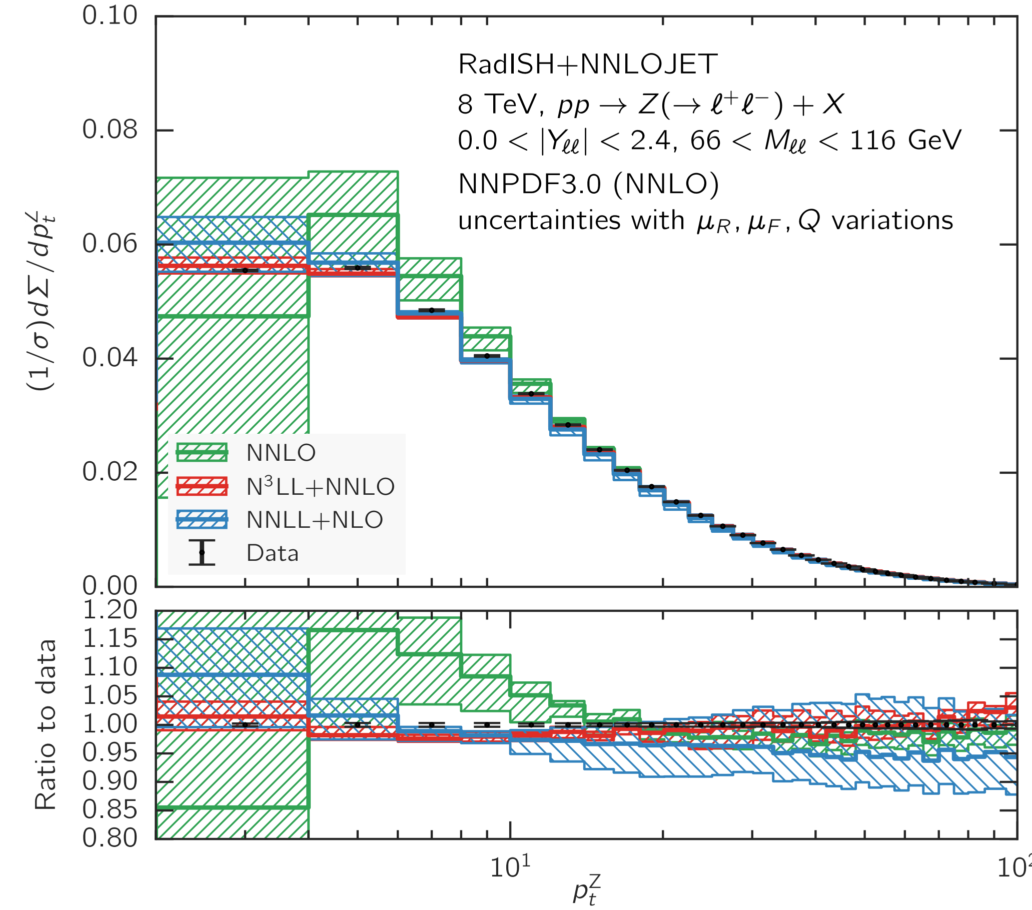
$$\mu_R = \mu_F = M_T = \sqrt{M_{\ell\ell}^2 + p_t^2},\ Q = \frac{M_{\ell\ell}}{2}$$

- We always work with 5 active (massless) flavours, i.e. neither HQ effects nor thresholds in PDFs
- Uncertainties are estimated as the envelope of the following variations :
    - 7-point variation of renormalisation/factorisation scales of a factor of two about their central value
    - For central renormalisation and factorisation scales, the resummation scale is varied by a factor of two around its central value
    - By default we set *p=4* , and checked that a variation by one unit does not produce significant differences

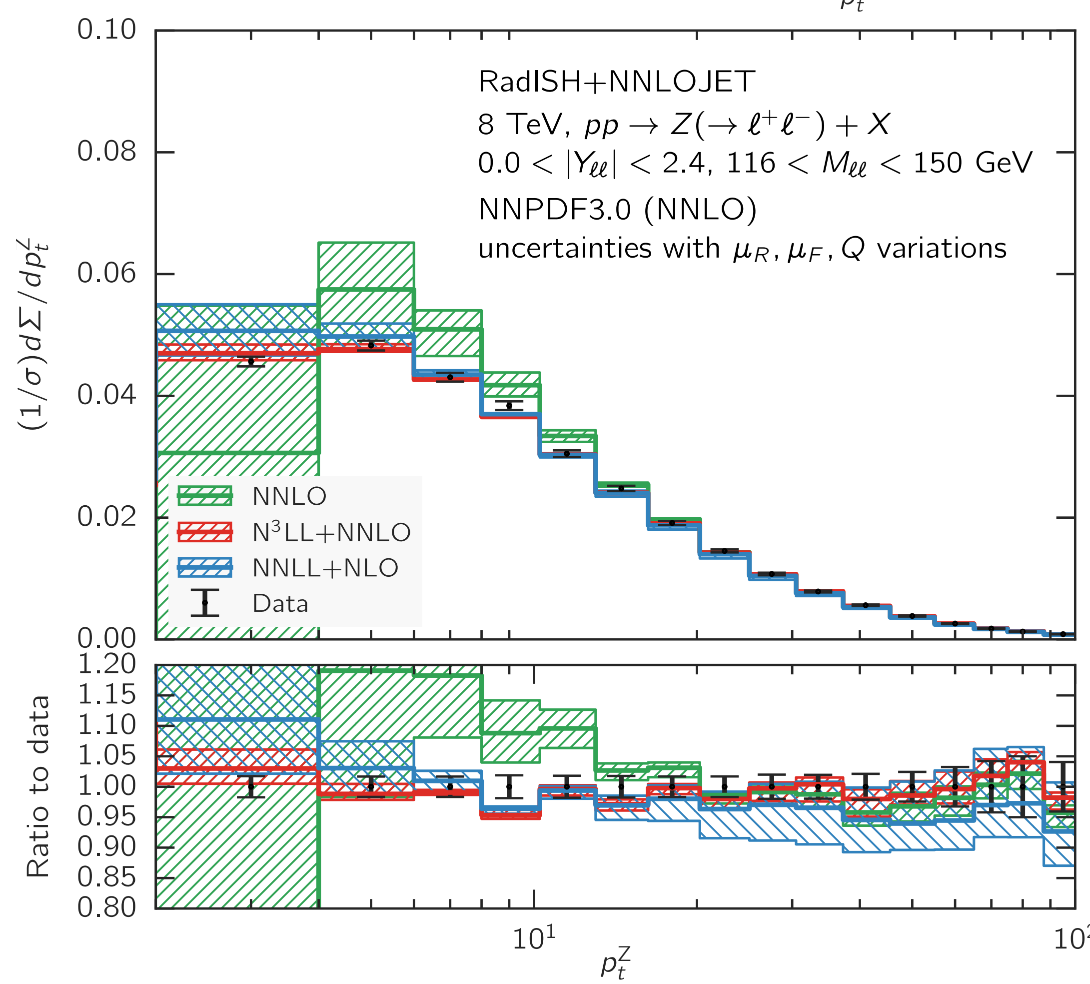
[data from ATLAS 1512.02192]

p_T^{ll} distribution



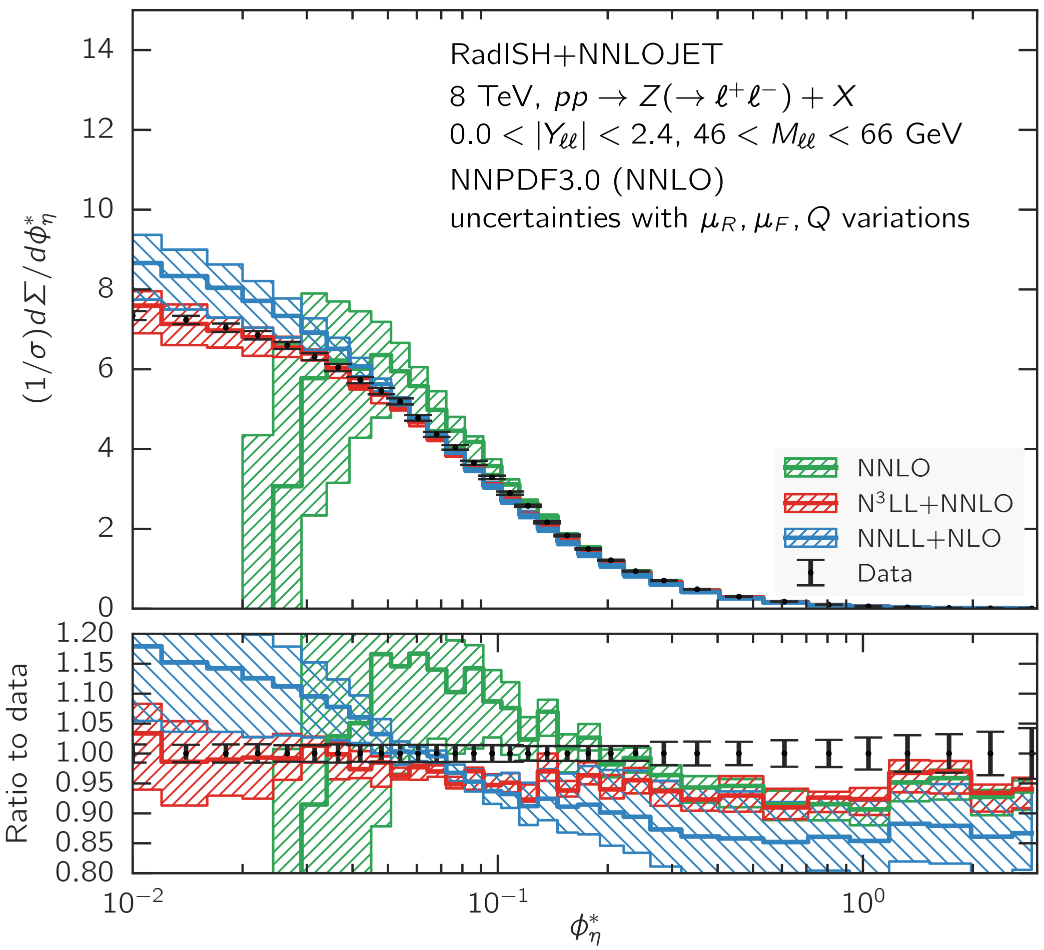


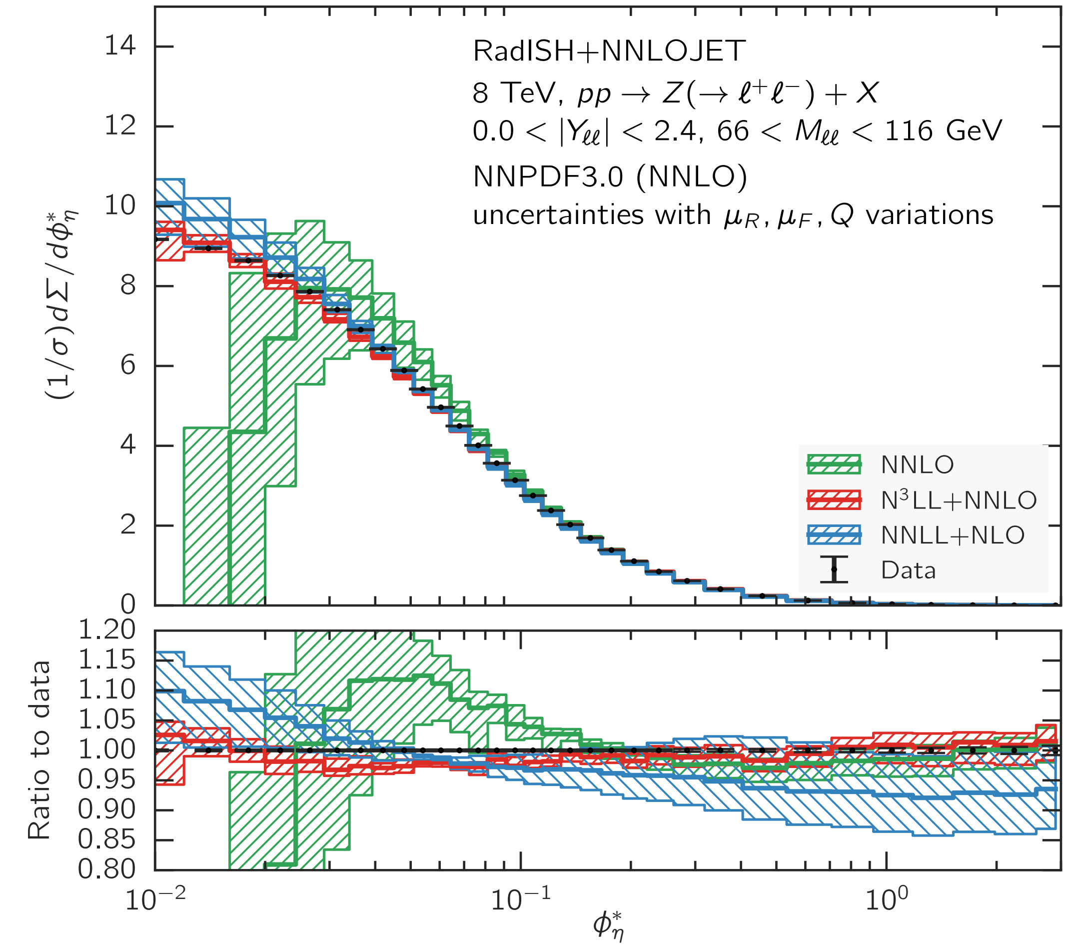
- ➡ Good convergence of perturbative series
- ➡ Residual uncertainty in the 3-5% range
- ➡ Inclusion of N^{3}LL+NNLO corrections leads to shape distortion and better agreement with data
- ➡ Similar conclusions when one gets more exclusive in the Z rapidity (see paper)

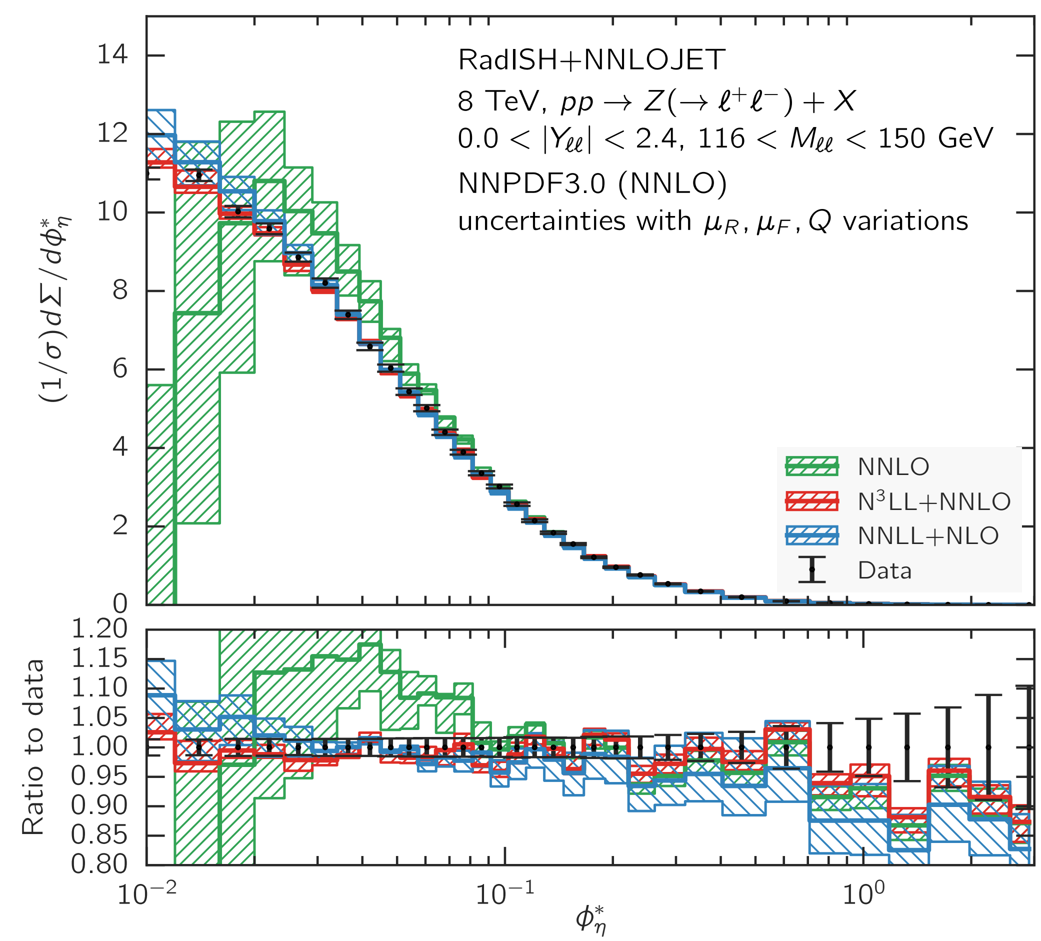


[data from ATLAS 1512.02192]

ϕ^*_η distribution





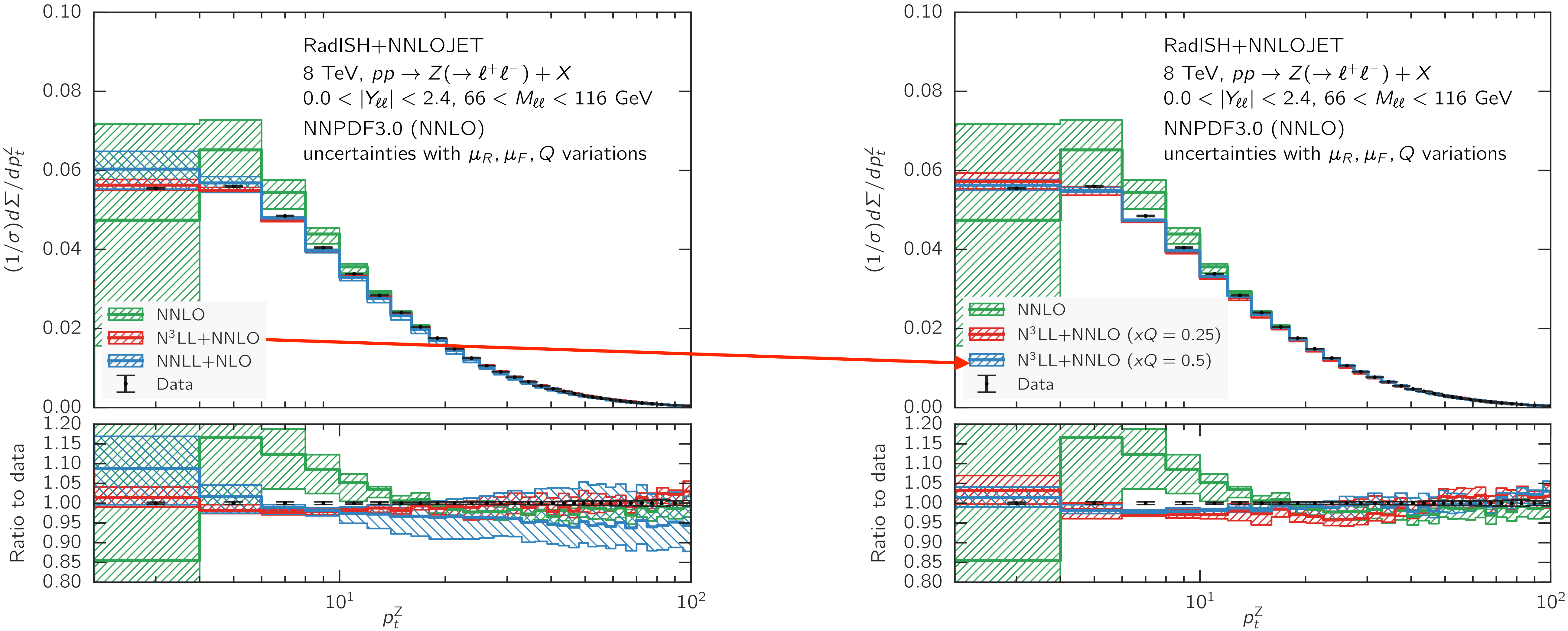


➡ Similar conclusions for angular observables, except for low invariant mass (already at NNLO)

➡ Small residual perturbative uncertainty

➡ Many more plots in the paper

Choice of the resummation scale

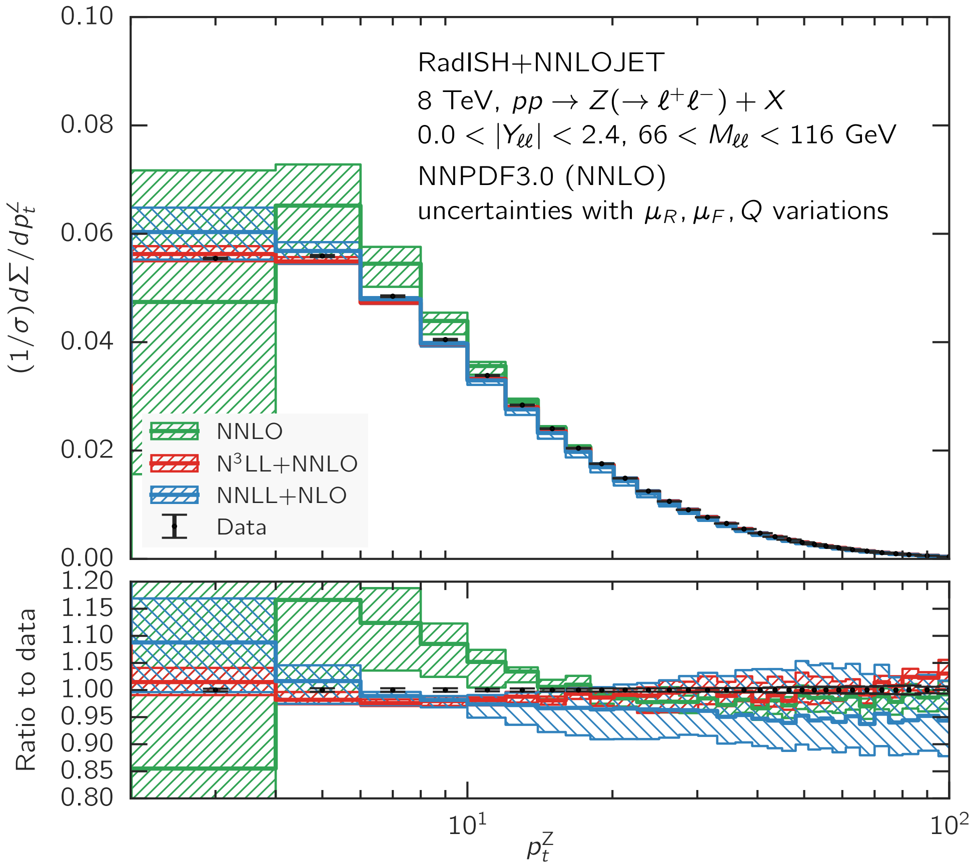


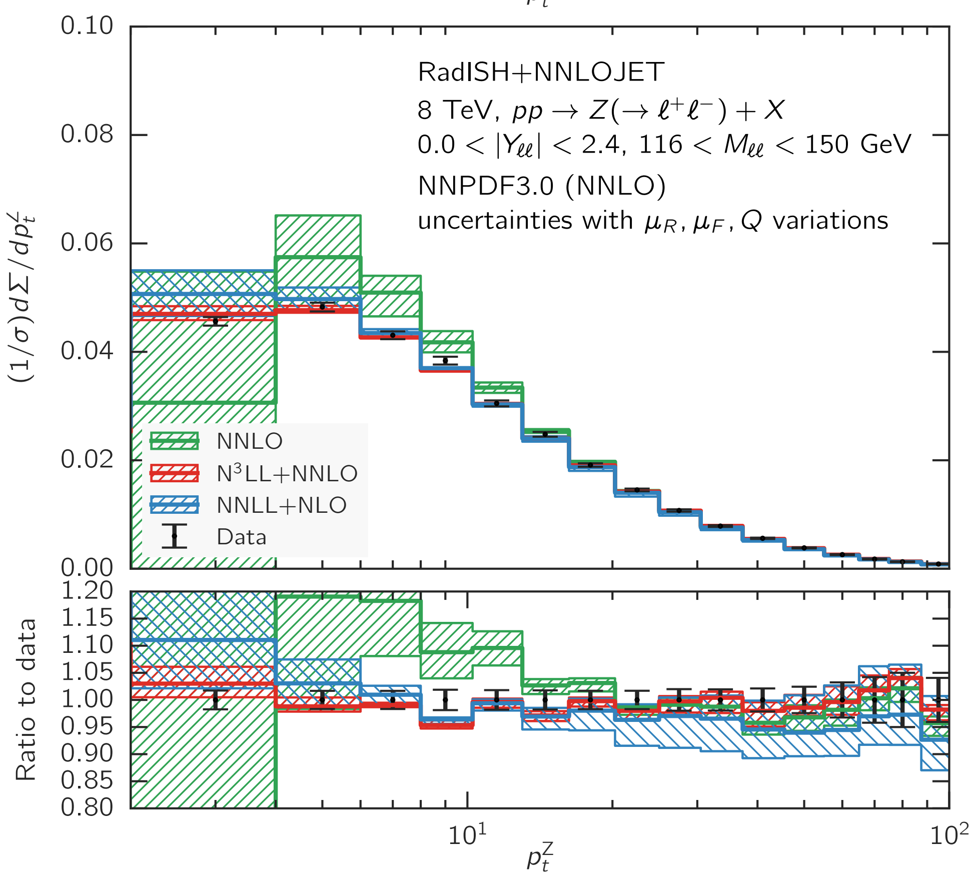
➡ Small dependence on the central resummation scale - agreement within uncertainties

➡The choice of p ensures that the resummation vanishes at a rate slightly faster than the fixed-order one

[data from ATLAS 1512.02192]

Matching vs. Resummation



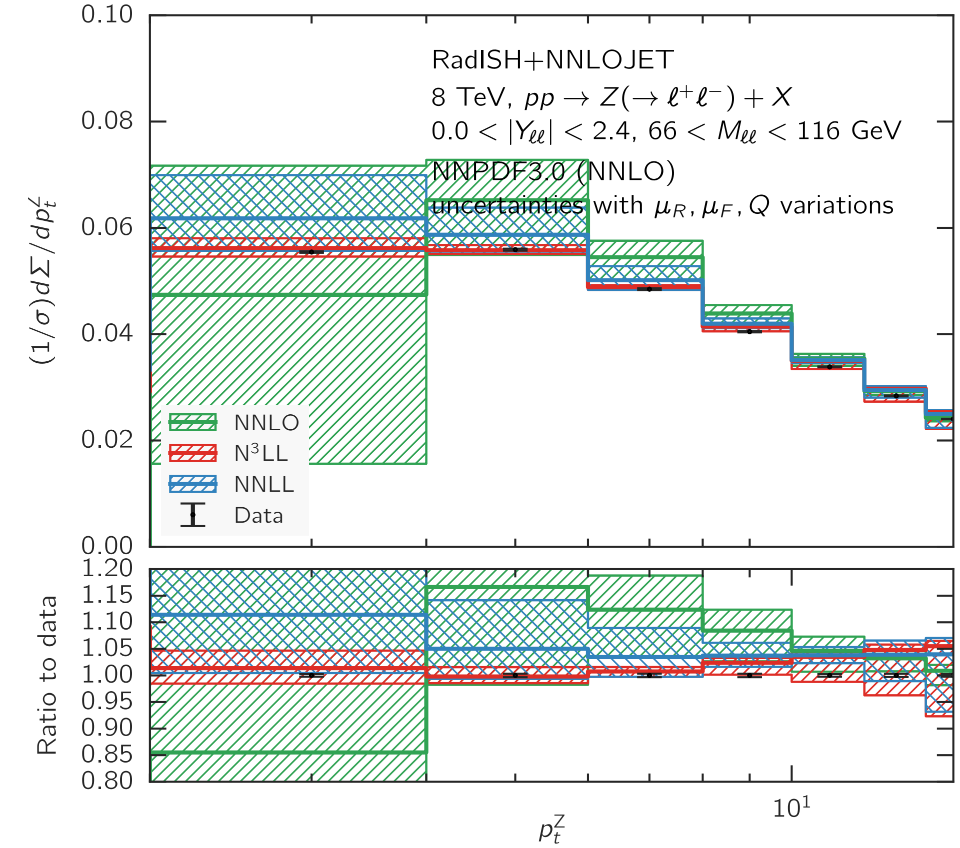


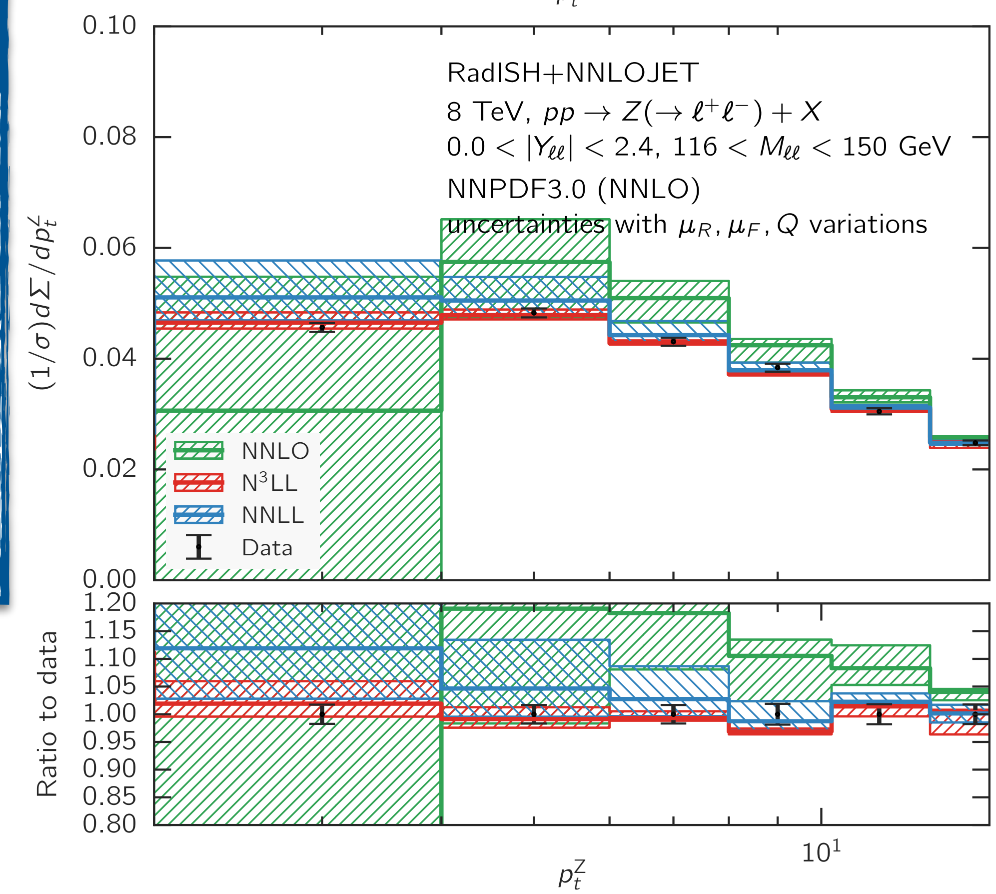
➡ NNLO alone works well above ~ 20 GeV, and it gives % corrections even for p_T < 10 GeV

➡ In presence of fiducial cuts, the F.O. converges to the logarithmic expansion very late (difference between Z+jets and Z phase space)

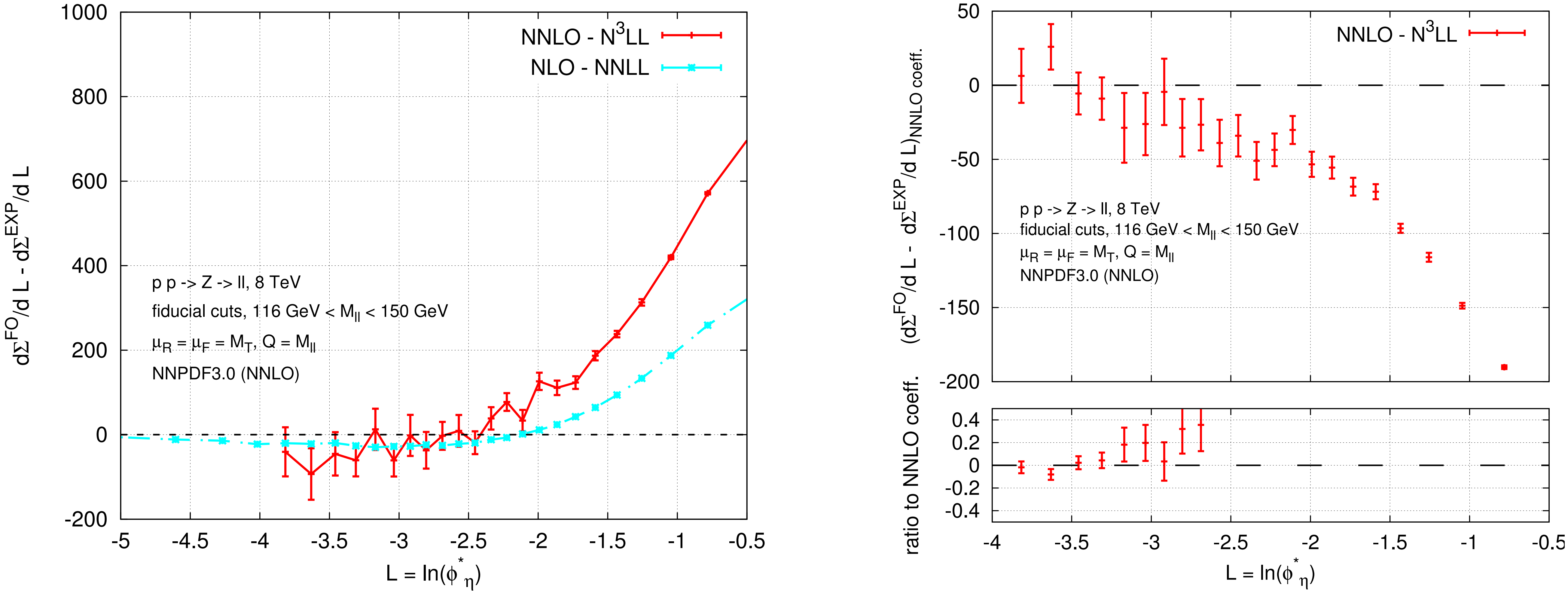
➡ A faster convergence has been observed at the level of inclusive Z (also w/ fixed scales)

➡ This effect is less pronounced at larger invariant masses (larger logarithms)





Fixed order vs. Resummation



➡ With fiducial cuts, the fixed-order converges to the logarithmic expansion very late (difference between Z+jets and Z phase space)

➡ NNLO coefficient converges slightly faster due to higher logarithmic powers

➡ Convergence could be shifted towards larger p_T values with an implementation of recoil in the lepton kinematics

Conclusions

- State of the art predictions for (Higgs and) DY distributions at N^{3}LL+NNLO obtained with `RadISH` matched to `NNLOJet`

- Residual perturbative uncertainties at the few-% level in different fiducial distributions, and perturbative results in good agreement with the data

  - We do not include effects of quark masses nor of non-perturbative corrections - both relevant at this level of precision

- The calculation is fully differential in the Born kinematics. However, for the time being the distribution of Born variables is by construction identical to the fixed-order prediction

  - The recoil due to the all-order radiation can be *propagated* to the final-state leptons, although the prescription is by definition ambiguous (needs inclusion of *full* power corrections). This will be done in future work

- Interesting to repeat the study for W production and p_T^Z/P_T^W ratio

Thank you for listening

Squared amplitude decomposition

- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

Real emissions

- Recast all-order squared ME for *n* real emissions as iteration of *correlated blocks*
- Scaling of the observable in the presence of radiation *must* preserve the above hierarchy

e.g. soft radiation (analogous considerations for hard-collinear)

$$|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \left(\frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right) + \left[\sum_{a>b} \frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 + \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots \right] + \left[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots \right] + \ldots \right\},$$

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$$\left[\sum_{a>b} \frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b)\right|^2 + \right.$$

$$\left. \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b)\right|^2 \left|\tilde{M}(k_c, k_d)\right|^2 + \ldots \right]$$

$$\left. + \left[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b, k_c)\right|^2 + \ldots \right] + \ldots \right\},$$

Squared amplitude decomposition

- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

Real emissions

- Recast all-order squared ME for *n* real emissions as iteration of *correlated blocks*
- Scaling of the observable in the presence of radiation *must* preserve the above hierarchy

e.g. soft radiation (analogous considerations for hard-collinear)

$$|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \underbrace{\left(\frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right)}_{\text{LL}} + \left[\sum_{a>b} \underbrace{\frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2}_{\text{NLL}} + \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \underbrace{\frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots}_{\text{NLL}} \right] + \left[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots \right] + \ldots \right\},$$

Squared amplitude decomposition

- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

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$$\left[\sum_{a>b} \underbrace{\frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2}_{\text{NLL}} + \right.$$

$$\left. \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \underbrace{\frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots}_{\text{NLL}} \right]$$

$$\left. + \left[\sum_{a>b>c} \underbrace{\frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots}_{\text{NNLL}} \right] + \ldots \right\},$$

Monte Carlo formulation

- One great simplification: choice of the resolution variable such that correlated blocks entering at N^kLL in the unresolved radiation only contribute at $N^{k+1}LL$ in the resolved case

- i.e. we can expand out the cutoff dependence and retain in the Radiator only the terms necessary to cancel the singularities in the resolved radiation

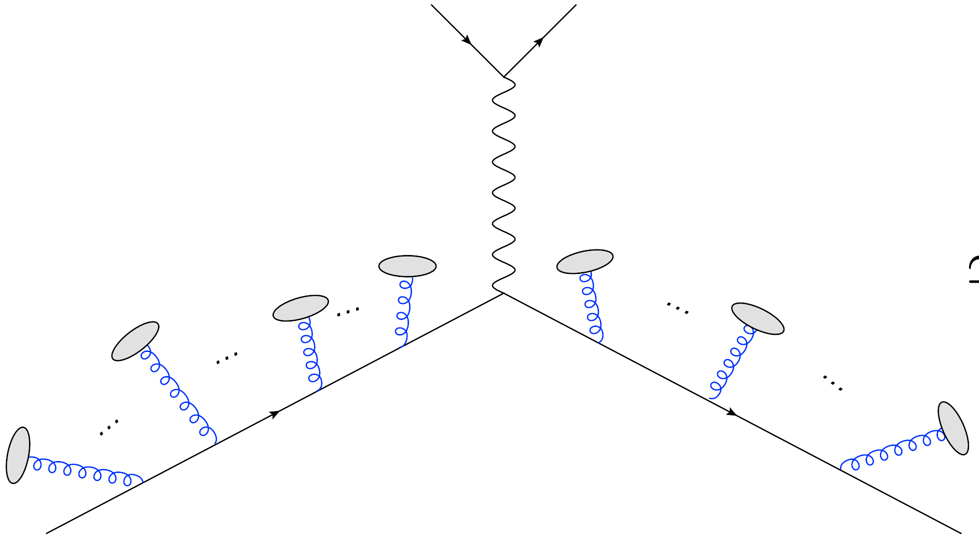
$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln \frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2 \frac{1}{\epsilon} + \ldots$$

$$R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln \frac{k_{t1}}{k_{ti}} + \ldots$$

Expansion is safe since in the resolved radiation

$k_{t1}/k_{ti} \sim 1$

e.g. at NLL



$$\simeq \int \frac{dk_{t1}}{k_{t1}} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}(k_{t1}) \right) \int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}))$$

$$\int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1})$$

Monte Carlo formulation

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$$R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln \frac{k_{t1}}{k_{ti}} + \dots$$

Expansion is safe since in the resolved radiation $k_{t1}/k_{ti} \sim 1$

- Corrections beyond NLL are obtained as follows
  - Add subleading effects in the Sudakov radiator and constants
  - Correct *a fixed number* of the NLL resolved emissions:
    - only one at NNLL
    - two at $\mathrm{N^3LL}$
    - …

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) = & \sum_{c,c'} \frac{d|M_B|^2_{cc'}}{d\Phi_B} \sum_{i,j} \int_{x_1}^{1} \frac{dz_1}{z_1} \int_{x_2}^{1} \frac{dz_2}{z_2} f_i\left(k_{t1}, \frac{x_1}{z_1}\right) f_j\left(k_{t1}, \frac{x_2}{z_2}\right) \\
& \Bigg\{ \delta_{ci}\delta_{c'j}\delta(1-z_1)\delta(1-z_2) \left(1 + \frac{\alpha_s(\mu_R)}{2\pi} H^{(1)}(\mu_R) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(2)}(\mu_R)\right) \\
& + \frac{\alpha_s(\mu_R)}{2\pi} \frac{1}{1 - 2\alpha_s(\mu_R)\beta_0 L} \left(1 - \alpha_s(\mu_R) \frac{\beta_1}{\beta_0} \frac{\ln(1 - 2\alpha_s(\mu_R)\beta_0 L)}{1 - 2\alpha_s(\mu_R)\beta_0 L}\right) \\
& \times \left(C_{ci}^{(1)}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \\
& + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1 - 2\alpha_s(\mu_R)\beta_0 L)^2} \Bigg(\left(C_{ci}^{(2)}(z_1) - 2\pi\beta_0 C_{ci}^{(1)}(z_1) \ln\frac{M^2}{\mu_R^2}\right) \delta(1-z_2)\delta_{c'j} \\
& + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\} \Bigg) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1 - 2\alpha_s(\mu_R)\beta_0 L)^2} \left(C_{ci}^{(1)}(z_1) C_{c'j}^{(1)}(z_2) + G_{ci}^{(1)}(z_1) G_{c'j}^{(1)}(z_2)\right) \\
& + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(1)}(\mu_R) \frac{1}{1 - 2\alpha_s(\mu_R)\beta_0 L} \left(C_{ci}^{(1)}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \Bigg\}
\end{aligned}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$k_{ti}/k_{t1} = \zeta_i = \mathcal{O}(1)$$

$$\int d\mathcal{Z}[\{R', k_i\}] G(\{\tilde{p}\}, \{k_i\}) = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1}) G(\{\tilde{p}\}, k_1, \ldots, k_{n+1})$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions dZ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NNLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \Big(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) \Big)$$

$$\times \left(R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2 \frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s} \right)$$

$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions *dZ* is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to N³LL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1})\right)$$

$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) \right)$$
$$\times \left(R''(k_{t1}) \ln\frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln\frac{1}{\zeta_s} \right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1})\right) \Big\}$$

$$+ \frac{1}{2} \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}} \frac{d\phi_{s1}}{2\pi} \int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}} \frac{d\phi_{s2}}{2\pi} R'(k_{t1})$$
$$\times \Bigg\{ \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \left(R''(k_{t1}) \right)^2 \ln\frac{1}{\zeta_{s1}} \ln\frac{1}{\zeta_{s2}} - \partial_L \mathcal{L}_{\mathrm{NLL}}(k_{t1}) R''(k_{t1}) \left(\ln\frac{1}{\zeta_{s1}} + \ln\frac{1}{\zeta_{s2}} \right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\}$$
$$\times \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1}, k_{s1}, k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1}, k_{s1})\right) -$$
$$\Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1}, k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1})\right) \Big\} + \mathcal{O}\left(\alpha_s^n \ln^{2n-6} \frac{1}{v} \right)$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions $d\mathcal{Z}$ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Equivalence to CSS formula

- Hard-collinear emissions off initial-state legs require some care in the treatment of kinematics. Final result reads

$$\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \frac{d\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)}{dp_t} \mathbf{f}_{N_2}(\mu_0)$$

$$\begin{aligned}\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = &\left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \\ &\times e^{-\mathbf{R}(\epsilon k_{t1})} \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\} \\ &\sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right) \\ &\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi}\mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right) \\ &\times \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right),\end{aligned}$$

- Formulation equivalent to b-space result, up to a scheme change. Using the delta representation for the distribution one finds

$$\delta^{(2)}(\vec{p_t} - (\vec{k}_{t1} + \ldots + \vec{k}_{tn})) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p_t}} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{ti}}$$

$$\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \frac{d\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)}{dp_t} \mathbf{f}_{N_2}(\mu_0) =$$

$$(1 - J_0(bk_t)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12} \frac{\partial^3}{\partial \ln(Mb/b_0)^3} \Theta(k_t - \frac{b_0}{b}) + \ldots$$

$$\begin{aligned}= &\sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b) \mathbf{C}^{c_1;T}_{N_1}(\alpha_s(b_0/b)) H(M) \mathbf{C}^{c_2}_{N_2}(\alpha_s(b_0/b)) \mathbf{f}(b_0/b) \\ &\times \exp\left\{-\sum_{\ell=1}^{2} \int_0^M \frac{dk_t}{k_t} \mathbf{R}'_\ell(k_t)(1 - J_0(bk_t))\right\}.\end{aligned}$$