NNLLA BFKL and Regge cuts

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Contents

The BFKL equation was derived for summation of radiative corrections to elastic scattering amplitudes of non-abelian gauge theories (NAGT) in the Regge limit ($s \gg |t|$) in the leading logarithmic approximation (LLA).

The equation was derived using the gluon Reggeization hypothesis, which was proved lately.

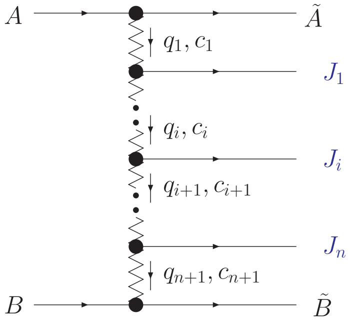
In the BFKL approach the primary Reggeon is the Reggeized gluon.
The Pomeron, which determines the high energy behaviour of cross sections, appears as a compound state of two Reggeized gluons, and the Odderon, responsible for the difference of particle and antiparticle cross sections, as a compound state of three Reggeized gluons.

The same is true in the NLLA.
Unfortunately, in the NNLLA it is not the case.

The derivation is based on the *s*-channel unitarity and analyticity and the Regge form of amplitudes with negative signature and adjoint representation of the gauge group in cross channels.
The unitarity was used for calculation of discontinuities of elastic amplitudes, and analyticity for their full restoration.

Use of the *s*-channel unitarity requires knowledge of multiple production amplitudes in the MRK.

The assumption was made that all amplitudes in the unitarity relations for elastic amplitudes, both elastic and inelastic, are determined by the Regge pole contributions. With this assumption, the *s*-channel discontinuities of the elastic amplitudes can be presented as the convolution in the transverse momentum space of energy independent impact factors of colliding particles, describing their interaction with Reggeons, and the Green's function *G* for two interacting Reggeons, which is universal (process independent).



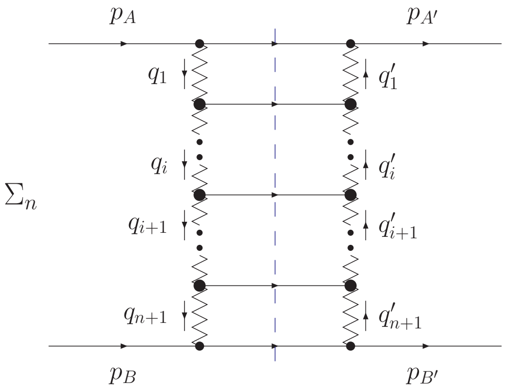
$$\Re\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n}$$

$$= 2s\Gamma_{\tilde{A}A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{J_i}(q_i, q_{i+1})\left(\frac{s_i}{s_0}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_0}\right)^{\omega(t_{n+1})}\Gamma_{\tilde{B}B}^{c_{n+1}}$$

The hypothesis is extremely powerful:

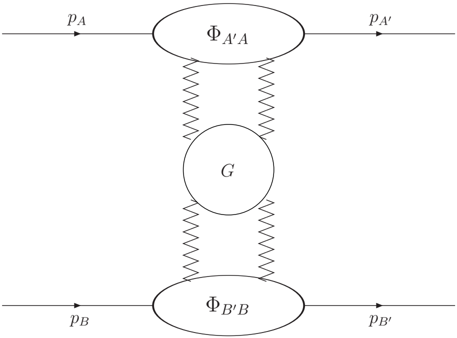
- It allows us to express scattering amplitudes only through several effective vertices and gluon trajectory.
- It creates the basis of the BFKL approach to the theoretical description of high energy scattering.
- The Pomeron and Odderon in QCD appear as the compound state of the Reggeized gluons.

s-channel discontunuities of elastic amplitudes are calculated using the *s*channel unitarity



They can be presented in the form :

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}.$$



Impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe transitions $A \to A'$ $B \to B'$,
G – Green's function for two interacting Reggeized gluons,

$$\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}},$$

$\hat{\mathcal{K}}$ – BFKL kernel, $Y = \ln(s/s_0)$,

$$\hat{\mathcal{K}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r$$

$$\hat{\mathcal{K}}_r = \hat{\mathcal{K}}_G + \hat{\mathcal{K}}_{Q\bar{Q}} + \hat{\mathcal{K}}_{GG}$$

Energy dependence of scattering amplitudes is determined by the BFKL kernel.

The virtual contrubution to the kernel

$\hat{\omega}_1 + \hat{\omega}_2$

is universal (does not depend on $\mathcal{R}$).

The contributions of real production $\hat{\mathcal{K}}_r$ differ for different representations by coefficients.

In the leading orders corresponding coefficients are

$$c_1 = 1\,, \quad c_{8_a} = c_{8_s} = \frac{1}{2}\,, \quad c_{10} = c_{\overline{10}} = 0\,, \quad c_{27} = -\frac{1}{4N_c}$$

In the next-to leading order there are two coefficients, corresponding contributions of planar a_R and nonplanar b_R diagrams. In particular,

$$a_0 = 1\,, \quad a_{8_a} = a_{8_s} = \frac{1}{4}, \quad b_1 = 1/2, \quad b_{8_a} = b_{8_s} = 0.$$

The last equality is especially important for the antisymmetric case, since the vanishing of b_{8_a} is crucial for the gluon Reggeization.

It turns out that all amplitudes in the unitarity relations are determined by the Regge pole contributions also in the NLLA.

The reason is that in this approximation one of two amplitudes in the unitarity relations can lose $\ln s$, while the second one must be taken in the LLA.

The LLA amplitudes are real, so that only real parts of the NLLA amplitudes are important in the unitarity relations. Since they have a simple pole Regge form, the scheme of derivation of the BFKL equation in the NLLA remains unchanged.

The only difference is that we have to know the Reggeon trajectory and Reggeon-Reggeon-gluon production vertex with higher accuracy and to know also effective Reggeon-Reggeon$\to$ gluon-gluon and Reggeon-Reggeon$\to$ quark-antiquark vertices.

Unfortunately, this scheme is completely violated in the NNLLA. In this approximation two powers of $\ln s$ can be lost compared with the LLA in the product of two amplitudes in the unitarity relations. It can be done losing either one $\ln s$ in each of the amplitudes or $\ln^2 s$ in one of them. In the first case, discontinuities receive contributions from products of real parts of amplitudes with negative signature in the NLLA, products of imaginary parts of amplitudes with negative signature in the LLA, and products of amplitudes with positive signature in the LLA. Of course, account of these contributions greatly complicates derivation of the BFKL equation.

In particular, since for amplitudes with positive signature there are different colour group representations in the t-channel for quark-quark, quark-gluon and gluon-gluon scattering, their account violates unity of consideration.

However, these complications do not seem to be as great as in the second case, when $\ln^2 s$ is lost in one of the amplitudes in the unitarity relations. In this case one of the amplitudes must be taken in the NNLLA and the other in the LLA. Since the amplitudes in the LLA are real, only real parts of the NNLLA amplitudes are important. But even for these parts the pole Regge form becomes inapplicable because of the contributions of the

three-Reggeon cuts

which appear in this approximation. Note that account of these contributions also violates unity of consideration of quark-quark, quark-gluon and gluon-gluon scattering because the cuts give contributions to amplitudes with different representations of the colour group in the t-channel for these processes.

It is necessary to say that, in general, breaking the pole Regge form is not a surprise.

It is well known that Regge poles in the complex angular momenta plane generate Regge cuts.

Moreover, in amplitudes with positive signature the Regge cuts appear already in the LLA. In particular, the BFKL Pomeron is the two-Reggeon cut in the complex angular momenta plane.

But in amplitudes with negative signature Regge cuts appear only in the NNLLA.

The violation of the pole Regge form was firstly noticed in the non-logarithmic two loops contributions to elastic parton scattering amplitudes by
V. Del Duca and E. W. N. Glover, JHEP **0110** (2001) 035 [hep-ph/0109028],
and then confirmed and generalized to the case of logarithmic terms in the three loop contributions
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, Phys. Lett. B **732** (2014) 233,
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, PoS RADCOR **2013** (2013) 046 [hep-ph/1312.5098],
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, JHEP **1502** (2015) 029,
could be explained by their contributions. It is shown
V. S. Fadin, AIP Conf. Proc. **1819** (2017) no.1, 060003 [arXiv:1612.04481 [hep-ph]],
S. Caron-Huot, E. Gardi and L. Vernazza, JHEP **1706** (2017) 016 [arXiv:1701.05241 [hep-ph]]
this is actually so.

In contrast to the Reggeon, the three-Reggeon cuts can contribute to amplitudes with various representations of the colour group in the *t*-channel.

Possible representations for quark-quark and quark-gluon scattering are only singlet (**1**) and octet (**8**).

For the gluon-gluon scattering there are also **10**, $\mathbf{10}^*$ and **27**. Due to Bose statistic for gluons, symmetry of the representations **1** and **27**, antisymmetry **10** and $\mathbf{10}^*$ and existence both symmetric $\mathbf{8_s}$ and antisymmetric $\mathbf{8_a}$ representations.

Therefore, besides amplitudes with colour octet in the *t*-channel the three-Reggeon cuts can contribute to amplitudes with the representations **1** for quark-quark-scattering and in the representation **10** and $\mathbf{10}^*$ for the gluon-gluon scattering.

Due to the signature conservation the cuts with negative signature has to be three-Reggeon ones. Since our Reggeon is the Reggeized gluon, the cuts start with the diagrams with three t-channels gluons. They are presented below, where particles A, A' and B, B' can be quarks or gluons.

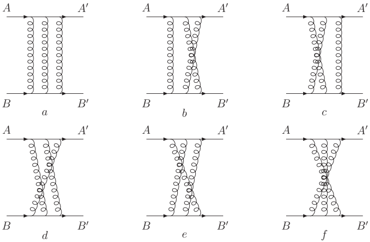


Figure 1: Three-gluon exchange diagrams

The octet part of the contribution of the diagrams with three-gluon exchanges can be written as

$$A_{ij}^{(8)} = \langle A'|\mathcal{T}^a|A\rangle\langle B'|\mathcal{T}^a|B\rangle \Bigg[C_{ij}A^{eik} + \frac{N_c^2}{8}\left(A_{ij}^a + A_{ij}^f\right)$$

$$+\delta_{i,q}\delta_{j,q}\frac{4-N_c^2}{8}\left(A_{ij}^a - A_{ij}^f\right)\Bigg] ,$$

where A_{ij}^{α} is the contribution of the diagram α with omitted colour factors and $A_{ij}^{eik} = \sum_{\alpha} A_{ij}^{\alpha}$. Note that A^{eik} is gauge invariant.

$$A_{gg}^{(10+10^*)} = \varepsilon_A^a \varepsilon_A^{*a'} \varepsilon_B^b \varepsilon_B^{*b'} \langle aa'|\hat{\mathcal{P}}_{10} + \hat{\mathcal{P}}_{10^*}|bb'\rangle \frac{-3}{4} N_c A^{(eik)} ,$$

where ε_A^a is the gluon A colour polarization vector, $\hat{\mathcal{P}}_R$ is the projection operator on the representation R, and

$$A_{qq}^{(1)} = \chi^*_{A'\alpha'}\chi_A^{\alpha}\chi_B^{\beta}\chi^*_{B'\beta'}\delta_{\alpha}^{\alpha'}\delta_{\beta}^{\beta'}\frac{(N_c^2-4)(N_c^2-1)}{16N_c^3}A^{(eik)} ,$$

χ_A^{α} is the quark A colour spinor.

The eikonal amplitude can be easily found:

$$A^{eik} = g^2 \frac{s}{t} \left(\frac{-4\pi^2}{3} \right) g^4 \, \vec{q}^{\,2} A_{\perp}^{(3)} \ ,$$

where $A_{\perp}^{(3)}$ is given by the integral

$$A_{\perp}^{(3)} = \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^2} =$$

$$= 3 C_{\Gamma}^2 \frac{4}{\epsilon^2} \frac{(\vec{q}^{\,2})^{2\epsilon}}{\vec{q}^{\,2}} \frac{\Gamma^2(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+3\epsilon)} \ ,$$

where

$$C_{\Gamma} = \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{(4\pi)^{2+\epsilon}\Gamma(1+2\epsilon)}$$

$$\frac{\Gamma^2(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+3\epsilon)} .$$

Note that we use the "infrared" ϵ, $\epsilon = (D-4)/2$, D is the space-time dimension.

The last term in $A_{ij}^{(8)}$ is not relevant; it is the contribution of positive signature in the quark-quark scattering. The second term does not violate the pole factorization and can be assigned to the Reggeized gluon contribution. This is not true for the first term, because

$$2C_{gq} \neq C_{qq} + C_{gg}\ ,\ \ 2C_{gq} - C_{qq} - C_{gg} = -\frac{1}{4}\left(1 + \frac{1}{N_c^2}\right)\ ,$$

which means violation of the pole factorizatio. It is not difficult to see that the nonvanishing in the limit $\epsilon \to 0$ part of the amplitudes $C_{ij}A^{(eik)}$ coincides with $g^2(s/t)(\alpha_s/\pi)^2\mathcal{R}_ij^{(2),0,[8]}$ of the paper

V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, JHEP **1502** (2015) 029.

The values $\mathcal{R}_ij^{(2),0,[8]}$ are given there in Eq. (4.35), $(\alpha_s/\pi)^2\mathcal{R}_ij^{(2),0,[8]}$ is the first (two-loop, non-logarithmic) contributions to the "non-factorizing remainder function" $\mathcal{R}_{ij}^{[8]}$ introduced in Eq. (3.1) of this paper.

This means that the violation of the pole factorization is due to the eikonal part of the contribution of the diagrams with three-gluon exchange.
However, one can not affirm that this part is given entire by the three-Reggeon cut. Indeed, it can contain also the Reggeized gluon contribution. In fact, a non-factorizing remainder function is not uniquely defined.
The problem of separation of the pole and cut contributions can be solved by consideration of logarithmic radiative corrections to them. In the case of the Reggeized gluon contribution the correction comes solely from the Regge factor, so that the first order correction (more strictly, its relative value; this is assumed also in the following) is $\omega(t)\ln s$, where $\omega(t)$ is the gluon trajectory. In the case of the three-Reggeon cut, one has to take into account the Reggeization of each of three gluons and the interaction between them. The Reggeization gives $\ln s$ with the coefficient $3C_R$,

where $A_\perp^{(3)} C_R$ is given by the integral

$$A_\perp^{(3)} C_R = -g^2 N_c C_\Gamma \frac{2}{\epsilon} \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^{1-\epsilon}}$$

$$= -g^2 N_c C_\Gamma \frac{4}{3\epsilon} (\vec{q}^{\,2})^\epsilon \frac{\Gamma(1-3\epsilon)\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)\Gamma(1+\epsilon)\Gamma(1+4\epsilon)} A_\perp^{(3)} .$$

Interaction between two Reggeons with transverse momenta $\vec{l}_1$ and $\vec{l}_2$ and colour indices c_1 and c_1 is defined by the real part of the BFKL kernel

$$\left[\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k})\right]_{c_1 c_2}^{c_1' c_2'} = T^a_{c_1 c_1'} T^a_{c_2 c_2'} \frac{g^2}{(2\pi)^{D-1}} \left[\frac{\vec{q}_1^{\,2} \vec{q}_2^{\,\prime\, 2} + \vec{q}_2^{\,2} \vec{q}_1^{\,\prime\, 2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right] ,$$

where $\vec{k}$ is the momentum transferred from one Reggeon to another in the interaction, $\vec{q}_1'$ and $\vec{q}_2'$ (c_1' and c_2') are the Reggeon momenta (colour indices) after the interaction, $\vec{q}_1' = \vec{q}_1 - \vec{k}$, $\vec{q}_2' = \vec{q}_2 + \vec{k}$, and $\vec{q} = \vec{q}_1 + \vec{q}_2 = \vec{q}_1' + \vec{q}_2'$.

Account of the interactions between all pairs of Reggeons leads in the sum to the colour coefficients which differ from the coefficients $C_{ij}^{(\mathcal{R})}$ only by the common factor $2N_c - C_{\mathcal{R}}$, where $C_{\mathcal{R}}$ is the value of the Casimir operator for the representation $\mathcal{R}$.

Now about the kinematic part of the kernel. The total contribution of the first two terms in it to the coefficient of $\ln s$ in the first order correction is $-4C_R$. The the contribution of the last term to the coefficient of $\ln s$ in the first order correction is $-C_3$, where

$$C_3 = g^2 N_c C_\Gamma \frac{4}{\epsilon} \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2 (l_1 + l_2)^{2\epsilon}}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^{1-2\epsilon}} \left(A_\perp^{(3)}\right)^{-1}$$

$$= g^2 N_c C_\Gamma \frac{32}{9\epsilon} (\vec{q}^{\,2})^\epsilon \frac{\Gamma(1-3\epsilon)\Gamma(1-\epsilon)\Gamma^2(1+3\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)\Gamma(1+4\epsilon)} \, .$$

Therefore, the first order correction in the case of Reggeized gluon is $\omega(t) \ln s$, and in the case of the three-Reggeon cut is $(-C_R - C_3) \ln s$. If to present the coefficients C_{ij} as the sum

$$C_{ij} = C_{ij}^R + C_{ij}^C \ ,$$

where C_{ij}^R correspond to the pole, so that

$$2C_{gq}^R = C_{qq}^R + C_{gg}^R \ ,$$

and C_{ij}^C correspond to the cut, we obtain that with the logarithmic accuracy the total three-loop contributions to the coefficient of $\ln s$ are where C_{ij}^R correspond to the pole, so that

$$2C_{gq}^R = C_{qq}^R + C_{gg}^R \ ,$$

and C_{ij}^C correspond to the cut, we obtain that with the logarithmic accuracy the total three-loop contributions to the coefficient of $\ln s$ are

$$A^{eik} \left(C_{ij}^R \omega(t) - C_{ij}^C (C_R + C_3) \right) \ln s \ .$$

The infrared divergent part of these contributions must be compared with the functions $g^2(s/t)\mathcal{R}_{ij}^{(3),1,[8]} \ln s$ of the paper V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza.

It is not difficult to see that with the accuracy with which the values $\mathcal{R}_i j^{(3),1,[8]}$ are known the equality

$$g^2(s/t)\mathcal{R}_{ij}^{(3),1,[8]} = A^{eik}\left(C_{ij}^R\omega(t) - C_{ij}^C(C_R + C_3)\right)$$

can be fulfilled if

$$C_{gg}^C = -\frac{3}{2}\,, \quad C_{gq}^C = -\frac{3}{2}\,, \quad C_{qq}^C = \frac{3(1-N_c^2)}{4N_c^2}\,,$$

and

$$aC_{gg}^R = 3\,, \quad C_{gq}^R = \frac{7}{4}\,, \quad C_{qq}^R = \frac{1}{2}\,.$$

It means that the restrictions imposed by the infrared factorization on the parton scattering amplitudes with the adjoint representation of the colour roup in the t-channel and negative signature can be fulfilled in the NNLLA at two and three loops if besides the Regge pole contribution there is the Regge cut contribution

$$A^{eik}C_{ij}^C\,(1 - (C_R + C_3)\ln s)\ .$$

But, evidently, it can not be considered as a proof that in the NNLLA the only singularities in the J plane are the Regge poles and three-Reggeon cuts.

It also leaves open the question of mixing of the Reggeized gluon and the three-Reggeon cut.
S. Caron-Huot, E. Gardi and L. Vernazza, JHEP **1706** (2017) 016 [arXiv:1701.05241 [hep-ph]]

This question can be clarified in four loops. The calculations are in progress.

Remind that the pole Regge form in the LLA and NLLA was proved in all loops using the bootstrap requirement (in the colour octet and negative signature two interacting Reggeized gluons must reproduce the Reggeized gluon again) and bootstrap conditions on the Reggeon vertices and trajectory derived from this requirement.

With account of the Reggeon cuts the bootstrap relations are not formulated yet.

Summary

- The gluon Reggeization is one of the remarkable properties of QCD.
- In the LLA and in the NLLA the Reggeization provides a simple factorized form of QCD amplitudes with octet representation of the colur group and negative signature in cross channels.
- It makes possible a simple derivation of the BFKL equation with use of s-channel unitarity.
- This form is violated in the NNLLA by the 3-Reggeon cuts
- The cuts strongly complicate derivation of the BFKL equation.
- Investigation of the cuts is only in the beginning.
- A great field for interesting work is open.