Amplitudes in the Multi-Regge Limit of $\mathcal{N} = 4$ SYM

Bram Verbeek

31/8/2018

work in collaboration with

V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca & G. Papathanasiou

Introduction

Goal: understand the mathematical structure of scattering amplitudes.

Planar $\mathcal{N} = 4$ Super Yang-Mills theory = perfect laboratory.

Further understanding required at higher numbers of particles $\rightarrow$ Multi-Regge kinematics as a simplification?

$\mathcal{N} = 4$ Super Yang-Mills

Maximally supersymmetric $SU(N_c)$ Yang-Mills in 4D.

Many special properties:

- Conformal symmetry (massless, $\beta(g) = 0$ to all orders).
- Maximal transcendentality $\to$ functions appearing in an L loop amplitude are always of 'transcendentality'/weight $2L$:

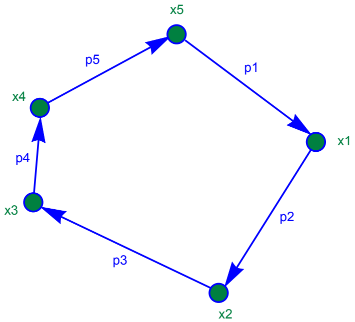
@ 1-Loop : $\text{Li}_2(x), \log^2(x), \zeta_2$

@ L-Loop : Generalised Polylogarithms: [Goncharov]

$$G(\underbrace{a_1, ..., a_{2L}}_{\text{weight 2L}}; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_{2L}; t).$$

The Planar Limit

In the limit $N_c \to \infty$ with $g^2 N_c \sim \mathcal{O}(1)$, a hidden symmetry appears



Define new coordinates x_i s.t.

$$x_i - x_{i-1} = p_i.$$

A seperate conformal group acts on x-space.
$\rightarrow$ Dual Conformal Symmetry

[Drummond, Henn, Korchemsky, Sokatchev]

$$z_i = \frac{(x_1 - x_{i+3})(x_{i+2} - x_{i+1})}{(x_1 - x_{i+1})(x_{i+2} - x_{i+3})}$$

The Planar Limit

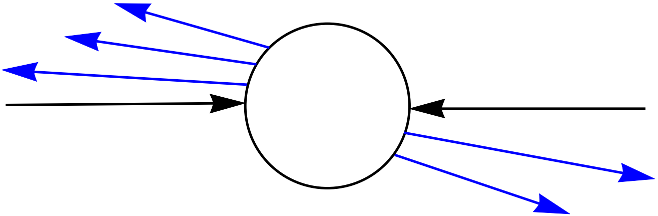
Dual Conformal Symmetry fixes 4 & 5 point amplitudes completely. [Anastasiou, Bern, Dixon, Smirnov; Bern, Dixon, Smirnov]

From 6 points onwards

- 6pt MHV $(--++++)$ @ 5 loops
  [Caron-Huot, Dixon, McLeod, Von Hippel]
- 6pt NMHV $(---+++)$ @ 4 loops
  [Dixon, Von Hippel, McLeod]
- 7pt MHV $(--+++++)$ @ 2 loops
  [Golden, Spradlin; Drummond, Papathansiou, Spradlin]

We want to push to higher points.
$\rightarrow$ Look at special kinematic limit.

The Multi-Regge Limit



Strong ordering in angle wrt beam axis ($p_1^+ = p_2^- = \mathbf{p}_1 = \mathbf{p}_2 = 0$)

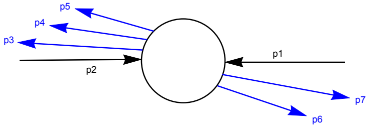
$$p_3^+ \gg p_4^+ \gg \dots p_{N-1}^+ \gg p_N^+ \,.$$

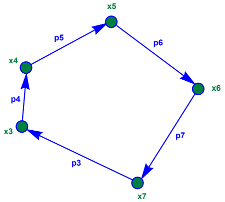
No hierarchy in transverse plane

$$|\mathbf{p}_3| \simeq \dots \simeq |\mathbf{p}_N| \,.$$

The Multi-Regge Limit

All nontrivial kinematics are in the transverse plane.





Dual conformal invariance restricts our variables to

$$\mathbf{z}_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}.$$

$\mathfrak{M}_{0,N-2}$

Kinematics are determined by $N-2$ points on the complex plane.

$$\begin{aligned}\mathfrak{M}_{0,N-2} &= \text{space of configurations for the Riemann sphere } (\simeq \mathbb{C}) \\ &\quad \text{with } N-2 \text{ marked points} \\ &= \text{the phase space for MRK.}\end{aligned}$$

$$\dim_{\mathbb{C}} \mathfrak{M}_{0,N-2} = N-5$$

$\rightarrow N-5$ dual conformal cross ratios $\{\mathbf{z}_i\}$.

Iterated Integrals on $\mathfrak{M}_{0,n}$

All iterated integrals on $\mathfrak{M}_{0,n}$ can be written in terms of generalised polylogarithms [Brown]

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_n; t).$$

Example:

$$G(a; z) = \log\left(1 - \frac{z}{a}\right)$$
$$G(0, 1; z) = -\mathrm{Li}_2(z).$$

We expect that amplitudes in MRK are made up of polylogarithms.

Single-Valued Polylogarithms

Branch cuts occur when

$$(x_i - x_j)^2 = 0.$$

For MRK we are restricted to the transverse plane, thus

$$|\mathbf{x}_i - \mathbf{x}_j|^2 = 0.$$

However, since the transverse plane has a Euclidean metric

$$|\mathbf{x}_i - \mathbf{x}_j|^2 \geq 0,$$

and thus amplitudes in the Multi-Regge limit are single-valued.

Single-Valued Polylogarithms

One can combine polylogarithms and their complex conjugates such that all branch cuts cancel.

Associate to each $G(a,b,...;z)$ a single-valued function $\mathcal{G}(a,b,...;z)$ such that

$$\partial_z \mathcal{G}(a,b,...;z) = \frac{1}{z-a}\mathcal{G}(b,...;z).$$

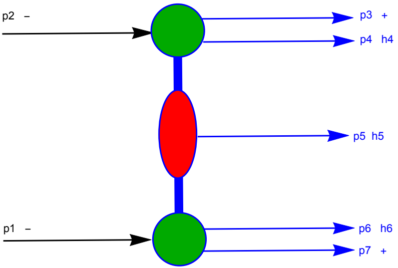
For example:

$$\mathcal{G}(a;z) = G(a;z) + G(\bar{a};\bar{z}) = \log\left|1+\frac{z}{a}\right|^2$$

$$\begin{aligned}\mathcal{G}(a,b;z) = & G(a,z)G\left(\bar{b},\bar{z}\right) + G(b,a)G\left(\bar{a},\bar{z}\right) + G\left(\bar{b},\bar{a}\right)G\left(\bar{a},\bar{z}\right) \\ & - G(a,b)G\left(\bar{b},\bar{z}\right) - G\left(\bar{a},\bar{b}\right)G\left(\bar{b},\bar{z}\right) \\ & + G\left(\bar{b},\bar{a},\bar{z}\right) + G(a,b,z).\end{aligned}$$

Convolution Structure

In MRK, large logarithms need to be resummed.
Upon doing this, the amplitude factorises. [Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]



For example, at 7 points LLA:

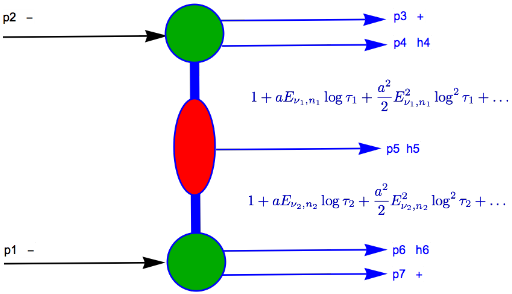
$$\sim a\mathcal{F}\left[\chi^{h_4}\tau_1^{aE_{\nu 1,n1}}C^{h_5}\tau_2^{aE_{\nu 2,n2}}\chi^{-h_6}\right]$$

where a is the coupling constant, $\log(\tau_i)$ are the large logs and

$$\mathcal{F}[f] = \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}}\frac{d\nu}{2\pi}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}|z|^{2i\nu}f(\nu,n)$$

denotes the Fourier-Mellin transform.

Convolution Structure



$$a\,\mathcal{F}\left[\chi^{h_4}\tau_1^{aE_{\nu 1,n1}}C^{h_5}\tau_2^{aE_{\nu 2,n2}}\chi^{-h_6}\right] \sim \sum_{ij}\frac{a^{i+j+1}}{i!j!}\log^i\tau_1\log^j\tau_2\,g^{(i,j)}_{h_4h_5h_6}$$

Convolution Structure

The Fourier-Mellin transform maps products into convolutions

$$\mathcal{F}[f \cdot g] = \mathcal{F}[f] * \mathcal{F}[g] = \int \frac{d^2\omega}{|\omega|^2} \mathcal{F}[f](\omega)\mathcal{F}[g]\left(\frac{z}{\omega}\right).$$

This means that we can compute

$$g^{(i,j)}_{h_4 h_5 h_6} \sim \mathcal{F}\left[\chi^{h_4} E^{i}_{\nu 1, n1} C^{h_5} E^{j}_{\nu 2, n2} \chi^{-h_6}\right]$$

by performing convolution integrals over a finite set of building blocks

$$\{\mathcal{F}[\chi], \mathcal{F}[C], \mathcal{F}[E]\}.$$

Stokes' Theorem

We are dealing with single-valued functions with isolated singularities on the Riemann sphere.

$$\downarrow$$

One can use Stokes' theorem to compute convolutions [Schnetz]

$$\int \frac{d^2 z}{\pi} f(z) = \text{Res}_{z=\infty} F(z) - \sum_i \text{Res}_{z=a_i} F(z)$$

where

$$\bar{\partial}_z F = f.$$

Convolutions reduce to residue computations.

Convolution Structure

$$
\begin{aligned}
g^{(1,0)}_{h_4 h_5 h_6} &\sim \mathcal{F}\left[\chi^{h_4} E_{\nu 1, n1} C^{h_5} \chi^{-h_6}\right] \\
&= \mathcal{F}\left[E_{\nu 1, n1}\right] * \mathcal{F}\left[\chi^{h_4} C^{h_5} \chi^{-h_6}\right] \\
&\sim \mathcal{F}\left[E_{\nu 1, n1}\right] * g^{(0,0)}_{h_4 h_5 h_6}
\end{aligned}
$$

Note that $\mathcal{F}\left[E_{\nu,n}\right] = -\frac{1}{2}(z_i + \bar{z}_i)/|1 - z_i|^2$ and so

$$
g^{(i,j)}_{h_4 h_5 h_6} \sim \int d^2 w \; (\text{RATIONAL FUNCTION}) \quad \underbrace{g^{(i-1,j)}_{h_4 h_5 h_6}}_{\text{SV-POLYLOGS}}
$$

→ Obtain higher-order coefficients by convoluting with $\mathcal{F}\left[E_{\nu,n}\right]$ from low-loop result.

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Convolution Structure

$$\begin{aligned} g^{(0,0)}_{-++} &\sim \mathcal{F}\left[\chi^{-} C^{+} \chi^{-}\right] \\ &= \mathcal{F}\left[\chi^{-}/\chi^{+}\right] * \mathcal{F}\left[\chi^{+} C^{+} \chi^{-}\right] \\ &\sim \mathcal{F}\left[\chi^{-}/\chi^{+}\right] * g^{(0,0)}_{+++} \end{aligned}$$

Note that $\mathcal{F}\left[\chi^{-}/\chi^{+}\right] = -z_i/(1-z_i)^2$ and so

$$g^{(i,j)}_{-++} \sim \int d^2 w \;(\text{RATIONAL FUNCTION}) \quad \underbrace{g^{(i,j)}_{+++}}_{\text{SV-POLYLOGS}}$$

→ Obtain any helicity configuration by performing a convolution on the MHV coefficients.

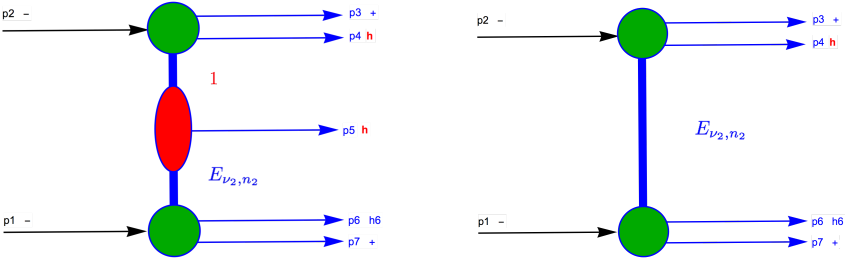
[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Factorisation

When expressed in terms of $\{\mathbf{x}_i\}$ we see that

$$g^{(0,1)}_{\mathbf{hh}h_6}(\mathbf{x}_1, \mathbf{x}_2) = g^{(1)}_{\mathbf{h}h_6}(\mathbf{x}_2).$$

Known at 2 loops [Bartels, Kormilitzin, Lipatov, Prygarin]



In general: propagators with no insertions where the neighbouring helicities are equal can be dropped.

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Factorisation

The factorisation allows us to determine all n-point MHV scattering amplitudes up to a certain loop order.

$$
\begin{aligned}
g_{+\dots+}^{(0,\dots,0,i_{a_1},0,\dots,0,i_{a_2},0,\dots,0,i_{a_k},0,\dots,0)}&(\mathbf{x}_1,\dots,\mathbf{x}_{N-5}) \\
&= g_{+\dots+}^{(i_{a_1},i_{a_2},\dots,i_{a_k})}(\mathbf{x}_{a_1},\dots,\mathbf{x}_{a_k})
\end{aligned}
$$

[Del Duca, Druc, Drummond, Duhr ,Dulat, Marzucca, Papathanasiou, BV]

Conclusion

- Higher loop and leg amplitudes at LLA in MRK in Planar $\mathcal{N} = 4$ SYM can be computed easily using convolutions of simple building blocks.
- Beyond LLA the formalism still holds. It has been applied as follows:
  - All MHV 5-loop amplitudes at LLA and 8-point LLA amplitudes for any helicity configuration up to 4 loops [1606.08807].
  - 7-point NLLA MHV up to 5 and NMHV up to 3 loops [1801.10605].
  - All MHV 3-loop amplitudes at NLLA [TBA].
- MRK offers an example where the function space can be understood to all orders.

Thank you