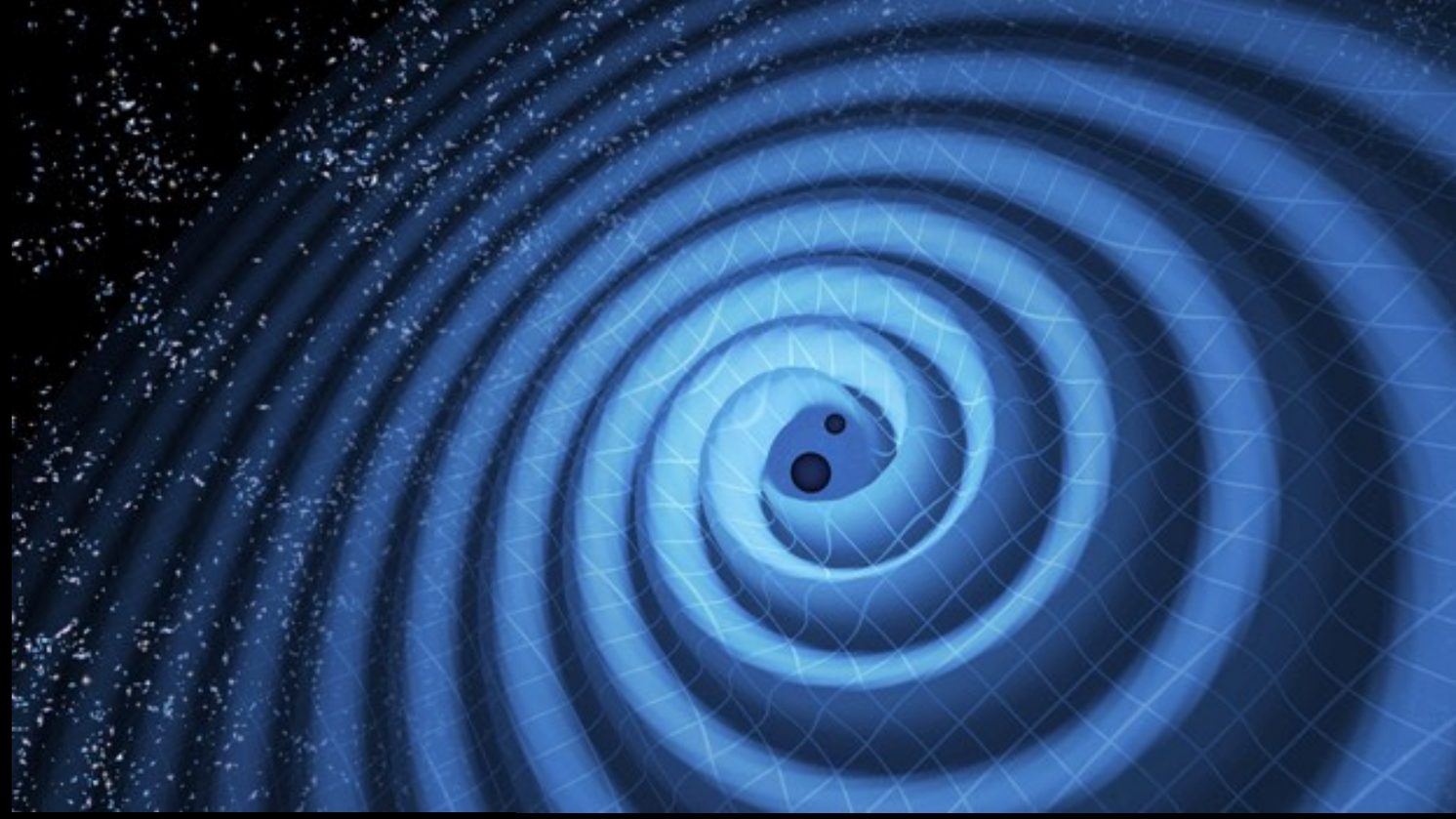
Analytic Calculation of Secondary Gravitational Waves and Primordial Black Holes

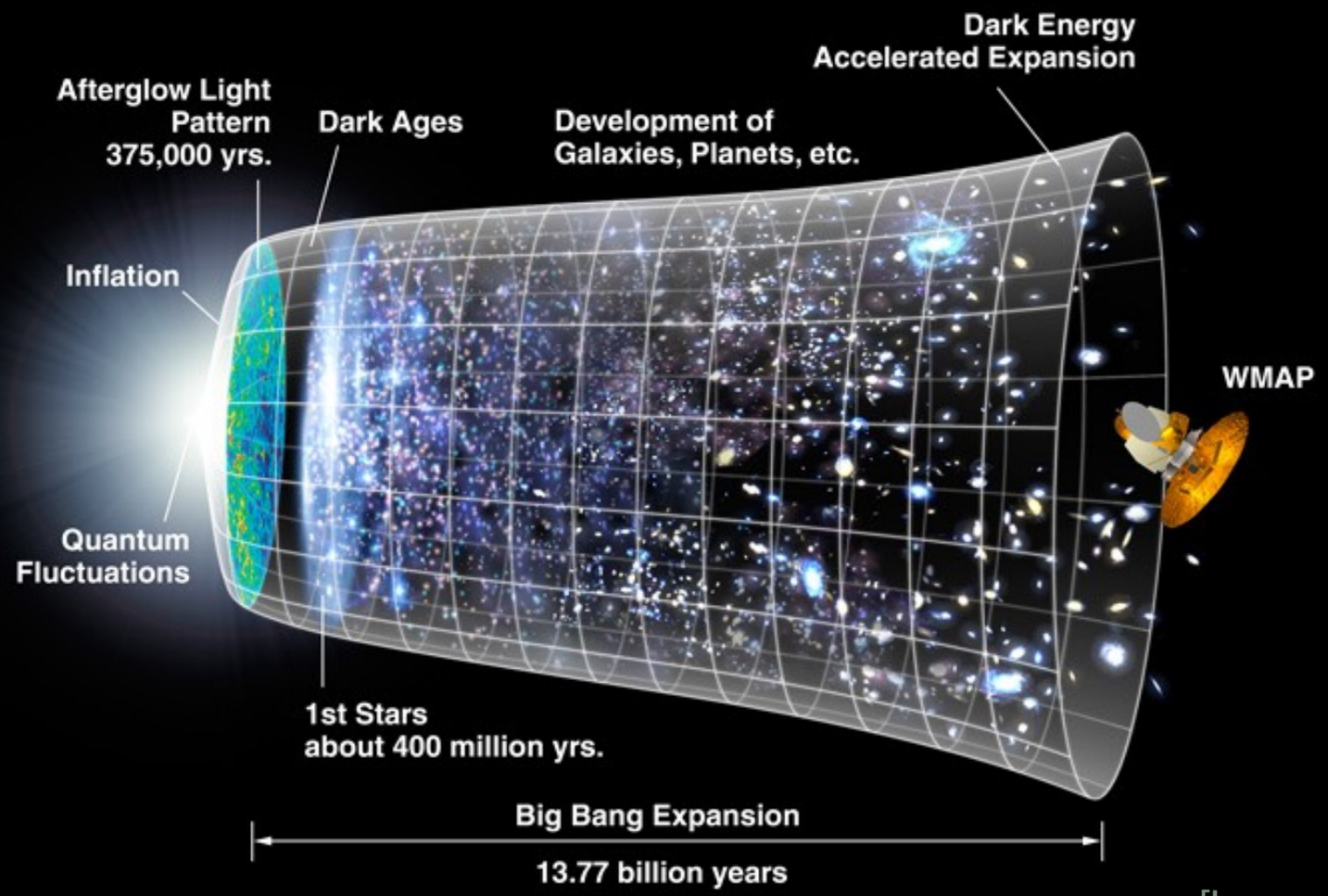
Takahiro Terada
(KEK, JSPS fellow)

[Kohri, Terada, 1804.08577]

GW Astrophysics
↓
GW Cosmology



[Image credit: LIGO/T. Pyle]



We can probe
the very early Universe
or High Energy Physics.

[Image credit: NASA/WMAP Science Team]

What is the secondary GW?

[Ananda, Clarkson, Wands, gr-qc/0612013]
[Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

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- Inflation produces
  scalar and tensor (1st-order) perturbations

$$\mathrm{d}s^2 = -a^2(1+2\Phi)\mathrm{d}\eta^2 + a^2\left((1-2\Phi)\delta_{ij} + \frac{1}{2}h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j$$

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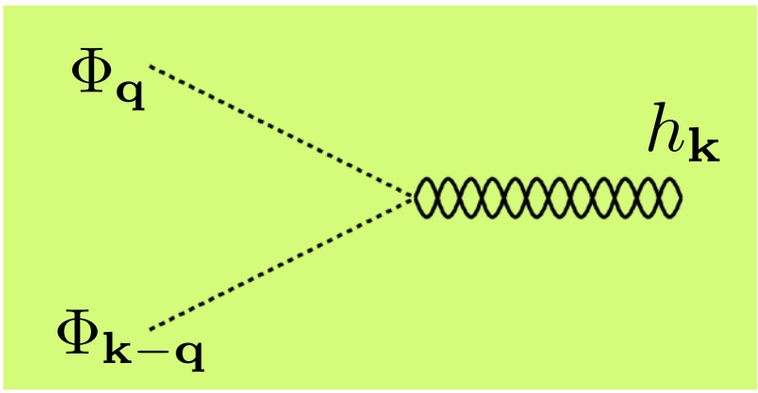

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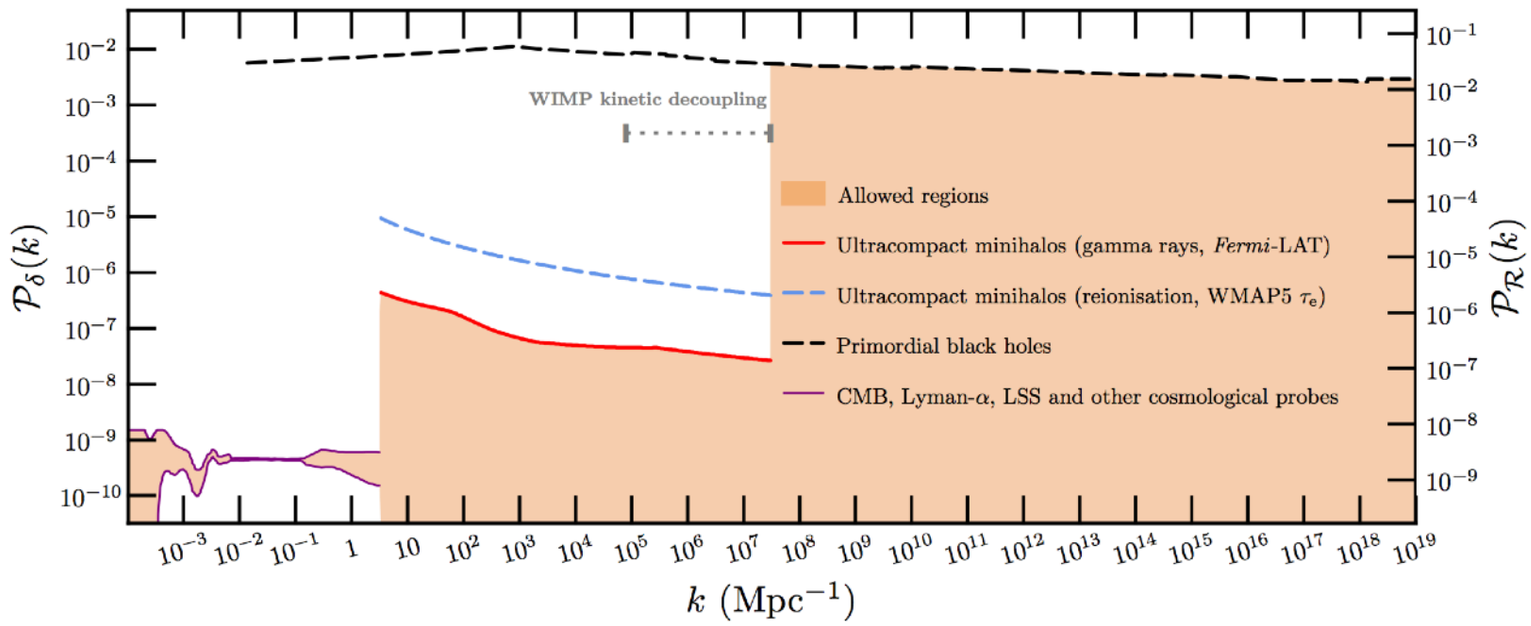
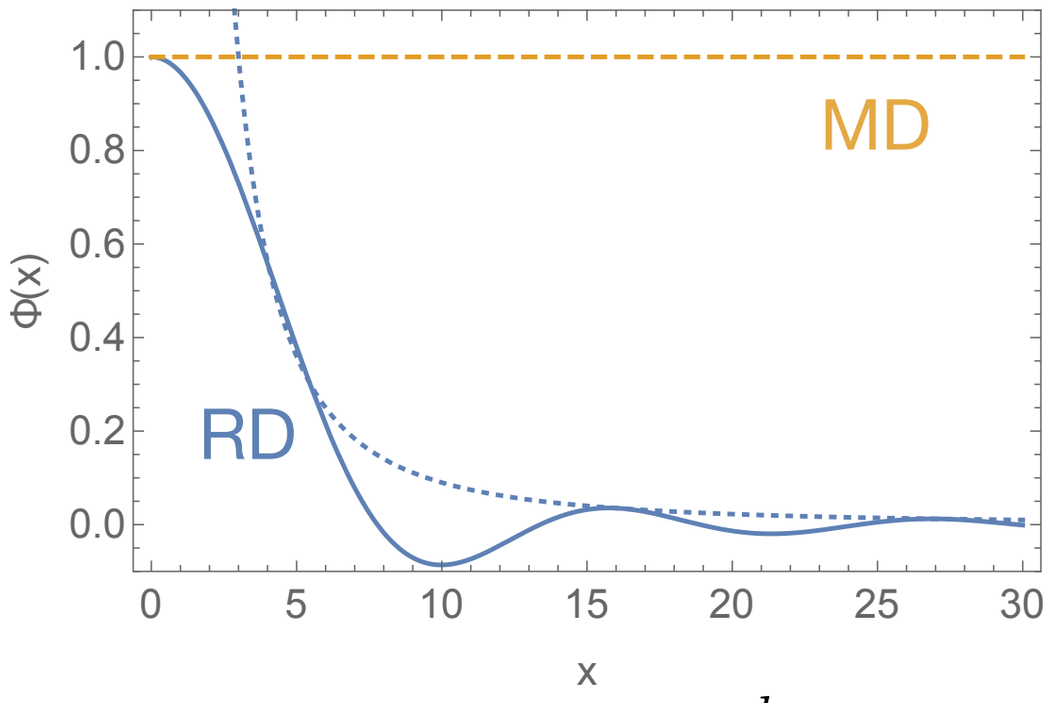
Enhanced scalar perturbations

[Bringmann, Scott, Akrami, 1110.2484]

Initial condition

Enhanced primordial curvature perturbations are allowed at small scales.





$x \equiv k\eta$

Dynamics

In a radiation-dominated (RD) era, the source decays.
In a matter-dominated (MD) era, the source does not decay.

Motivation (or What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

$$\mathcal{P}_h \sim \int \mathrm{d}k \int \mathrm{d}k' \left(\int \mathrm{d}t f(k, k', t) \right)^2 \mathcal{P}_\zeta(k) \mathcal{P}_\zeta(k')$$

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primordial quantity

Motivation (or What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

“universal part”

$$\mathcal{P}_h \sim \int \mathrm{d}k \int \mathrm{d}k' \left(\int \mathrm{d}t\, f(k, k', t) \right)^2 \mathcal{P}_\zeta(k) \mathcal{P}_\zeta(k')$$

describing time evolution　primordial quantity

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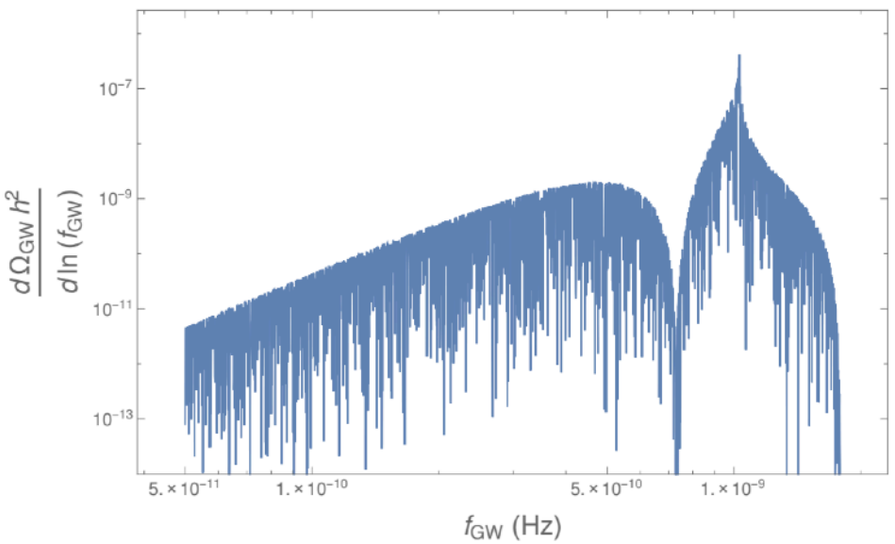
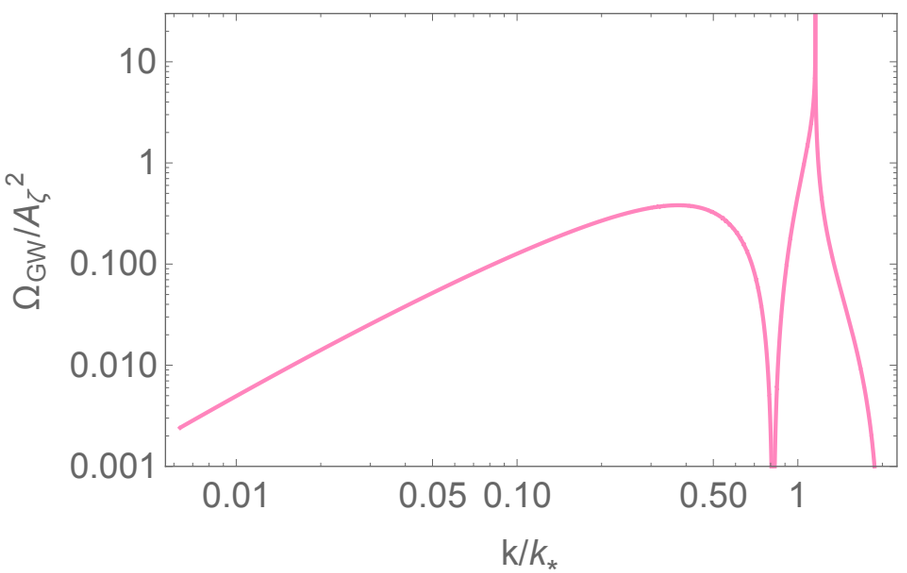
[Orlofsky, Pierce, Wells, 1612.05279]





More quantitative explanations

- Basic things
- Radiation-dominated (RD) era
- Matter-dominated (MD) era
- More general cases

Observable & Basic definitions

Observable & Basic definitions

Energy fraction

$$\Omega_{\rm GW}(\eta,k) = \frac{1}{\rho_{\rm tot}(\eta)}\frac{{\rm d}\rho_{\rm GW}(\eta)}{{\rm d}\ln k} = \frac{1}{24}\left(\frac{k}{a(\eta)H(\eta)}\right)^2 \overline{\mathcal{P}_h(\eta,k)}$$

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Power spectrum

$$\langle h^{\lambda}_{\mathbf{k}}(\eta)h^{\lambda'}_{\mathbf{k}'}(\eta)\rangle=\delta_{\lambda\lambda'}\delta^3(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_h(\eta,k)$$

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Gravitational field: Green's function method

$$a(\eta)h_{\mathbf{k}}(\eta)=4\int^{\eta}{\rm d}\bar{\eta}\,G_{\mathbf{k}}(\eta,\bar{\eta})a(\bar{\eta})S_{\mathbf{k}}(\bar{\eta})$$

$$G''_{\mathbf{k}}(\eta,\bar{\eta})+\left(k^2-\frac{a''(\eta)}{a(\eta)}\right)G_{\mathbf{k}}(\eta,\bar{\eta})=\delta(\eta-\bar{\eta})$$

Power spectrum & “universal part”

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$$\mathcal{P}_h(\eta,k) = 4\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \left(\frac{4v^2-(1+v^2-u^2)^2}{4vu}\right)^2 I^2(v,u,x)\mathcal{P}_\zeta(kv)\mathcal{P}_\zeta(ku)$$

where $x \equiv k\eta \qquad u = |\mathbf{k}-\tilde{\mathbf{k}}|/k \qquad v = \tilde{k}/k$

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We want to analytically calculate this function.

Definitions: $I(v,u,x) = \int_0^x \mathrm{d}\bar{x}\frac{a(\bar{\eta})}{a(\eta)}kG_k(\eta,\bar{\eta})f(v,u,\bar{x})$

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$$f(v,u,\bar{x})=\frac{6(w+1)}{3w+5}\Phi(v\bar{x})\Phi(u\bar{x})+\frac{6(1+3w)(w+1)}{(3w+5)^2}\left(\bar{x}\partial_{\bar{\eta}}\Phi(v\bar{x})\Phi(u\bar{x})+\bar{x}\partial_{\bar{\eta}}\Phi(u\bar{x})\Phi(v\bar{x})\right)$$
$$+\frac{3(1+3w)^2(1+w)}{(3w+5)^2}\bar{x}^2\partial_{\bar{\eta}}\Phi(v\bar{x})\partial_{\bar{\eta}}\Phi(u\bar{x})$$

We have used $\mathcal{H}=aH=2/((1+3w)\eta) \qquad \bar{x}\equiv k\bar{\eta} \qquad w\equiv P/\rho$

Analytic calculation

For definiteness, consider w = 1/3 (RD era).

$$I(v,u,x) = \int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta,\bar{\eta}) f(v,u,\bar{x})$$

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$$\Phi(x) = \frac{9}{x^2}\left(\frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3})\right)$$

$$f_{\mathrm{RD}}(v,u,x) = \frac{12}{u^3v^3x^6}\left(18uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} + (54-6(u^2+v^2)x^2+u^2v^2x^4)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\right.$$
$$\left. +2\sqrt{3}ux(v^2x^2-9)\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 2\sqrt{3}vx(u^2x^2-9)\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}}\right)$$

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$$
\begin{aligned}
I_{\rm RD}(v,u,x) =& \frac{3}{4u^3v^3x}\left(-\frac{4}{x^3}\left(uv(u^2+v^2-3)x^3\sin x - 6uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}}\right.\right.\\
&\left.+6\sqrt{3}ux\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 6\sqrt{3}vx\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} - 3(6+(u^2+v^2-3)x^2)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\right)\\
&+(u^2+v^2-3)^2\left(\sin x\left(\mathrm{Ci}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right)+\mathrm{Ci}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\right.\right.\\
&-\mathrm{Ci}\left(\left|1-\frac{v+u}{\sqrt{3}}\right|x\right)-\mathrm{Ci}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right)+\log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\Big)\\
&+\cos x\left(-\mathrm{Si}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right)-\mathrm{Si}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\right.\\
&\left.\left.\left.+\mathrm{Si}\left(\left(1-\frac{v+u}{\sqrt{3}}\right)x\right)+\mathrm{Si}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right)\right)\right)
\end{aligned}
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$$\mathrm{Si}(x) = \int_0^x \mathrm{d}\bar{x}\frac{\sin\bar{x}}{\bar{x}}$$

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$$

Oscillation average in the late-time limit

$$
\begin{aligned}
\overline{I^2_{\rm RD}(v,u,x\to\infty)} =& \frac{1}{2}\left(\frac{3(u^2+v^2-3)}{4u^3v^3x}\right)^2\left(\left(-4uv+(u^2+v^2-3)\log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\right)^2\right.\\
&\left.+\pi^2(u^2+v^2-3)^2\Theta(v+u-\sqrt{3})\right)
\end{aligned}
$$

Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider w = 0 (MD era).

$$I(v,u,x) = \int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta,\bar{\eta}) f(v,u,\bar{x})$$

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$$kG_{\mathbf{k}}(\eta,\bar{\eta}) = \frac{1}{x\bar{x}}\left((1+x\bar{x})\sin(x-\bar{x}) - (x-\bar{x})\cos(x-\bar{x})\right)$$

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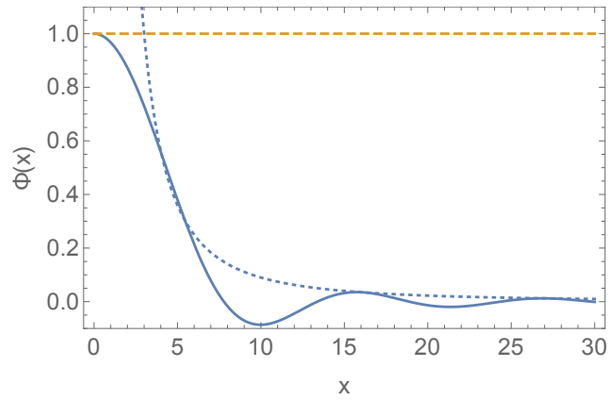
$$f_{\mathrm{MD}}(v,u,x) = \frac{6}{5}$$



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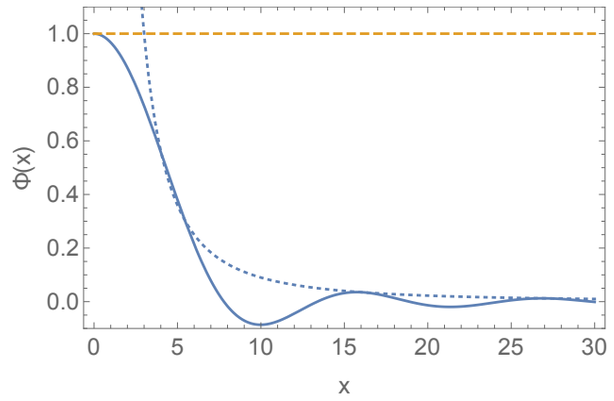

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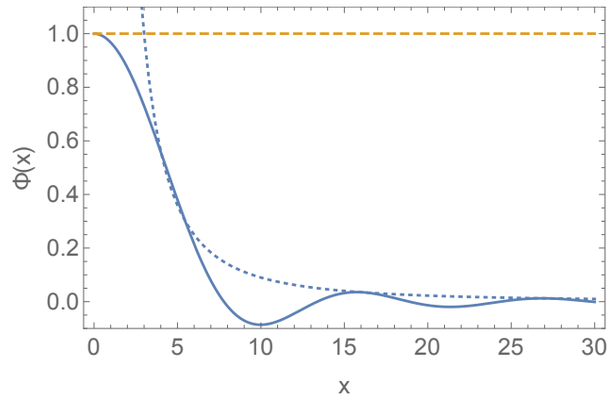

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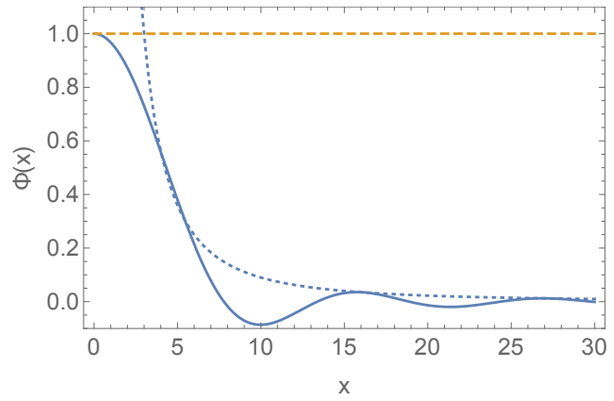

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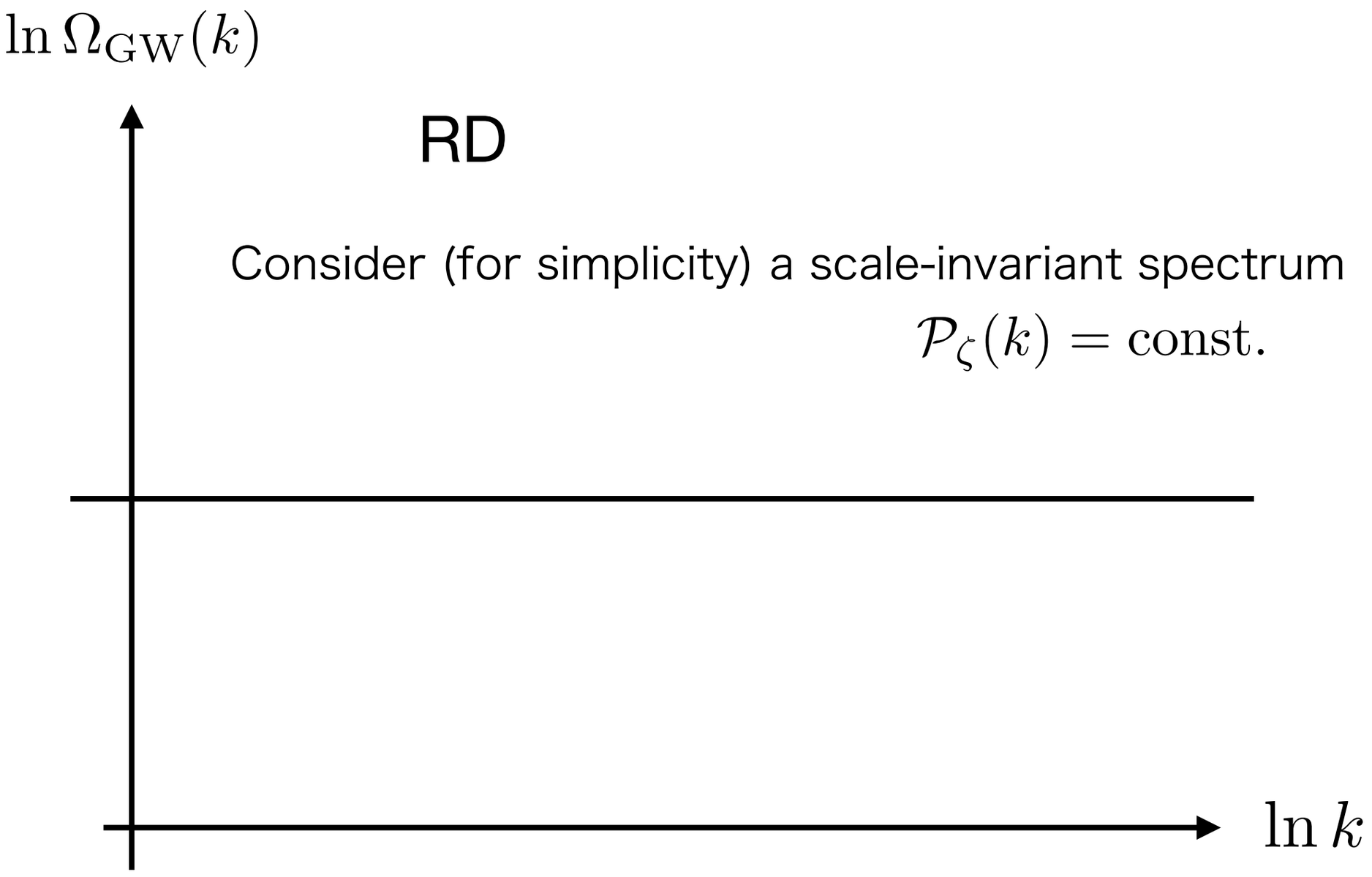
Constant metric distortion → After MD → **Propagating GW**

General cases



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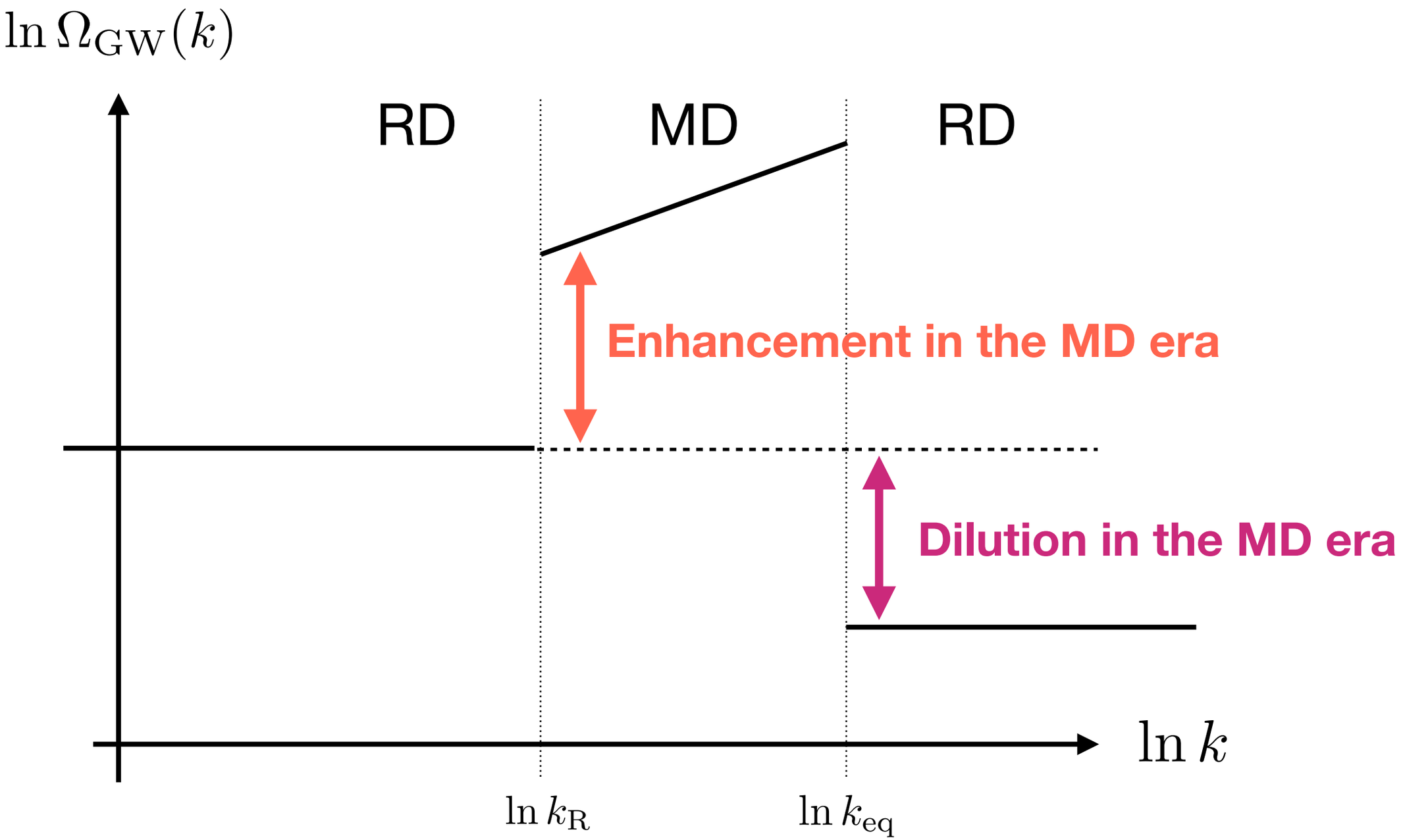

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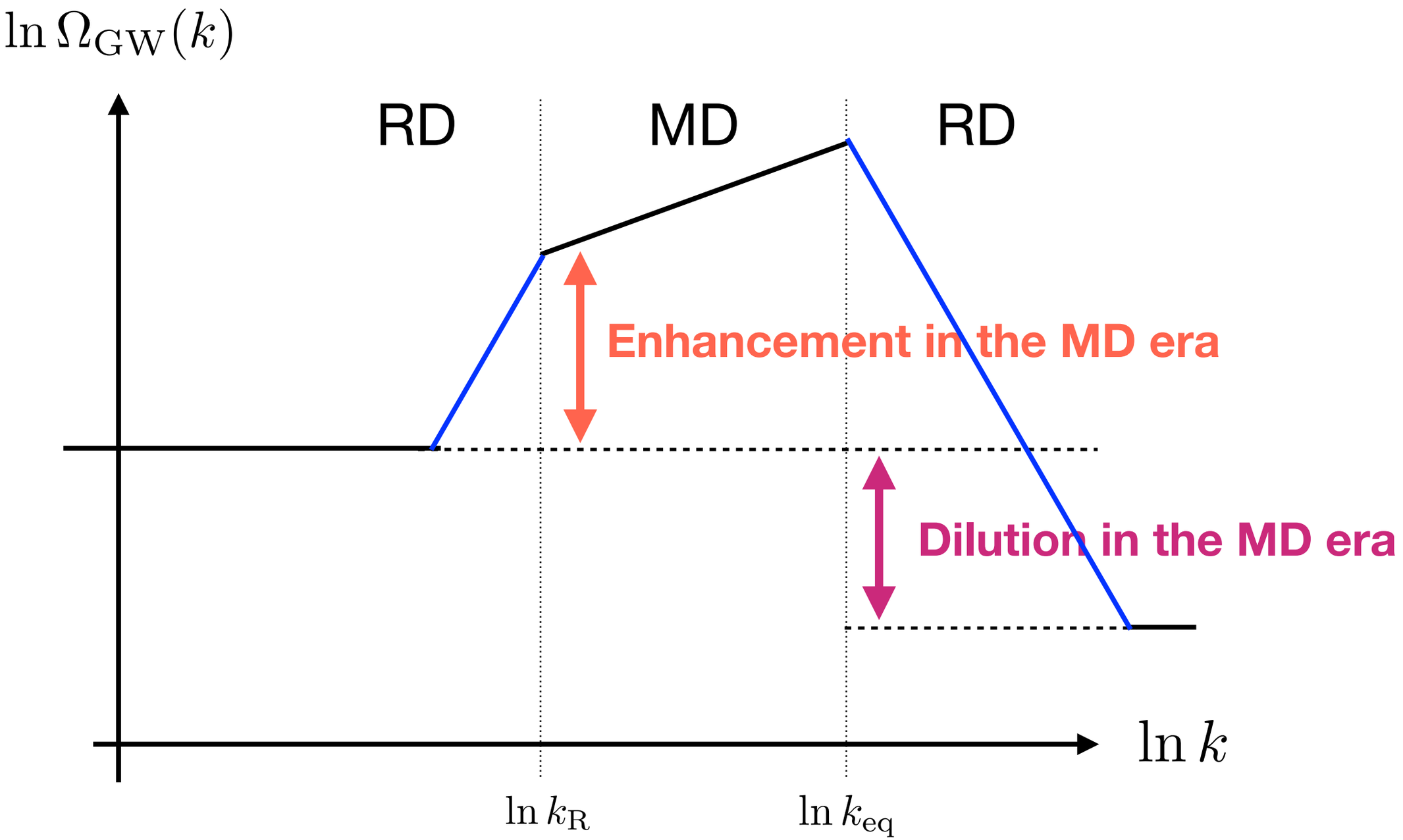
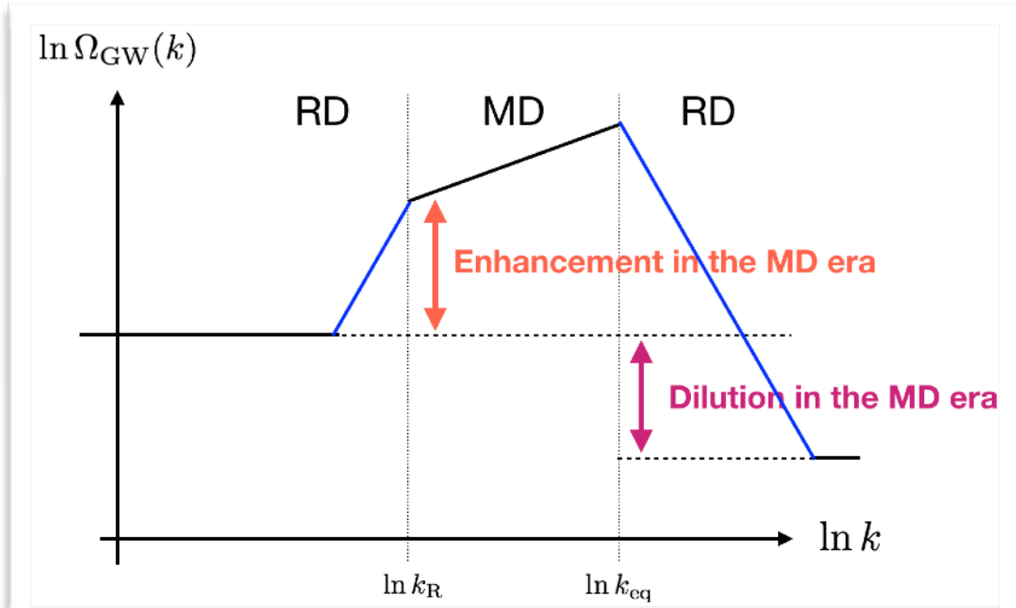


General cases

The MD → RD transition

$$I(v,u,x) = \int_0^{x_{\rm R}} \mathrm{d}\bar{x} \left(\frac{x_{\rm R}}{x}\right) \left(\frac{\bar{x}}{x_{\rm R}}\right)^2 k G_k^{\rm MD\to RD}(\eta,\bar{\eta}) f_{\rm MD}(v,u,\bar{x}) + \int_{x_{\rm R}}^{x} \mathrm{d}\bar{x} \left(\frac{\bar{x}}{x}\right) k G_k^{\rm RD}(\eta,\bar{\eta}) f_{\rm MD\to RD}(v,u,\bar{x})$$



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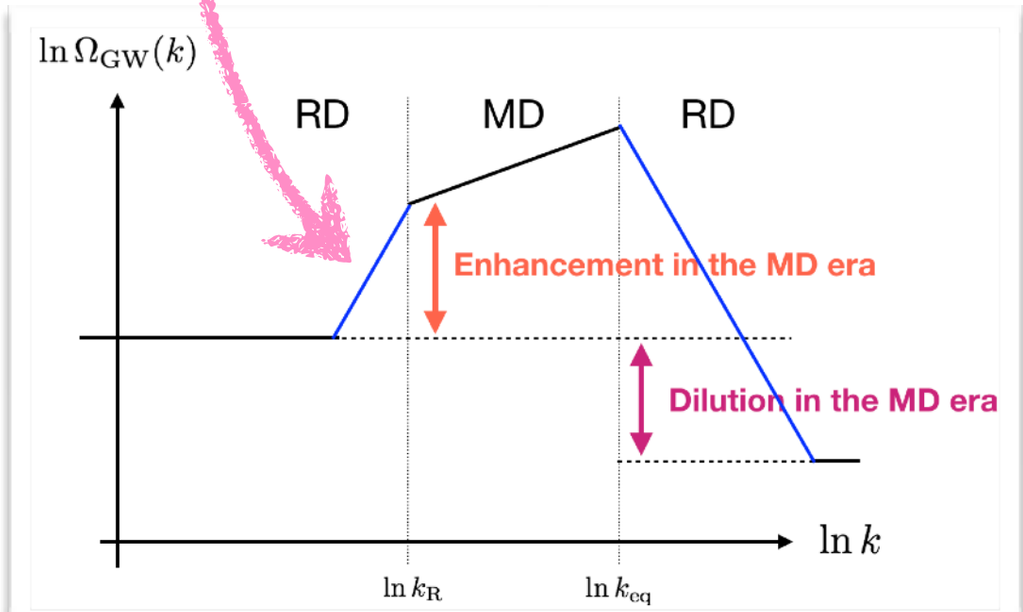

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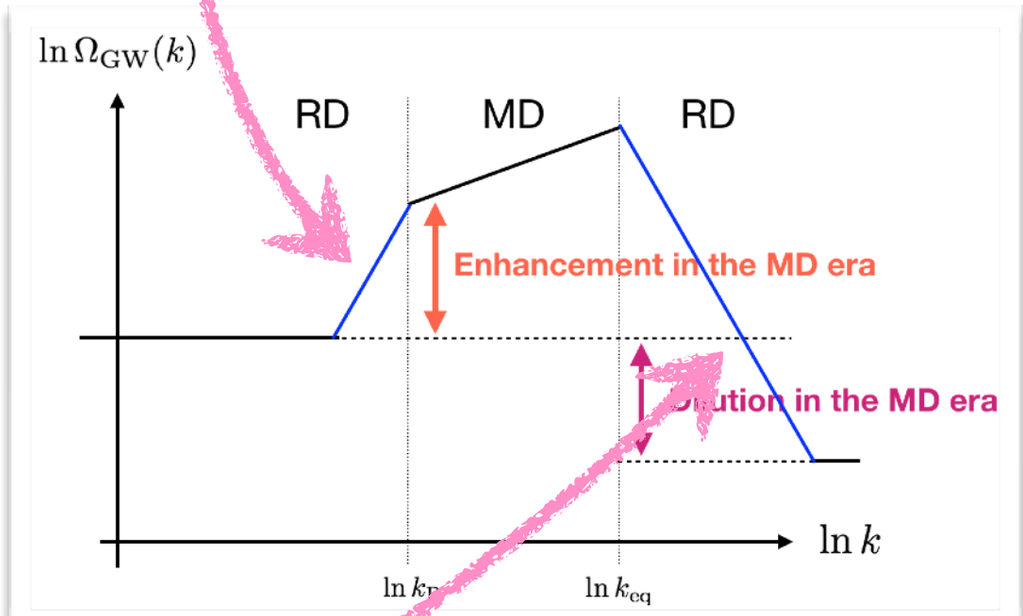

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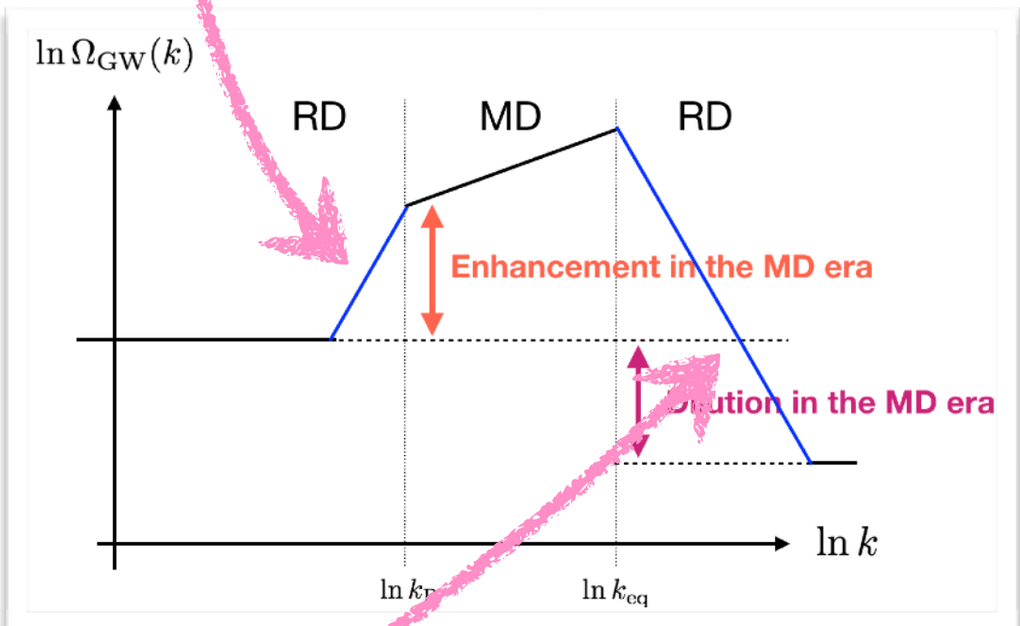
General cases

The MD → RD transition

$$I(v,u,x) = \int_{x_{\rm R}} \cdots$$

$$\frac{3}{5x x_{\rm R}^3}\left(3(2x_{\rm R}^2-1)\cos x - 6x_{\rm R}\sin x + 2x_{\rm R}^4\cos(x-x_{\rm R}) + 4x_{\rm R}^3\sin(x-x_{\rm R}) + 3\cos(x-2x_{\rm R})\right)$$

$$\cdots (x)\, kG_k^{\rm RD}(\eta,\bar{\eta}) f_{\rm MD\to RD}(v,u,\bar{x})$$



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General cases

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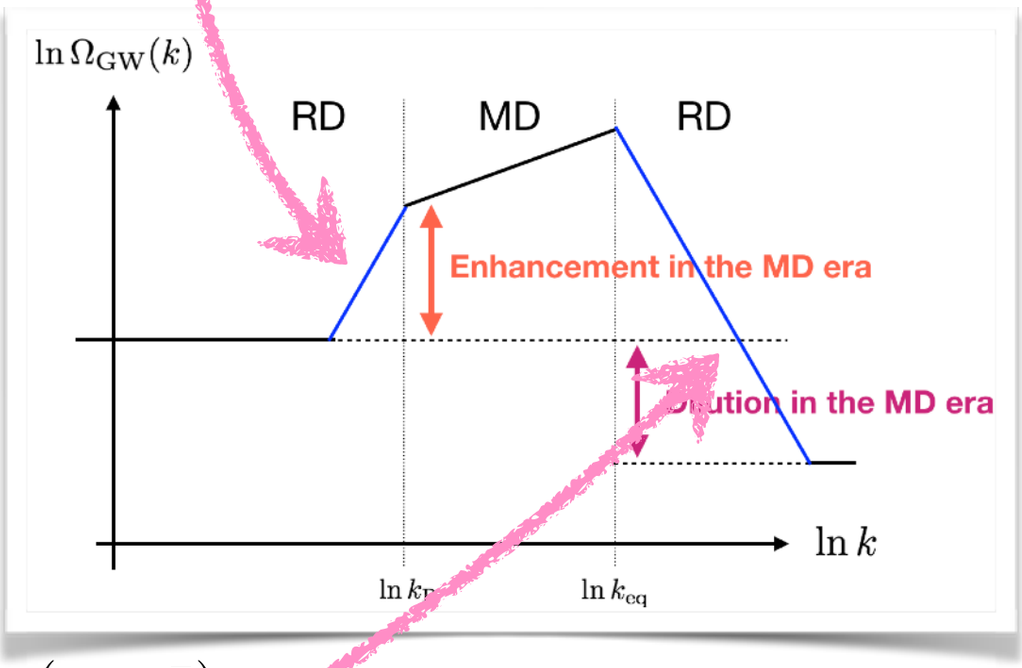

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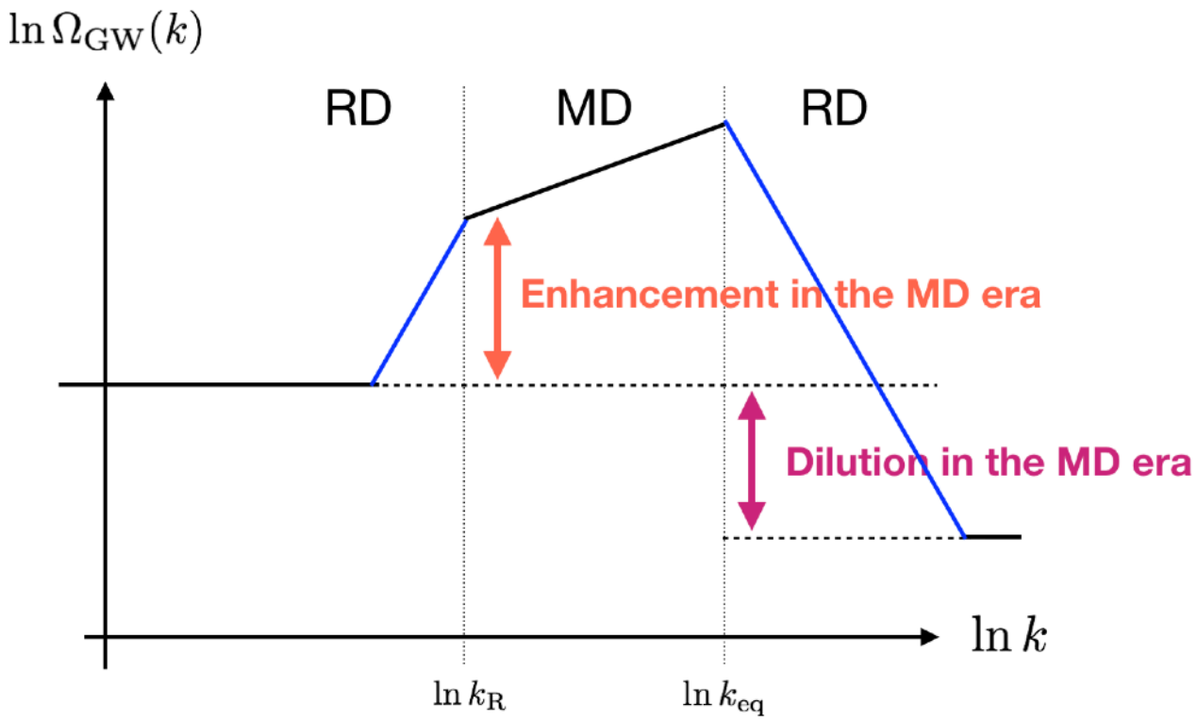
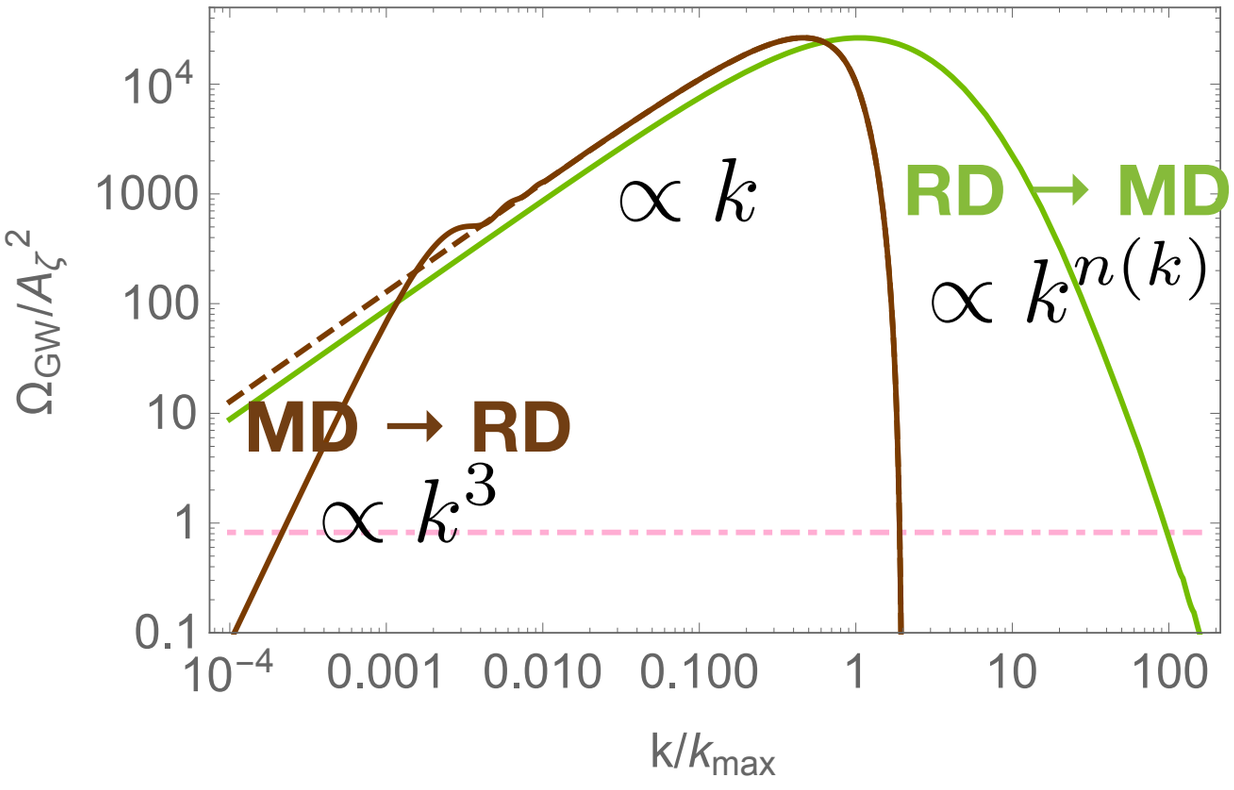
$$\frac{(x^3 - 3(x-x_{\rm eq}) - x x_{\rm eq}^2)\cos(x-x_{\rm eq}) - (3 + 3x x_{\rm eq} - x_{\rm eq}^2)\sin(x-x_{\rm eq})}{x^3} \times \frac{6}{5}\frac{\ln(c_1 u x_{\rm eq})}{(c_2 u x_{\rm eq})^2}\frac{\ln(c_1 v x_{\rm eq})}{(c_2 v x_{\rm eq})^2}$$

General cases

Schematic



Precise

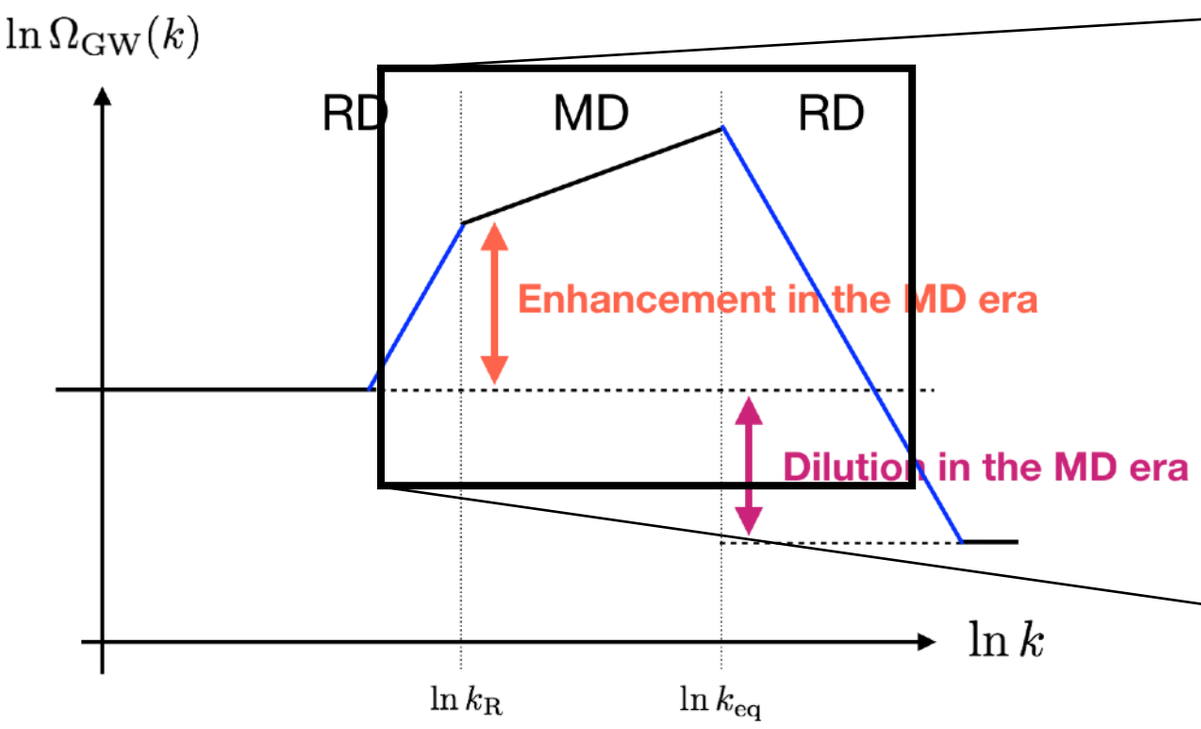


$$-6 \lesssim n(k) \lesssim -4$$

Scale-invariant primordial spectrum is assumed.

General cases

Schematic



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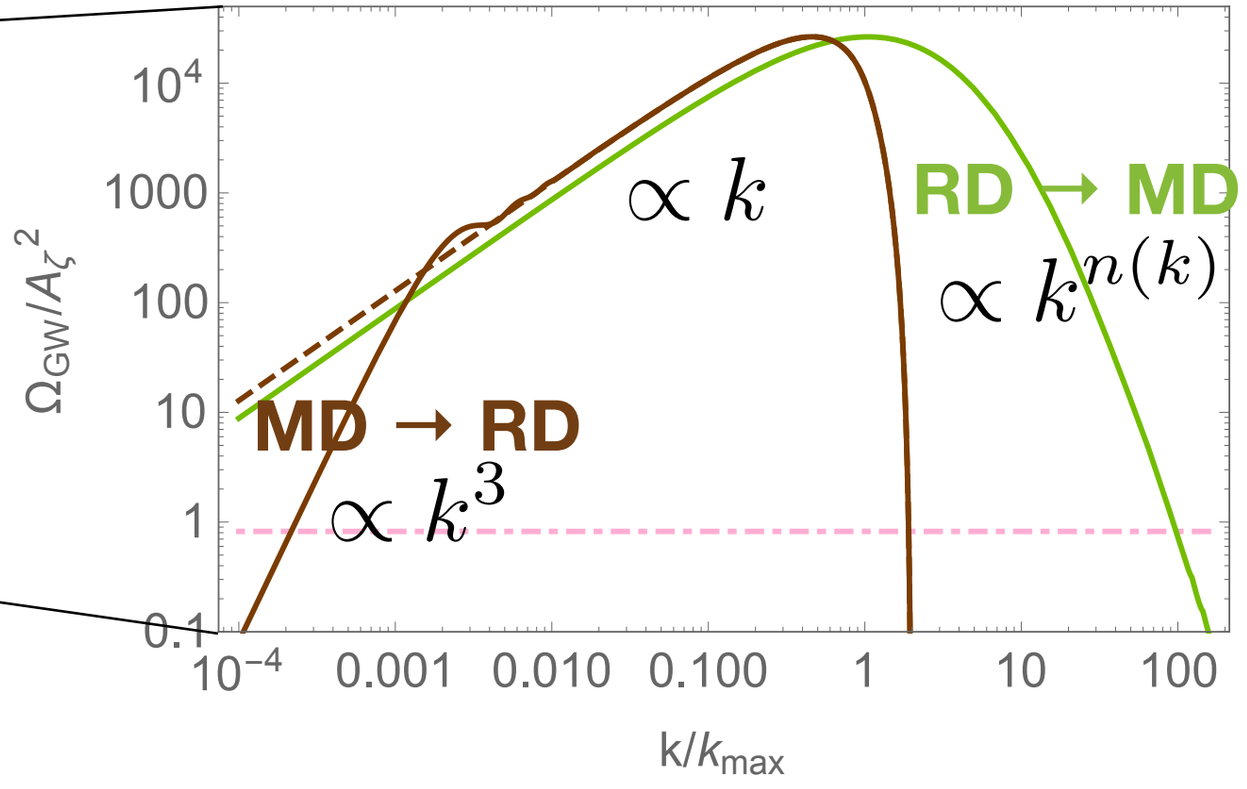

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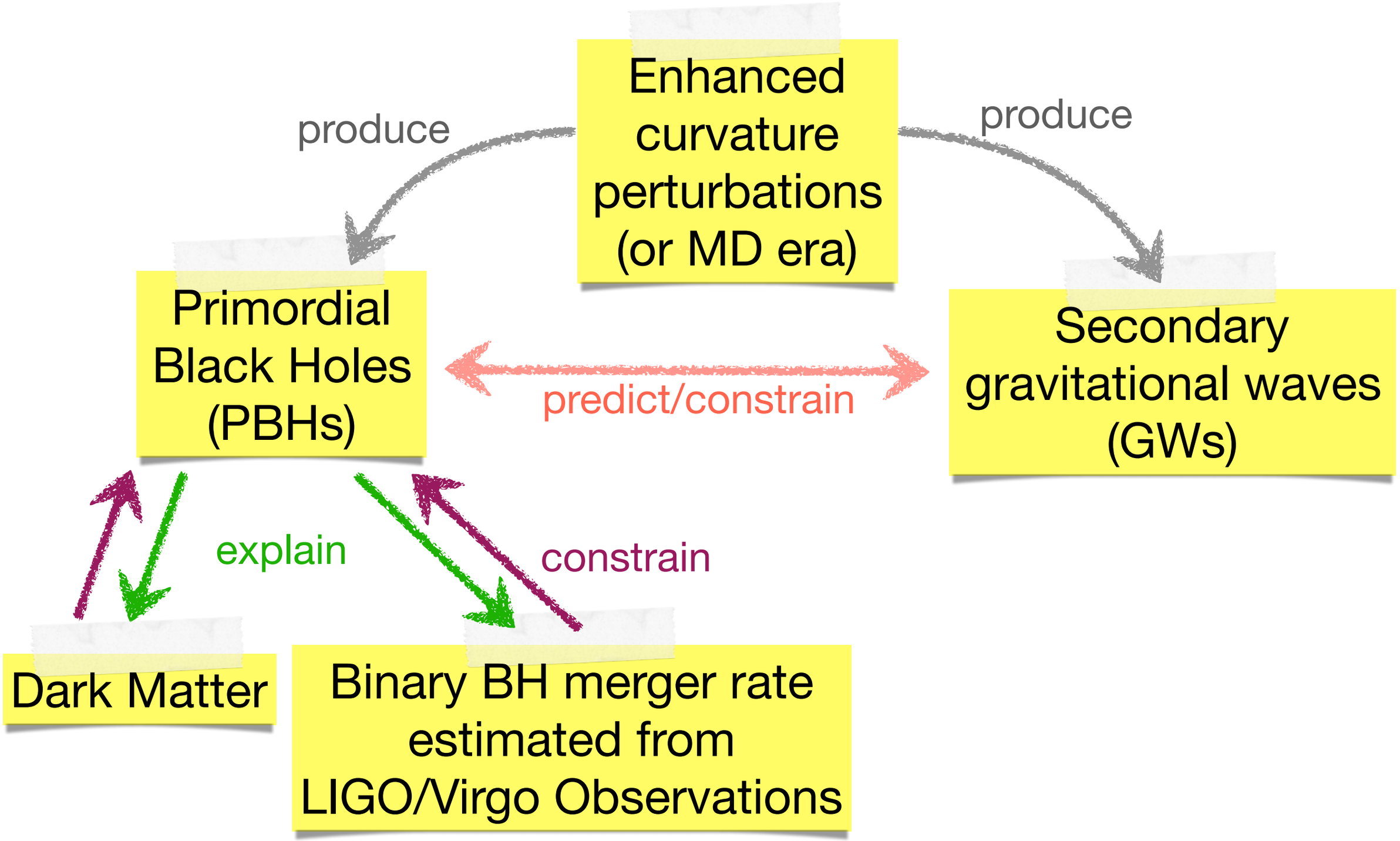
APPLICATION:
PRIMORDIAL BLACK HOLE SCENARIOS

[Kohri, Terada, 1802.06785]

Motivation & Relevance



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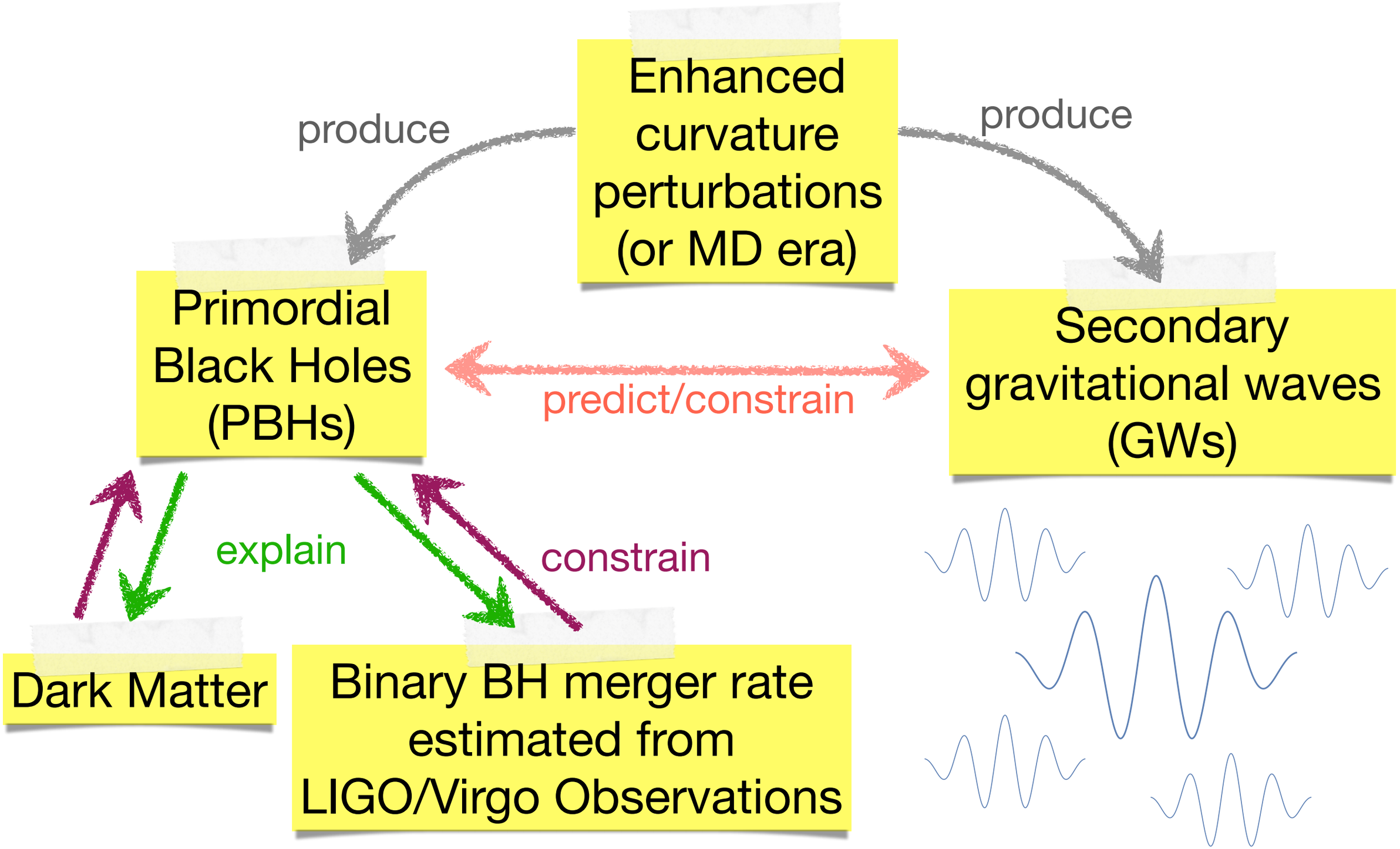


Motivation & Relevance

Enhanced curvature perturbations (or MD era)

produce

produce

Primordial Black Holes (PBHs)

predict/constrain

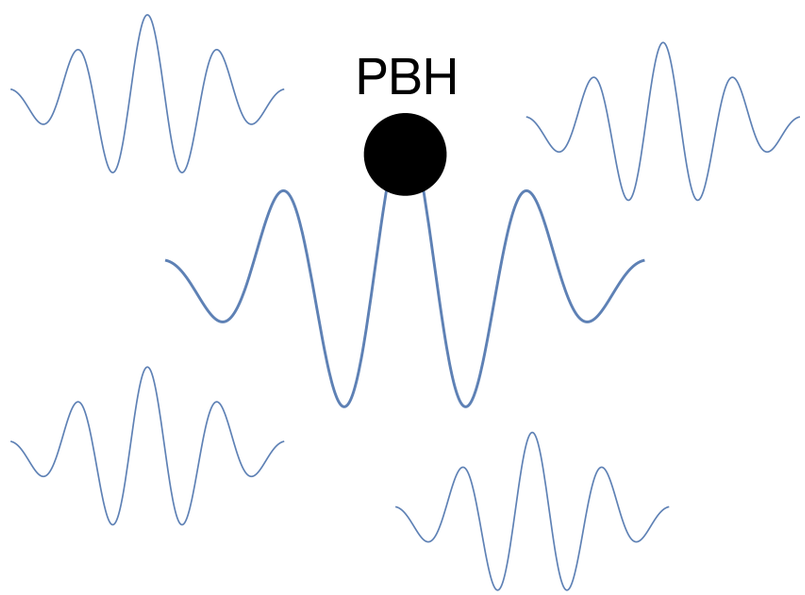
Secondary gravitational waves (GWs)

explain

constrain

Dark Matter

Binary BH merger rate estimated from LIGO/Virgo Observations



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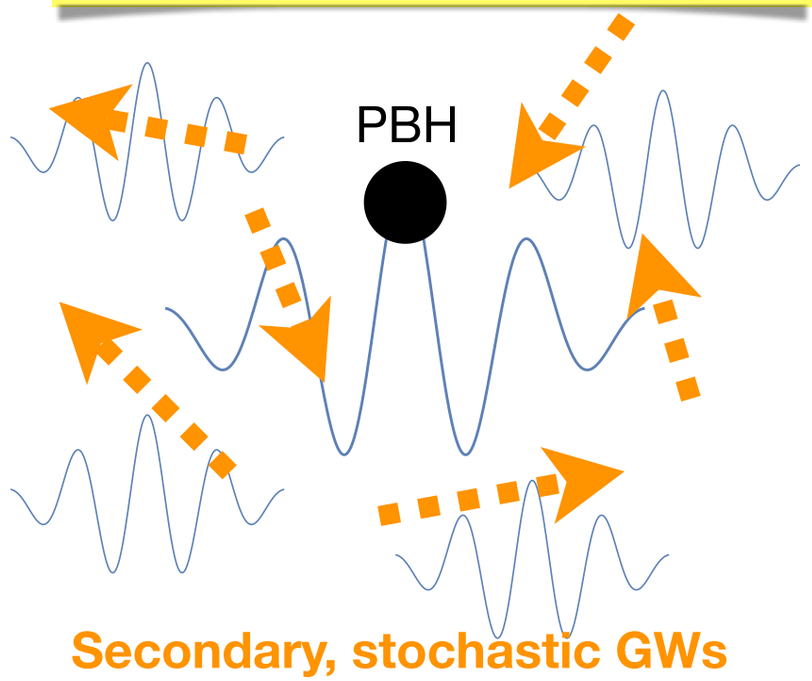

Current PBH abundance

PBH fraction in CDM

$$f_{\rm PBH}(M) = \frac{1}{\rho_{\rm CDM}} \frac{{\rm d}\rho_{\rm PBH}}{{\rm d}\ln M}$$

$$= \left(\frac{g_*(T)}{g_*(T_{\rm eq})} \frac{g_{*,s}(T_{\rm eq})}{g_{*,s}(T)} \frac{T}{T_{\rm eq}} \gamma \beta(\sigma(k(M))) \right) \Big|_{T={\rm Min}[T_M, T_{\rm R}]} \frac{\Omega_{\rm m}}{\Omega_{\rm CDM}}$$

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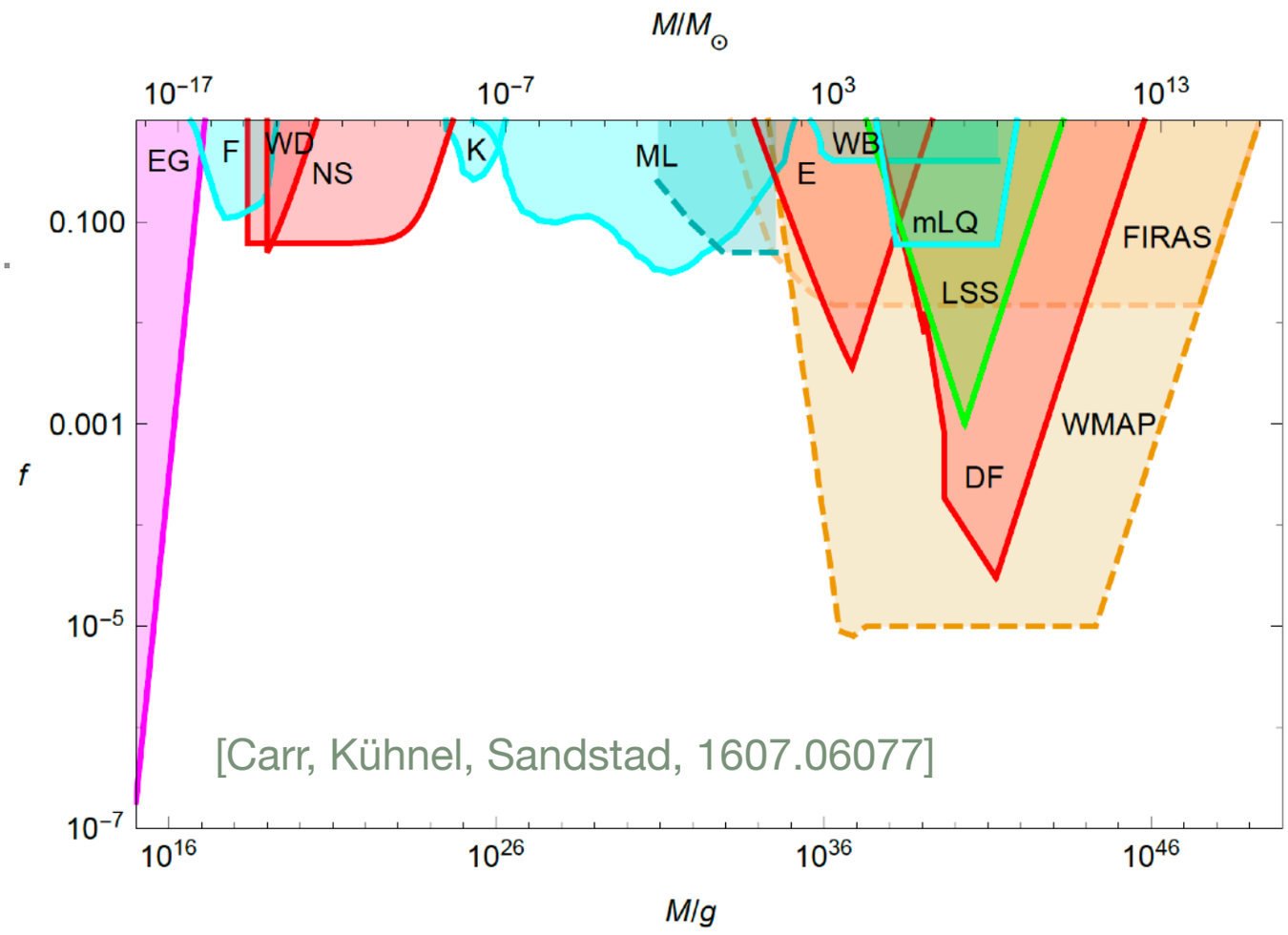
PBH formation probability

Current PBH abundance

Observational constraints

Note: These are not the latest constraints.



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PBH formation probability

PBH formation probability

Coarse-grained perturbations

[Young, Byrnes, Sasaki, 1405.7023]
see also [Ando, Inomata, Kawasaki, 1802.06393]

$$\sigma^2(k) = \int_{-\infty}^{\infty} \mathrm{d}\ln q\, w^2\left(\frac{q}{k}\right) \frac{4(1+w_{\rm eos})^2}{(5+3w_{\rm eos})^2} \left(\frac{q}{k}\right)^4 T^2(q/k) P_\zeta(q)$$

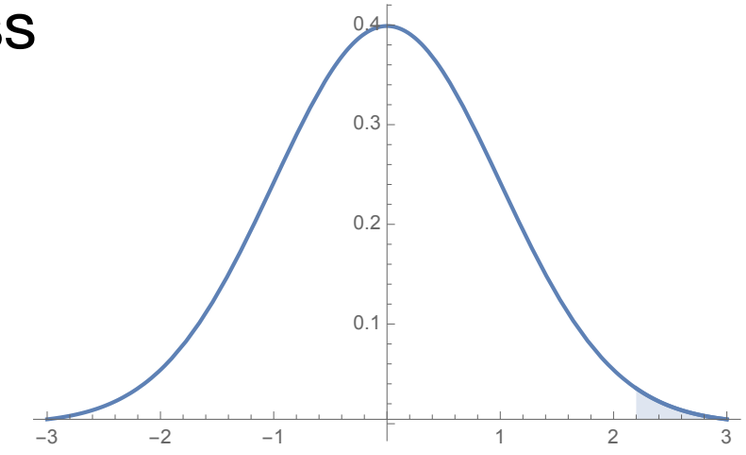
$$\sim \frac{2(1+w_{\rm eos})^2}{(5+3w_{\rm eos})^2} P_\zeta(k)$$

Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_{\rm c}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{\delta}{2\sigma^2}\right) \mathrm{d}\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

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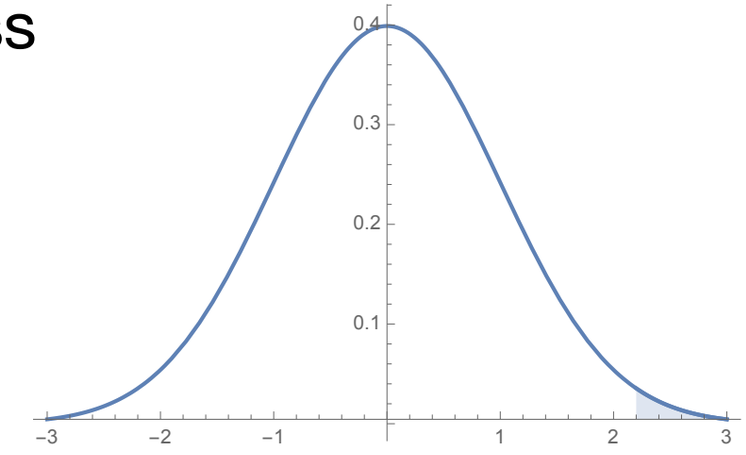

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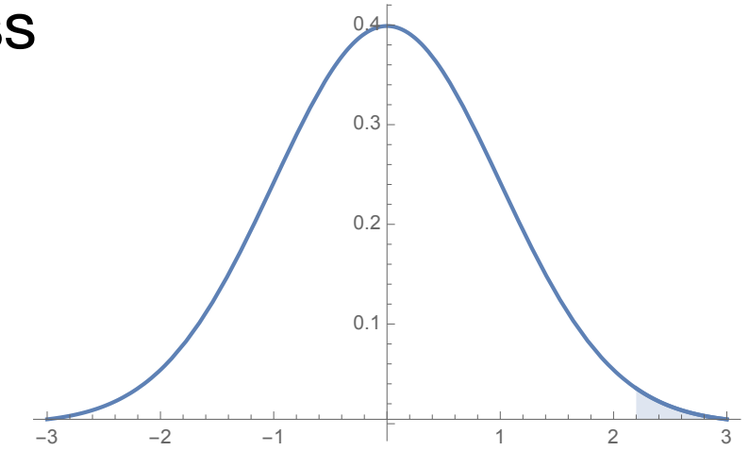

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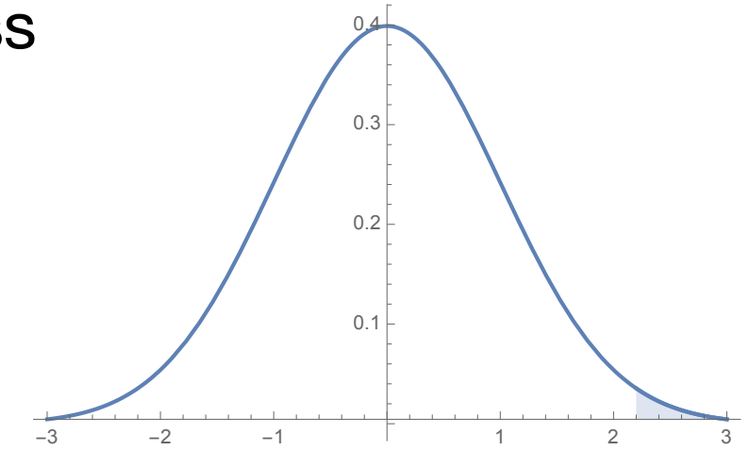

(exaggerated figure for illustration)

PBH formation probability

Formation probability in RD

Pressure prevents the formation → rare process

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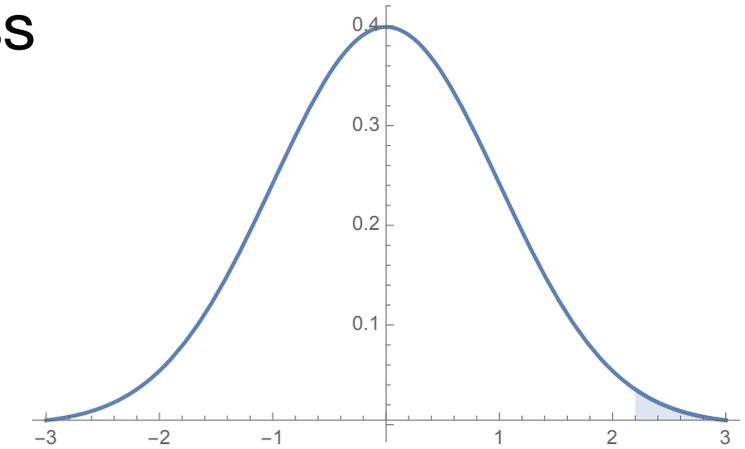

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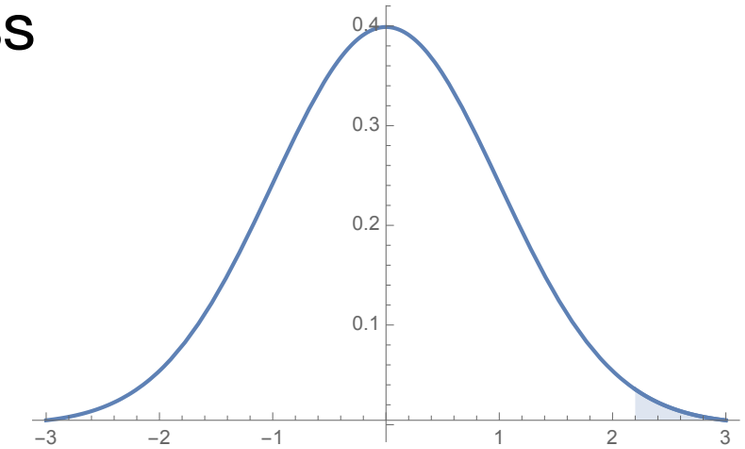

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Formation probability in MD

[Khlopov, Polnarev, 1980, 1985]

$$\frac{\delta\rho}{\rho} \propto a(t)$$

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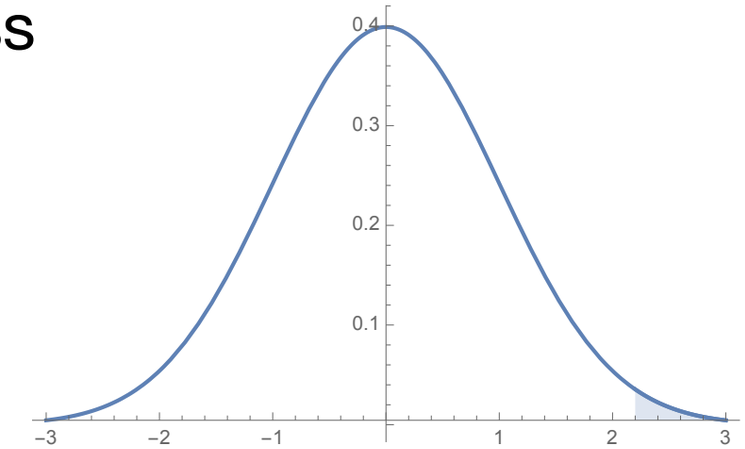

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[Harada, Yoo, Kohri, Nakao, Jhingan, 1609.01588]
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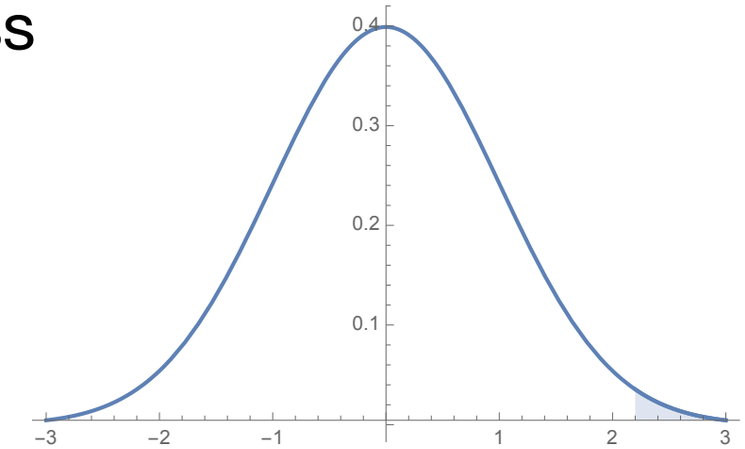

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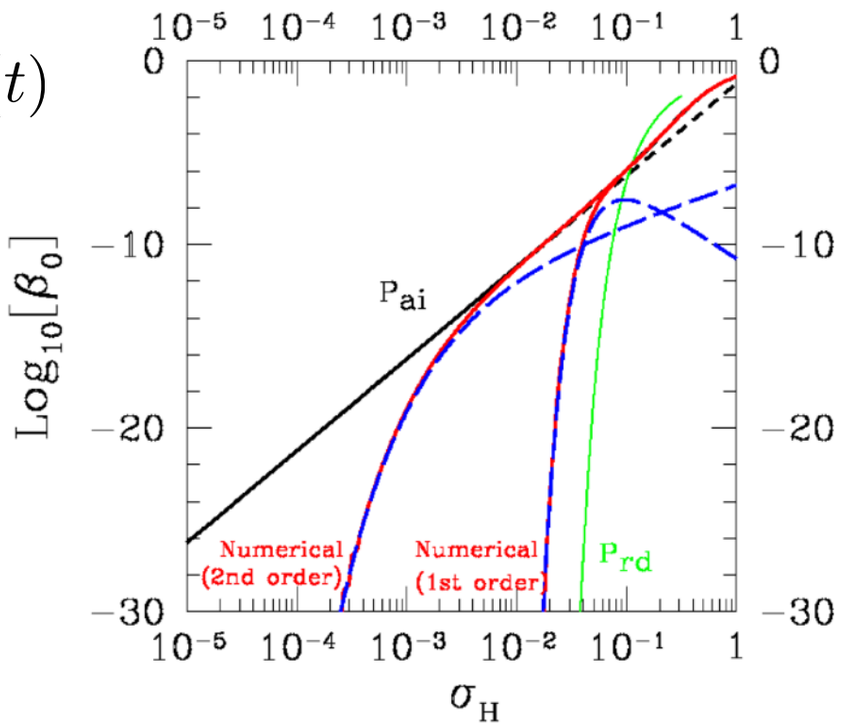


[Harada, Yoo, Kohri, Nakao, 1707.03595]

Inflation with running spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{\alpha_s}{2} \ln \frac{k}{k_*} + \frac{\beta_s}{6} \left(\ln \frac{k}{k_*}\right)^2 + \cdots}$$

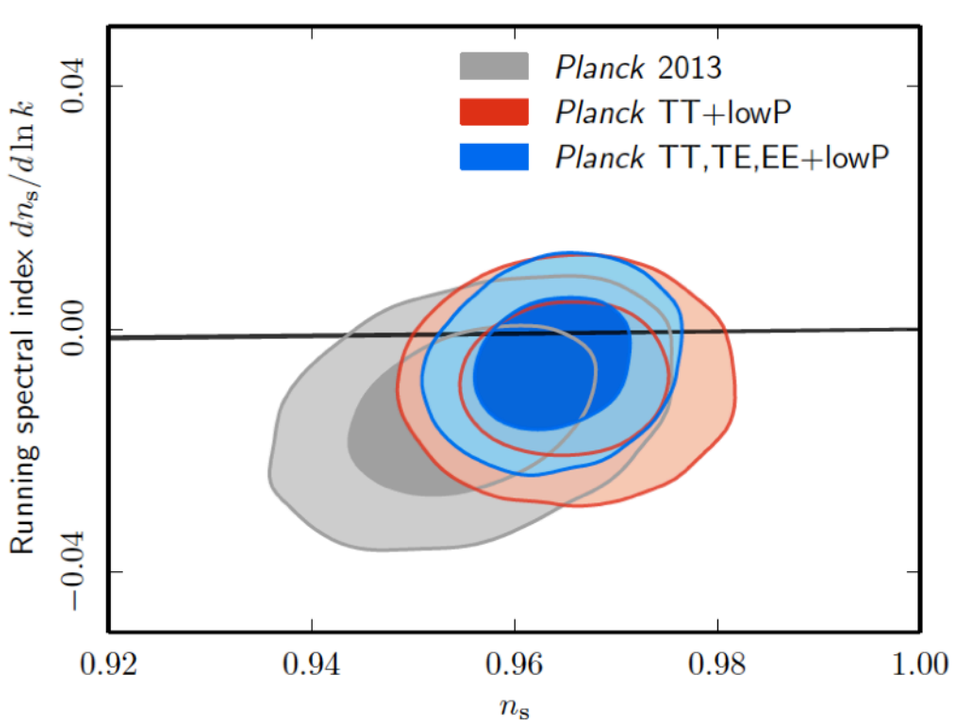


Fig. 4. Marginalized joint 68 % and 95 % CL for $(n_s\,, \mathrm{d}n_s/\mathrm{d}\ln k)$ using *Planck* TT+lowP and *Planck* TT,TE,EE+lowP. For comparison, the thin black stripe shows the prediction for single-field monomial chaotic inflationary models with $50 < N_* < 60$.

Planck TT+lowP (TT,TE,EE+lowP)

$$n_s = 0.9569 \pm 0.0077 \;\; (0.9586 \pm 0.0056)\,,$$
$$dn_s/d\ln k = 0.011^{+0.014}_{-0.013} \;\; (0.009 \pm 0.010)\,,$$
$$d^2 n_s/d\ln k^2 = 0.029^{+0.015}_{-0.016} \;\; (0.025 \pm 0.013)\,,$$

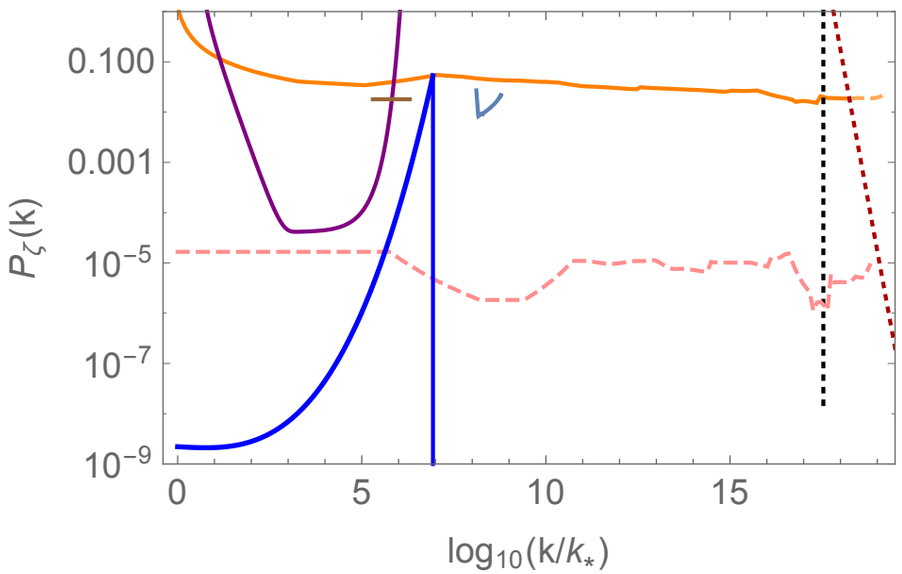
at 68% CL at the pivot scale k*=0.05/Mpc.

[Planck 2015, “Constraints on Inflation”]

Interplay between PBH & GW

PBH-for-LIGO scenario

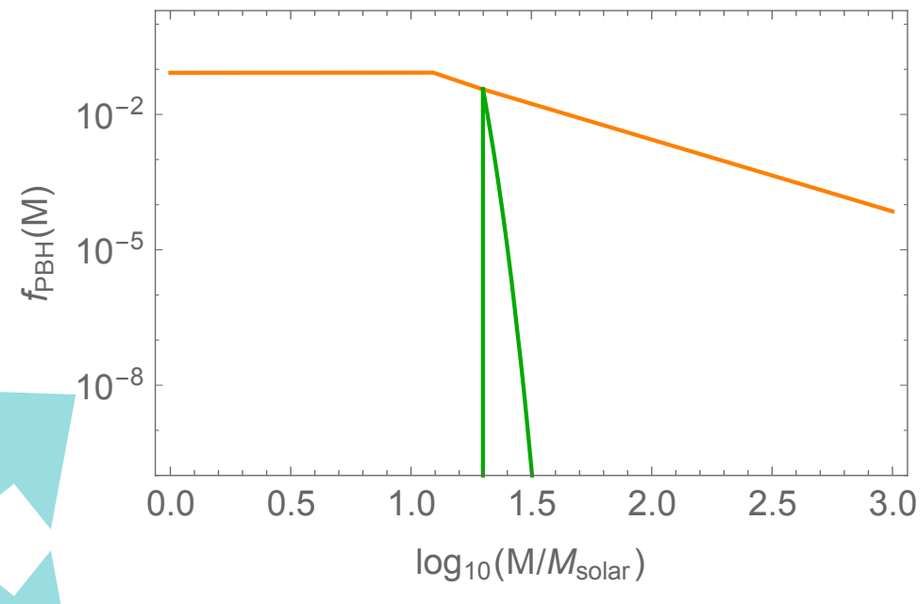
Curvature perturbations



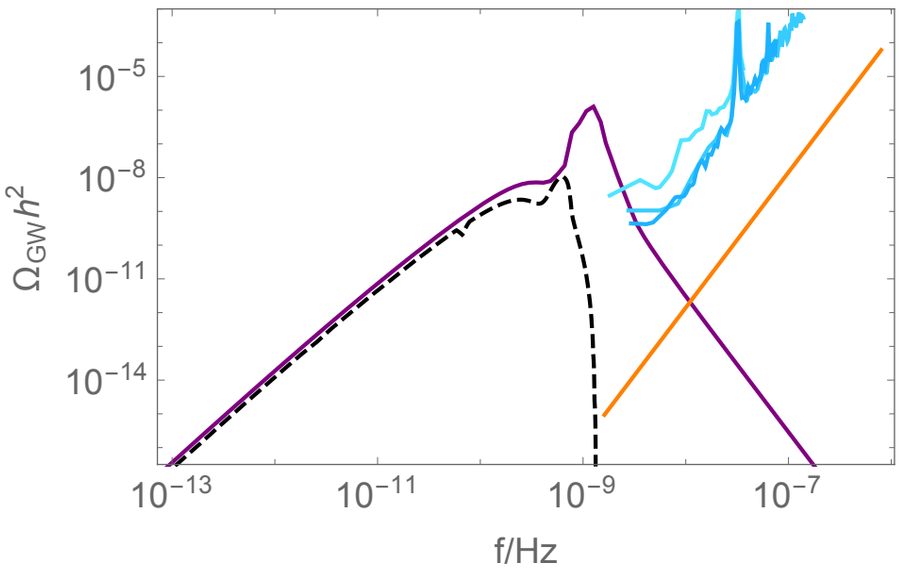
$$n_{\rm s} = 0.96$$
$$\alpha_{\rm s} = 0$$
$$\beta_{\rm s} = 0.026$$

$$T_{\rm R} = 10^9\,{\rm GeV}$$

PBH mass spectrum



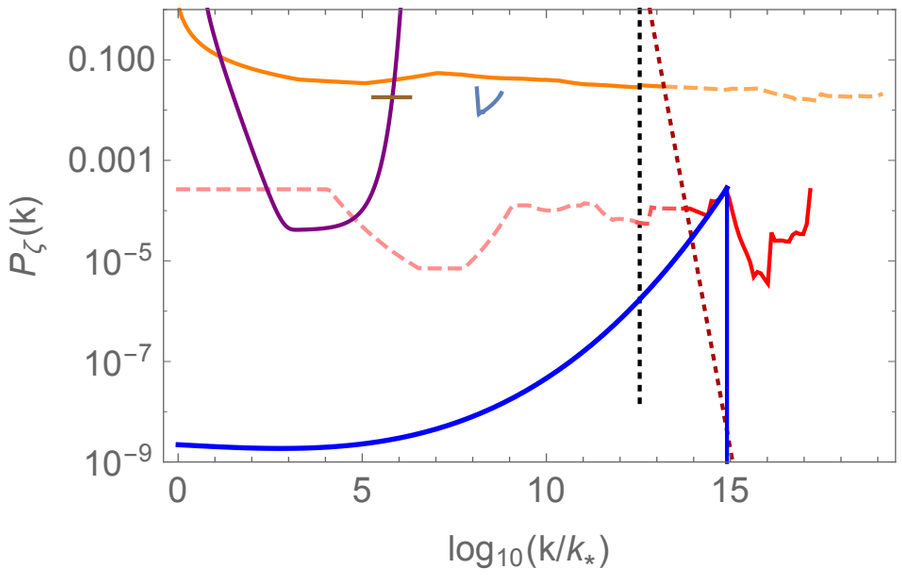
Secondary GW spectrum



Interplay between PBH & GW

PBH-DM scenario

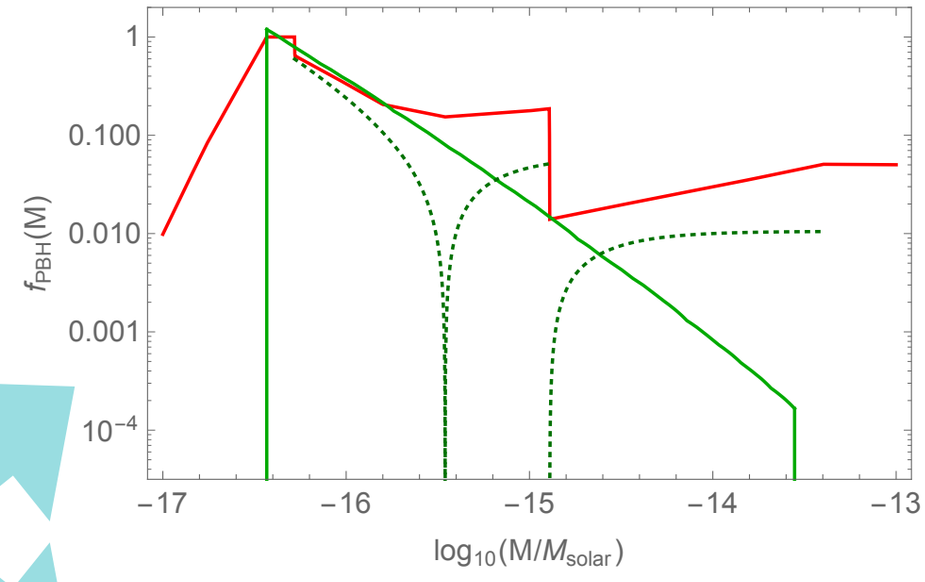
Curvature perturbations



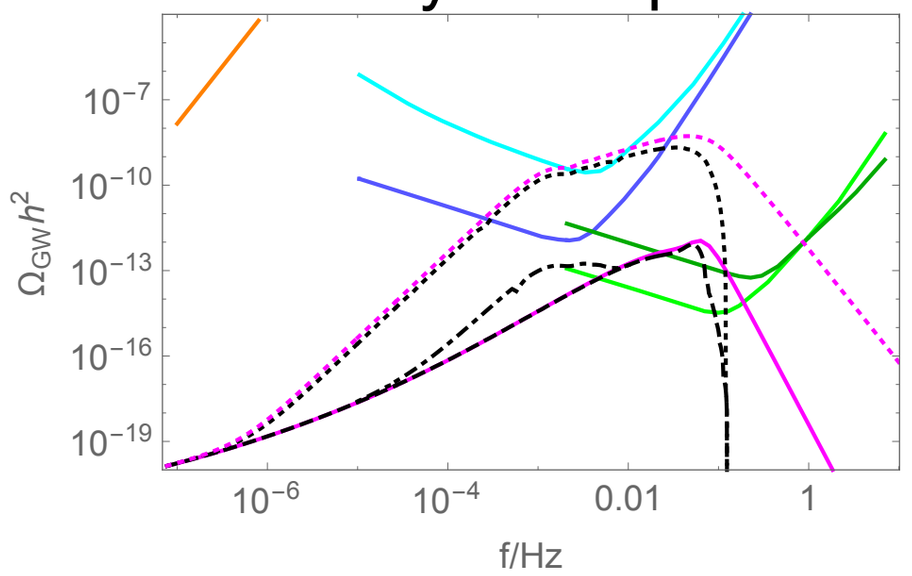
$$n_{\rm s} = 0.96$$
$$\alpha_{\rm s} = 0$$
$$\beta_{\rm s} = 0.0019485$$

$$T_{\rm R} = 10^4\,{\rm GeV}$$

PBH mass spectrum



Secondary GW spectrum



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- **Please use our results to write your papers!**

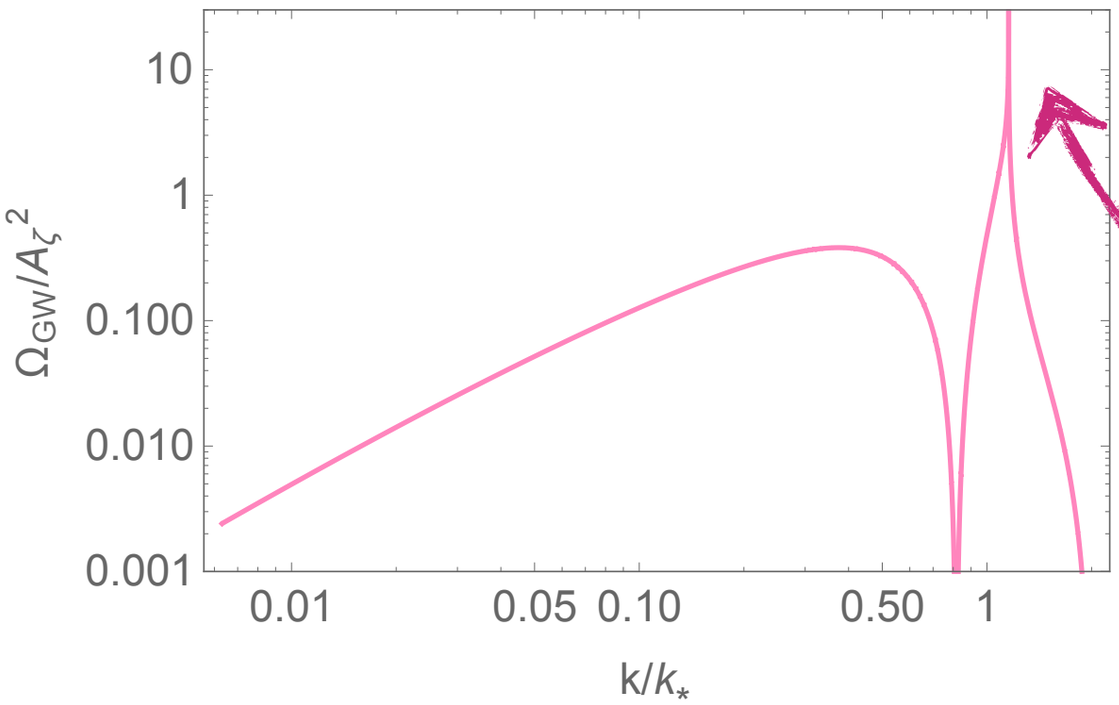
Appendix

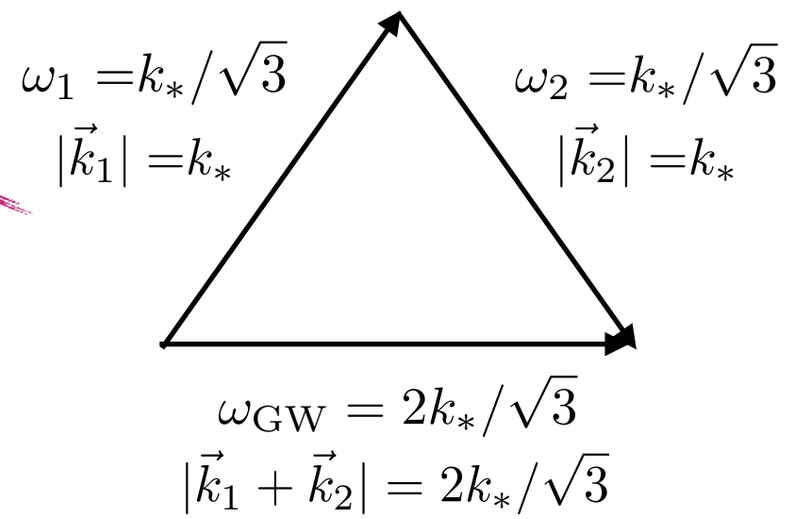
Calculable GW spectrum

Example 1 in RD: Monochromatic case $\quad \mathcal{P}_\zeta(k) = A_\zeta \delta(\log k/k_*)$

$$\Omega_{\rm GW}(\eta,k) = \frac{3A_\zeta^2}{64}\left(\frac{4-\tilde{k}^2}{4}\right)^2 \tilde{k}^2\left(3\tilde{k}^2-2\right)^2 \qquad \tilde{k} \equiv k/k_*$$

$$\times\left(\pi^2(3\tilde{k}^2-2)^2\Theta(2\sqrt{3}-3\tilde{k}) + \left(4+(3\tilde{k}^2-2)\log\left|1-\frac{4}{3\tilde{k}^2}\right|\right)^2\right)\Theta(2-\tilde{k})$$





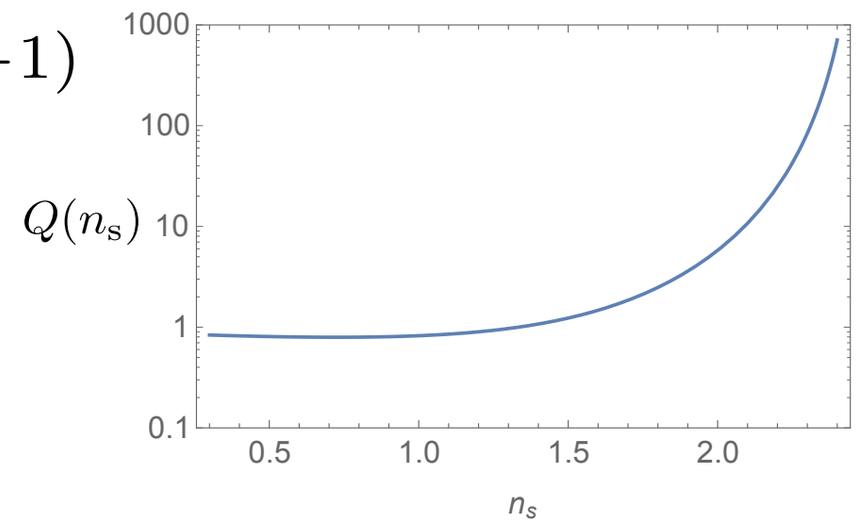
Calculable GW spectrum

Example 2 in RD: Scale-invariant case $\mathcal{P}_\zeta(k) = A_\zeta$

$$\Omega_{\rm GW}(\eta, k) \simeq 0.8222 A_\zeta^2$$

Example 3 in RD: Power-law case $\mathcal{P}_\zeta = A_\zeta \left(\frac{k}{k_*}\right)^{n_{\rm S}-1}$

$$\Omega_{\rm GW}(\eta, k) = Q(n_{\rm S}) A_\zeta^2 \left(\frac{k}{k_*}\right)^{2(n_{\rm S}-1)}$$



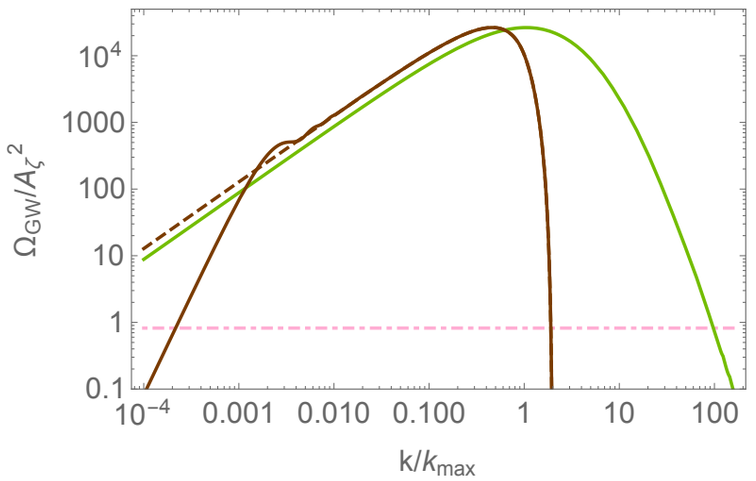
Calculable GW spectrum

Example 1 in MD: Monochromatic case $\mathcal{P}_\zeta(k) = A_\zeta \delta(\log k/k_*)$

$$\Omega_{\rm GW} = \frac{3}{25}\left(\frac{k_*}{aH}\right)^2 \left(1-\left(\frac{k}{2k_*}\right)^2\right)^2 A_\zeta^2 \Theta(2k_* - k)$$

Example 2 in MD: Scale-invariant case with a cutoff

$$\mathcal{P}_\zeta(k) = A_\zeta \Theta(k_{\rm max} - k)$$

$$\Omega_{\rm GW} = \frac{A_\zeta^2}{14000}\left(\frac{k}{aH}\right)^2 \times \begin{cases} \left(1792\tilde{k}^{-1} - 2520 + 768\tilde{k} + 105\tilde{k}^2\right) & (0 < k \le k_{\rm max}) \\ \left(1-2\tilde{k}^{-1}\right)^4 \left(105\tilde{k}^2 + 72\tilde{k} + 16 - 32\tilde{k}^{-1} - 16\tilde{k}^{-2}\right) & (k_{\rm max} < k \le 2k_{\rm max}) \end{cases}$$

$$\tilde{k} \equiv k/k_*$$