

# Complex Langevin simulations and the QCD phase diagram

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**D·IAS**

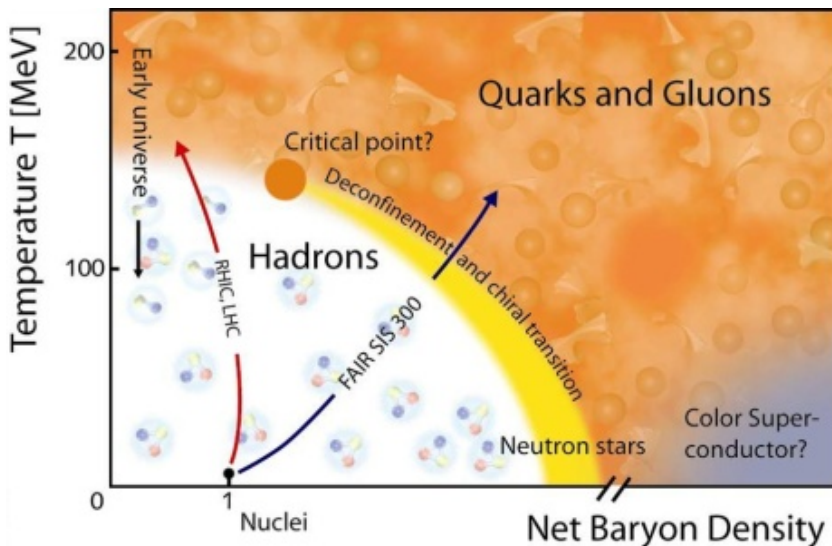
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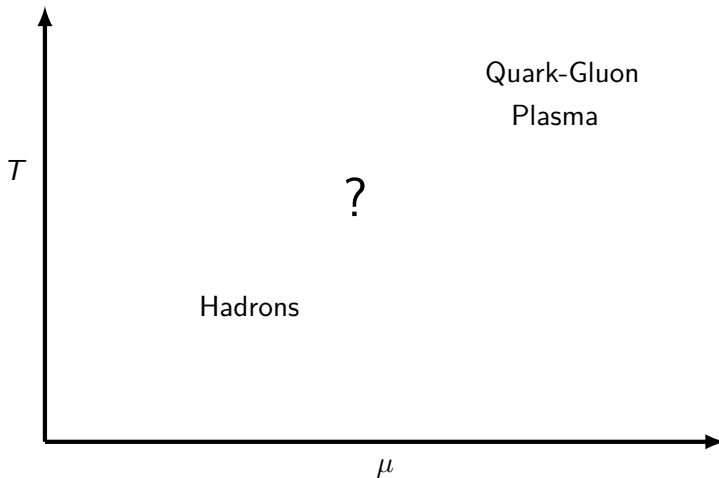
# Outline

- 1 Introduction
- 2 Pure Yang-Mills
- 3 Heavy Dense QCD
- 4 XY model
- 5 Full QCD

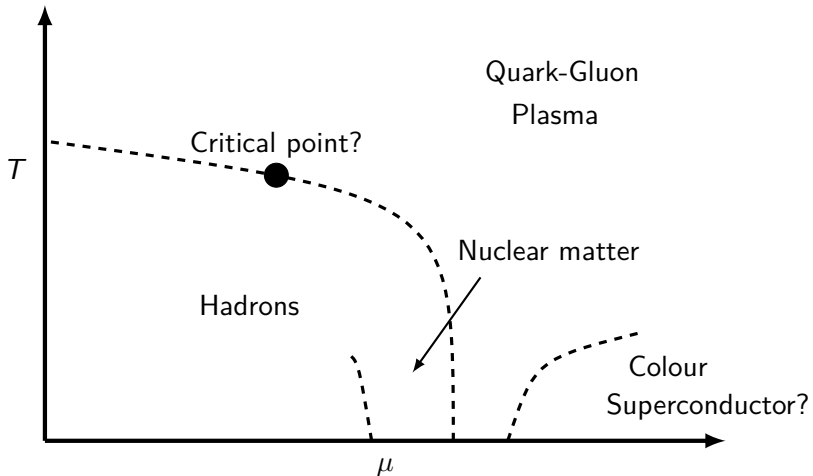
# Phase diagram for QCD



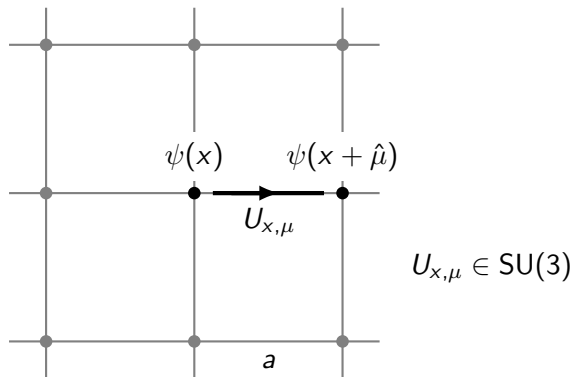
# Phase diagram for QCD



# Phase diagram for QCD



## Overview on Lattice QCD



- Discretize Euclidean space-time by a hypercubic lattice  $\Lambda$
- Quantize QCD using Euclidean path integrals
- Calculate expectation values using Monte Carlo techniques:

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}[U] A[U] (\det D)^{N_f} e^{-S_G[U]} \quad U_{x,\mu} \in \text{SU}(3)$$

- **Ab initio** method: Only QCD parameters  $m_q \rightarrow \kappa$ ,  $g_0 \rightarrow \beta$ .

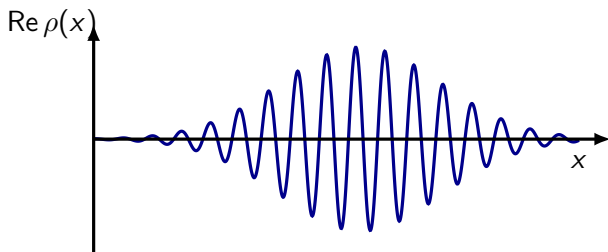
# The SIGN problem

- Finite chemical potential  $\rightarrow$  SIGN problem

$$(\det D(\mu))^* = \det D(-\mu^*) \rightarrow \det D(\mu \neq 0) \in \mathbb{C}.$$

- Importance Sampling based Monte Carlo methods fail

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U] O(U) |\det D| e^{i\phi} e^{-S_G(U)}$$



# How bad is the SIGN problem?!

- Overlap problem

$$\langle O \rangle = \frac{\int \mathcal{D}[U] O |\det D| e^{i\phi} e^{-S_G(U)}}{\int \mathcal{D}[U] |\det D| e^{i\phi} e^{-S_G(U)}} = \frac{\langle O e^{i\phi} \rangle_{pq}}{\langle e^{i\phi} \rangle_{pq}}$$

- The average phase **vanishes** in the thermodynamic limit:  
 $V \rightarrow \infty$

$$\langle e^{i\phi} \rangle_{pq} = e^{-V \cdot \Delta F} \rightarrow 0$$



# Alternatives?

## Taylor expansion

- Expand around small  $\mu$

$$\frac{T_c(\mu)}{T_c(0)} = 1 + c_2 \cdot \left(\frac{\mu}{T_c(0)}\right)^2 + c_4 \cdot \left(\frac{\mu}{T_c(0)}\right)^4 + \dots$$

- Radius of convergence?

## Reweighting

- Absorb phase in the observable

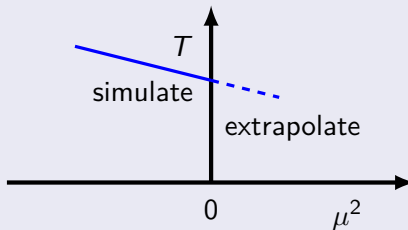
$$\langle O \rangle = \frac{\int \mathcal{D}[U] O |\det D| e^{i\phi} e^{-S_G(U)}}{\int \mathcal{D}[U] |\det D| e^{i\phi} e^{-S_G(U)}} = \frac{\langle O e^{i\phi} \rangle_{pq}}{\langle e^{i\phi} \rangle_{pq}}$$

- Overlap problem

# Alternatives?

## Analytic continuation

- Continue from imaginary  $\mu$  to real  $\mu$



- How far can one extrapolate?

Many more ...

## Here: Complex Langevin

- Stochastic simulation with complexified degrees of freedom

## Complex Langevin - Idea

- Consider a simple Gaussian integral

$$Z(a, b) = \int dx e^{-S}, \text{ with } S = \frac{1}{2}ax^2 + ibx$$

- Solution: Complete the square or complexify  $x \rightarrow z = x + iy$
- Consider **Real action**
- Langevin equation: Stochastic process (Brownian motion)

$$\frac{\partial x}{\partial t} = \frac{\partial S}{\partial x} + \eta(t)$$

where  $\langle \eta(t)\eta(t') \rangle = 2\delta(t - t')$  Gaussian white noise

- The associated Fokker-Plank equation

$$\langle O(x) \rangle_\eta = \int dx \rho(x, t) O(x)$$

- Stationary solution  $\rho(x) = e^{-S}$

# Complex Langevin - Idea

- Consider **Complex action**
- Complexify  $x \rightarrow z = x + iy$  Langevin equation

$$\frac{\partial x}{\partial t} = \text{Re} \frac{\partial S}{\partial z} + \eta(t) \quad \text{and} \quad \frac{\partial y}{\partial t} = \text{Im} \frac{\partial S}{\partial z}$$

- The associated Fokker-Plank equation

$$\langle O(x + iy) \rangle_\eta = \int dx dy \tilde{\rho}(x + iy) O(x + iy)$$

- Convergence:
  - Imaginary direction  $y$  is compact (No boundary terms)
  - Action  $S$  is holomorphic

$$\Rightarrow \int dx dy \tilde{\rho}(x + iy) O(x + iy) = \int dx \rho(x, t) O(x)$$

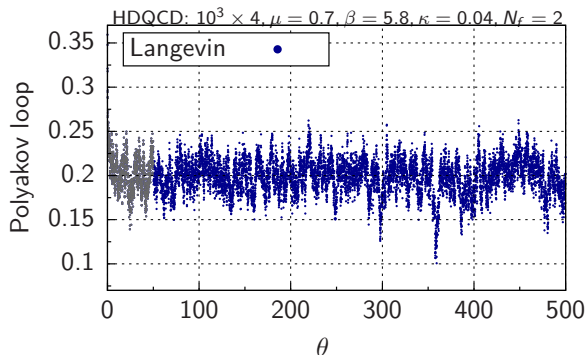
## Complex Langevin - Gauge theories

- Complexify degrees of freedom  $SU(3) \rightarrow SL(3, \mathbb{C})$

$$U_{x,\mu} = \exp \left[ i a \lambda^c \left( A_{x,\mu}^c + i B_{x,\mu}^c \right) \right]$$

- Evolve links according (1st order) Langevin equation

$$U_{x,\mu}(\theta + \varepsilon) = \exp \left[ i \lambda^a \left( -\varepsilon D_{x,\mu}^a S + \sqrt{\varepsilon} \eta_{x,\mu}^a \right) \right] U_{x,\mu}(\theta)$$



# Complex Langevin simulations

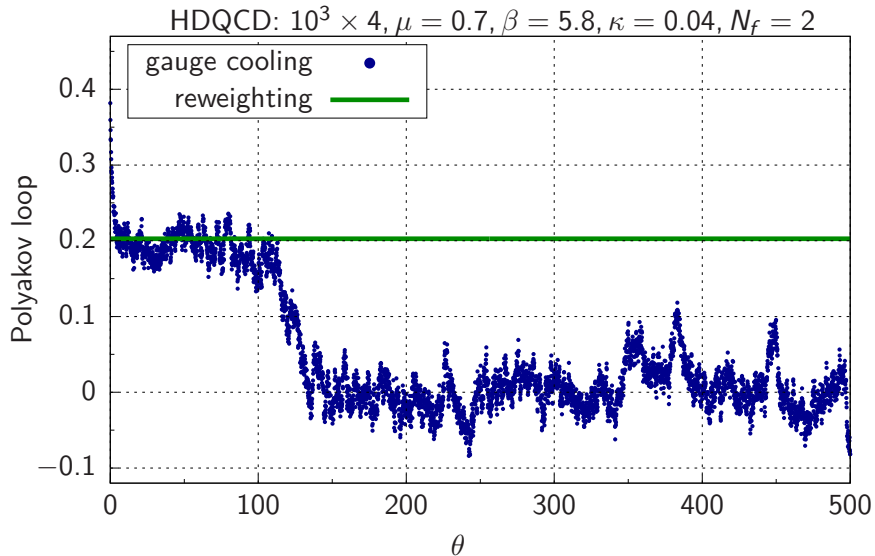
- However,  $SL(3, \mathbb{C})$  is **not** a compact group. . .
- Convergence  $\Leftrightarrow$ 
  - Action  $S$  is holomorphic
  - "Imaginary" direction of  $SL(3, \mathbb{C})$  falls off quickly enough
- Measure distance to  $SU(3)$  manifold

$$\text{unitnorm} = \text{Tr} \left( U_{x,\mu} U_{x,\mu}^\dagger - 1 \right)^2$$

- Gauge cooling is essential, but **sometimes** not sufficient. . .

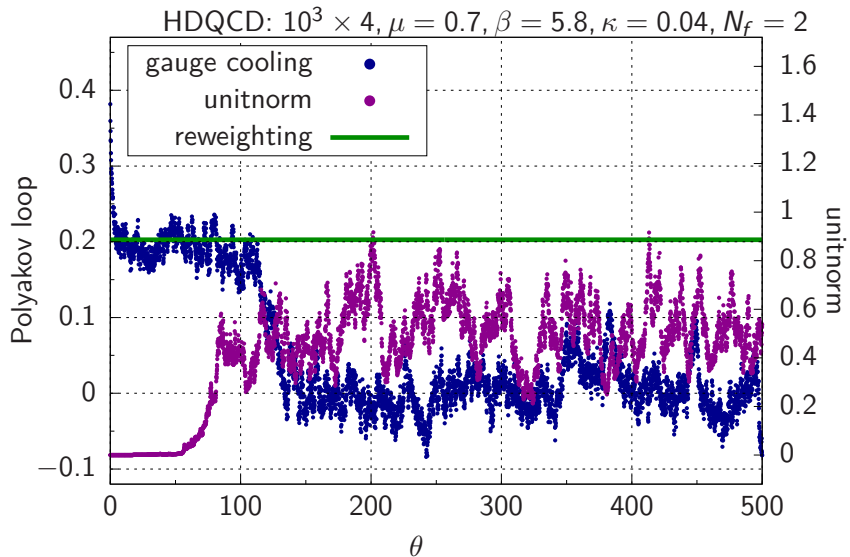
$$U_{x,\mu} \rightarrow \Omega_x U_{x,\mu} \Omega_{x+\mu}^{-1}$$

# Gauge cooling



- Tunneling to wrong results.

# Gauge cooling



- Tunneling to wrong results, when unitnorm grows too large.



# Dynamic stabilization

- Adding a trivial force to the Langevin dynamics

$$U_{x,\nu}(\theta + \varepsilon) = \exp \left[ i\lambda^a \left( \varepsilon K_{x,\nu}^a + i\varepsilon \alpha_{DS} M_x^a + \sqrt{\varepsilon} \eta_{x,\nu}^a \right) \right] U_{x,\nu}(\theta)$$

where

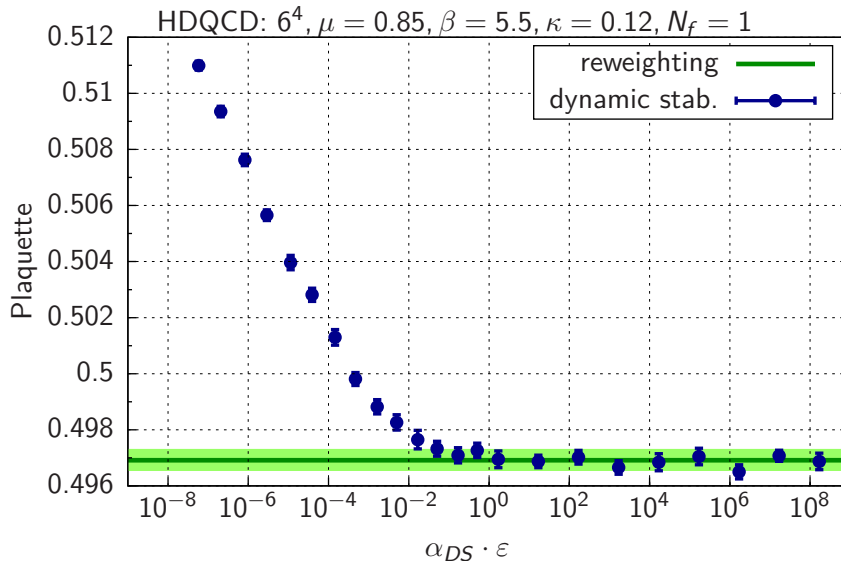
$$M_x^a = i b_x^a \left( \sum_c b_x^c b_x^c \right)^3 \text{ and } b_x^a = \text{Tr} \left[ \lambda^a \sum_\nu U_{x,\nu} U_{x,\nu}^\dagger \right].$$

- Expanding the force in terms of gauge fields  $A$  and  $B$

$$M_x^a \sim a^7 \left( \bar{B}_y^c \bar{B}_y^c \right)^3 \bar{B}_x^a + \mathcal{O}(a^8).$$

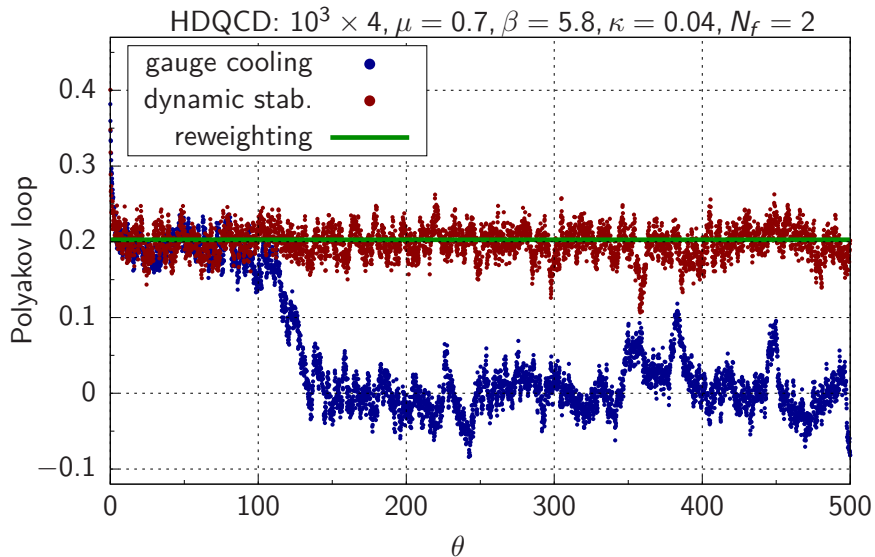
- Dynamic stabilization is numerically cheap and can be combined with gauge cooling (Here: 1 step)

# Dynamic stabilization



- For sufficient large  $\alpha_{DS}$  we find agreement with reweighting.

# Dynamic stabilization

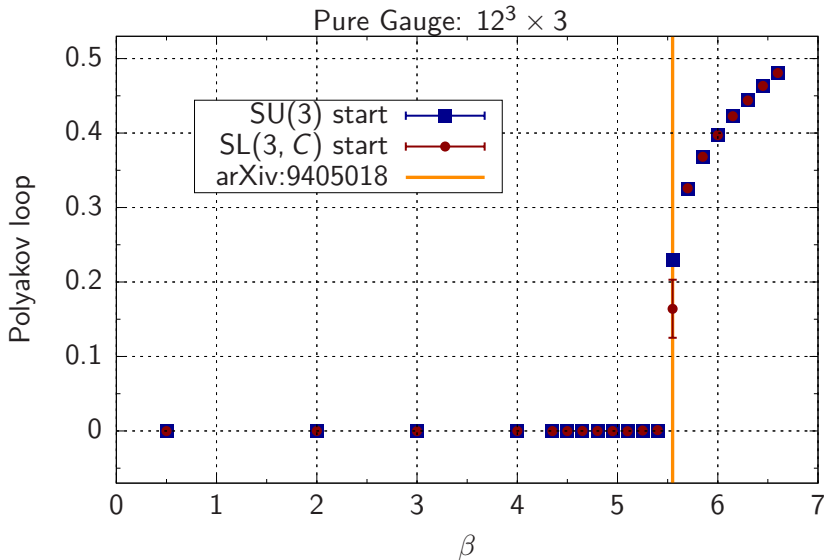


- Improved stability using dynamic stabilization

# Results

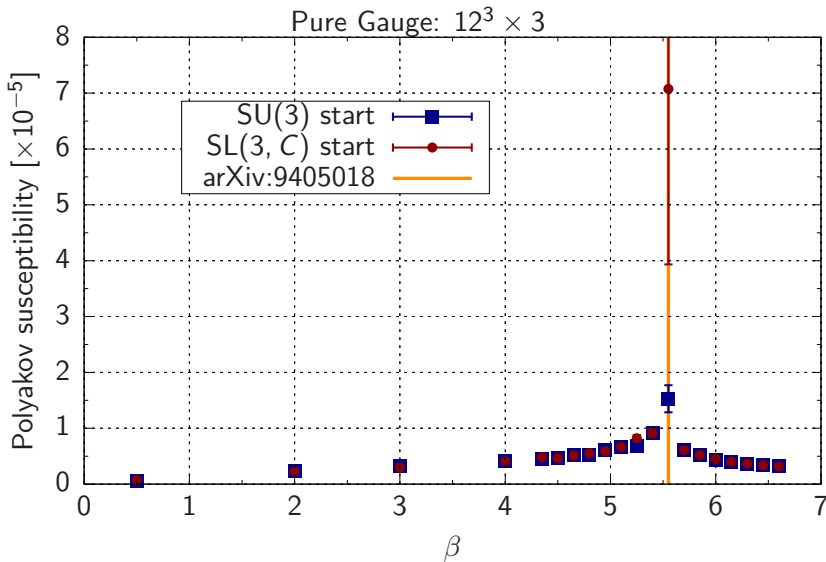
- **Pure Yang-Mills**  $\Leftrightarrow$  No sign problem
  - Checking Langevin simulations
  - Compare to standard Hybrid Monte Carlo methods
- **Heavy dense QCD (HDQCD)**
  - Heavy and dense approximation of QCD
  - Has a sign problem and phase transitions
  - Numerical cheap :)
- **XY model with finite  $\mu$** 
  - Toy model for QCD
- **Full (Staggered) QCD**
  - Including light dynamical fermions
  - Very preliminary results

# Pure Yang-Mills



- The correct transition is obtained, even for SL(3,  $\mathbb{C}$ ) start.

# Pure Yang-Mills



- The correct transition is obtained, even for SL(3, C) start.

# Heavy Dense QCD

- **Here: QCD in the limit of heavy quarks (HDQCD).**

- Fermion determinant simplifies with the Polyakov loops  $P_{\vec{x}}$  and  $P_{\vec{x}}^{-1}$  as

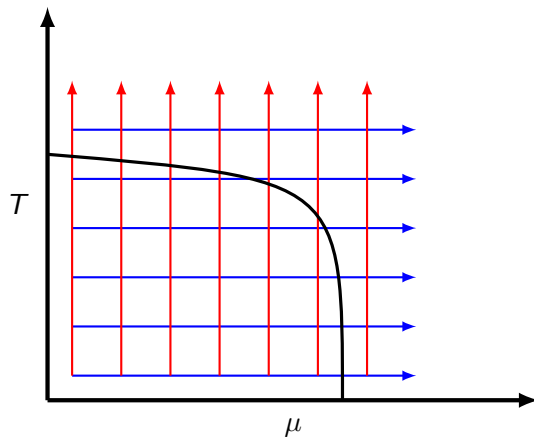
$$\det D(\mu) = \prod_{\vec{x}} \det (1 + C P_{\vec{x}})^2 \det (1 + C' P_{\vec{x}}^{-1})^2,$$

where the Polyakov loop  $P_{\vec{x}}$  is defined as

$$P_{\vec{x}} = \frac{1}{V} \sum_t U_0(\vec{x}, t)$$

- For the gluonic part, we use the full Wilson gauge action.
- Map out the **phase diagram** for HDQCD.
- Expected transition:  $\mu_c^0 = -\ln(2\kappa)$

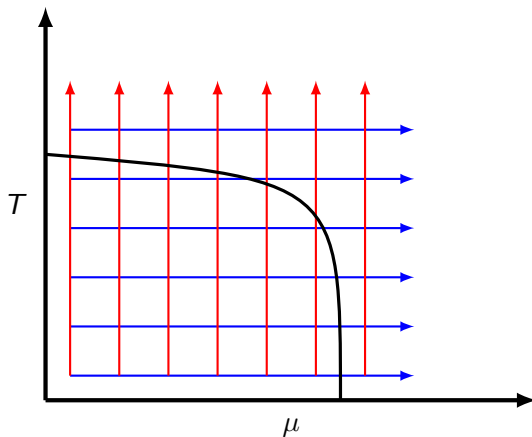
# Heavy Dense QCD



- Strategy:
  - Determine  $\mu$ -transition in Fermion density
  - Determine  $T$ -transition in Polyakov loop

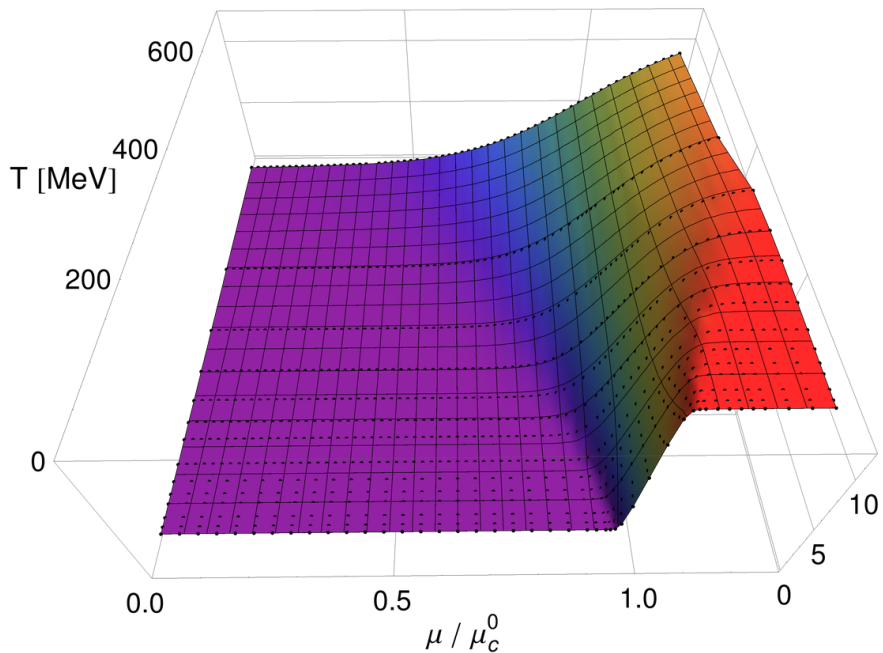


# Heavy Dense QCD



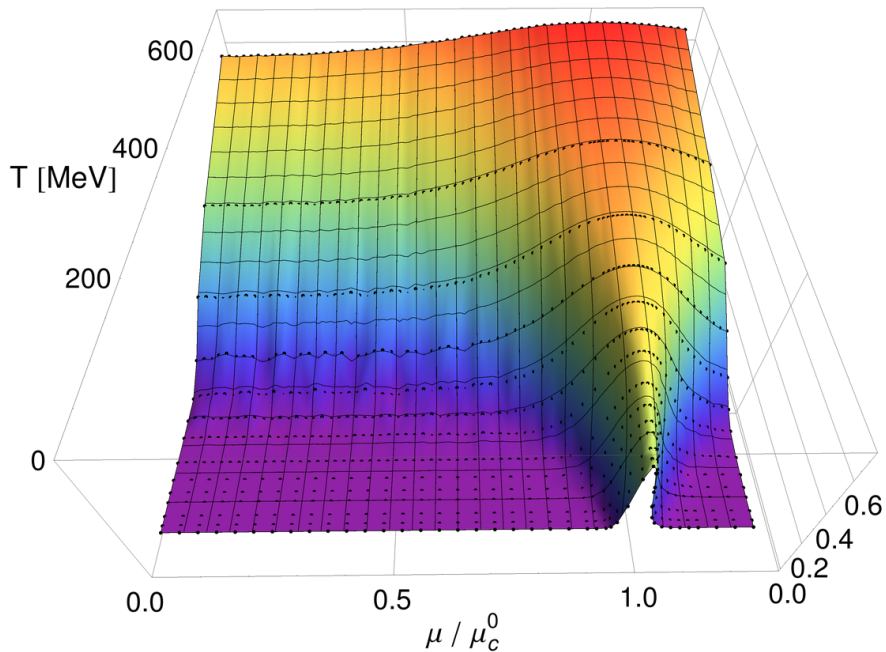
$\beta = 5.8$	$V = 6^3, 8^3, 10^3$	$a \sim 0.15 \text{ fm}$
$\kappa = 0.04$	$N_f = 2$	$\mu_c^0 = 2.53$
$N_\tau$	28 - 2	
$T \text{ [MeV]}$	48 - 671	

# Heavy Dense QCD



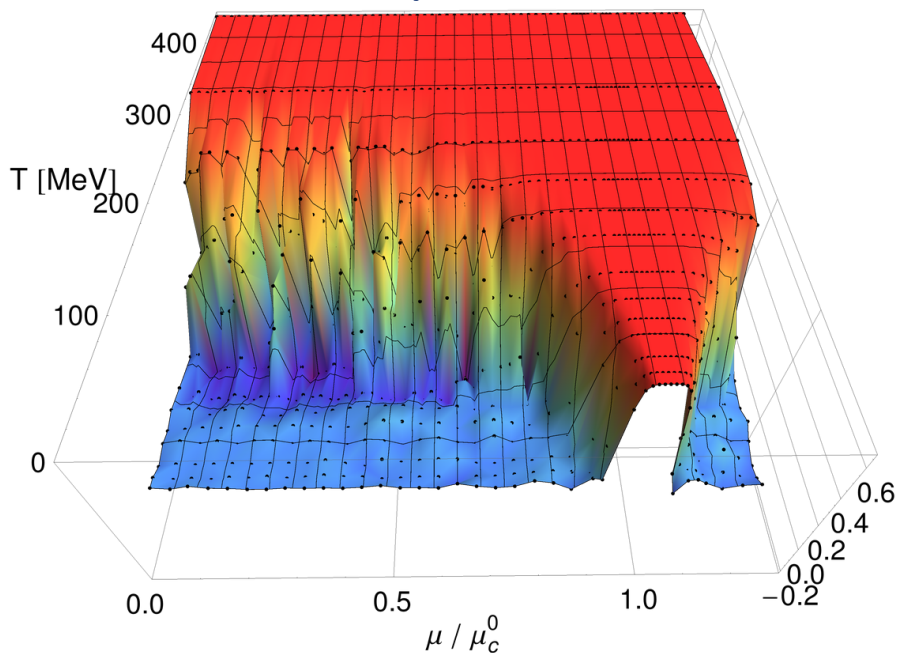
- Fermion density:  $n = \frac{1}{N_t} \frac{\partial \ln Z}{\partial \mu}$

# Heavy Dense QCD



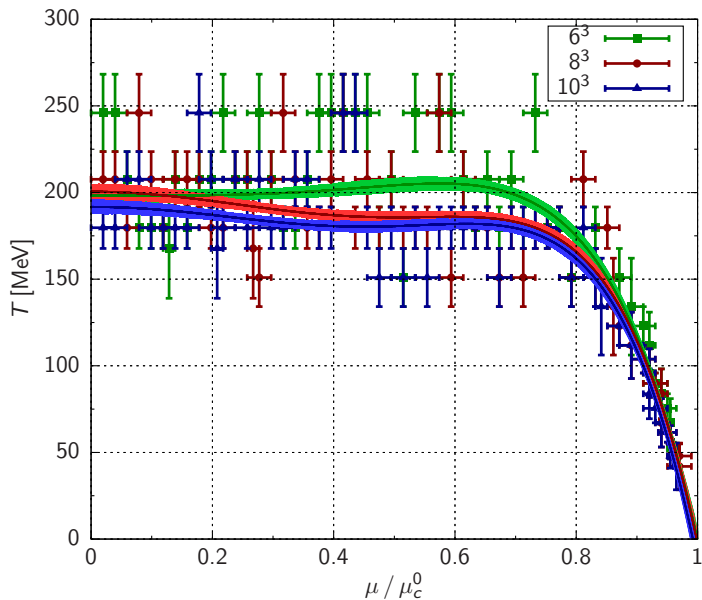
- Polyakov loop

# Heavy Dense QCD



- Binder cumulant of the Polyakov loop  $B = 1 - \frac{\langle P^4 \rangle}{3 \langle P^2 \rangle^2}$

# Heavy Dense QCD



- Fit the phase boundary using  $T_c(\mu) = \sum_k b_k (1 - \mu/\mu_c)^k$

# XY model with finite $\mu$

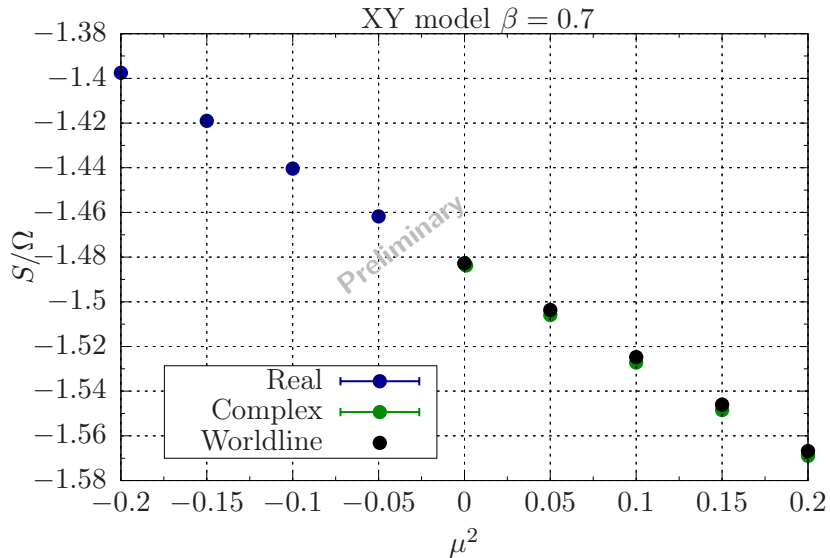
[Aarts, James, 2005]

- **XY model with finite  $\mu$**

- Toy model to test Complex Langevin
- Has a phase transition at  $\beta_c \sim 0.45$
- Ordered phase:  $\beta > \beta_c$ , disordered phase:  $\beta < \beta_c$

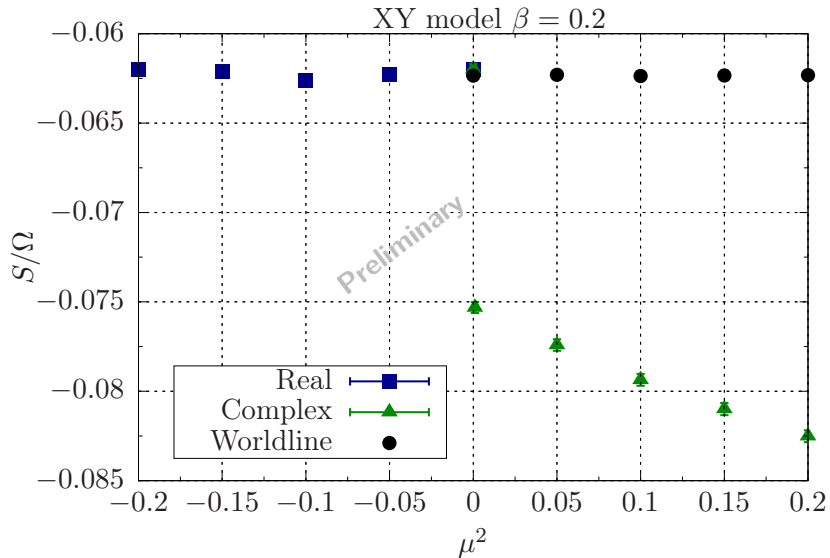
$$S = -\beta \sum_x \sum_{\nu=0}^2 \cos(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0})$$

# XY model with finite $\mu$



- In the ordered phase Complex Langevin agrees with (dual) Worldline formulation

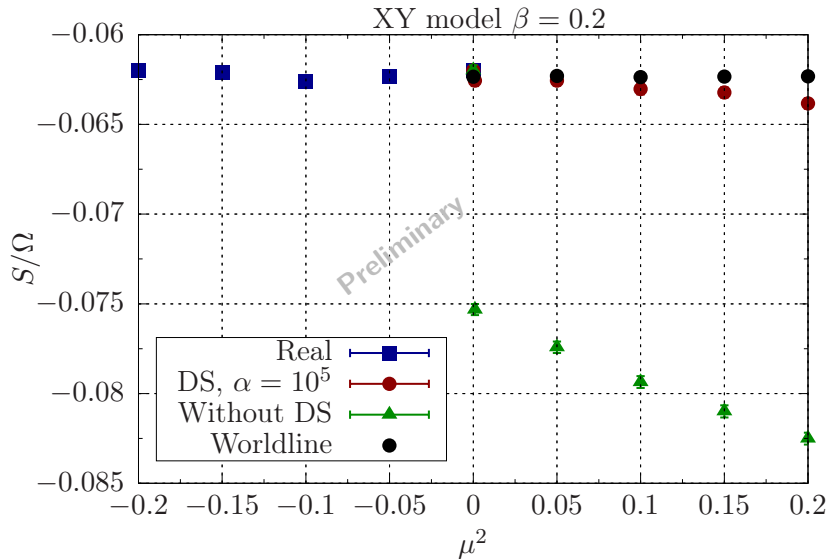
# XY model with finite $\mu$



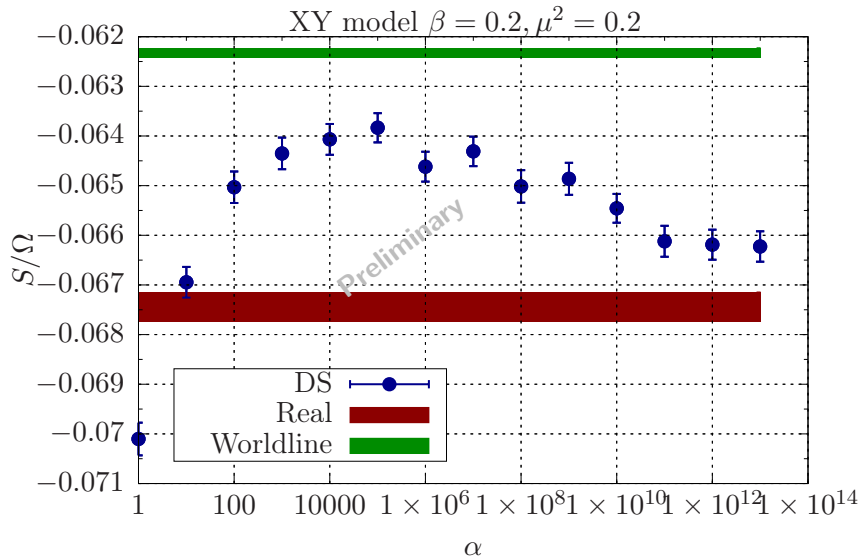
- In the disordered phase Complex Langevin fails spectacularly



# XY model with finite $\mu$



- Adding dynamic stabilization

XY model with finite  $\mu$ 

- For large  $\mu^2$  a discrepancy remains

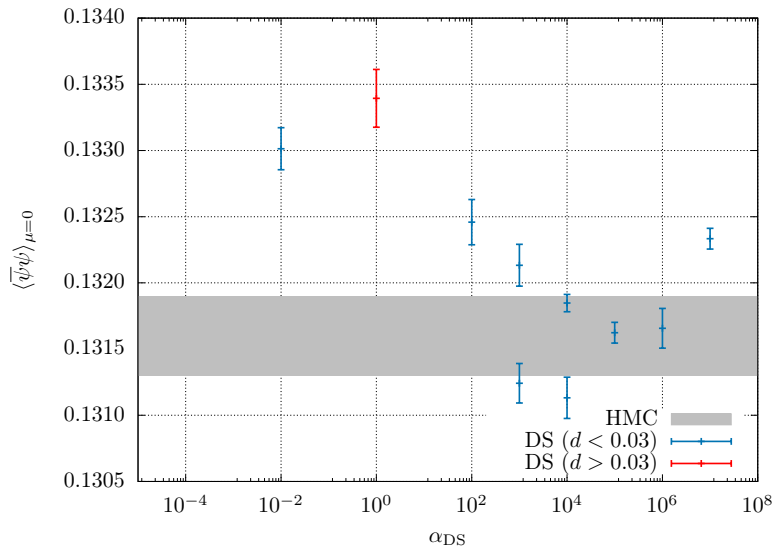
# Naïve Staggered QCD

- **Here: QCD using unimproved Staggered quarks.**
  - Setup:  $\beta = 5.6$ ,  $am_q = 0.025$
  - Every Langevin update needs inversion of the Dirac operator. . .

$$D_{x,\mu}^a S = D_{x,\mu}^a S_{YM} - \frac{N_f}{4} \text{Tr} [M^{-1} D_{x,\mu}^a M]$$

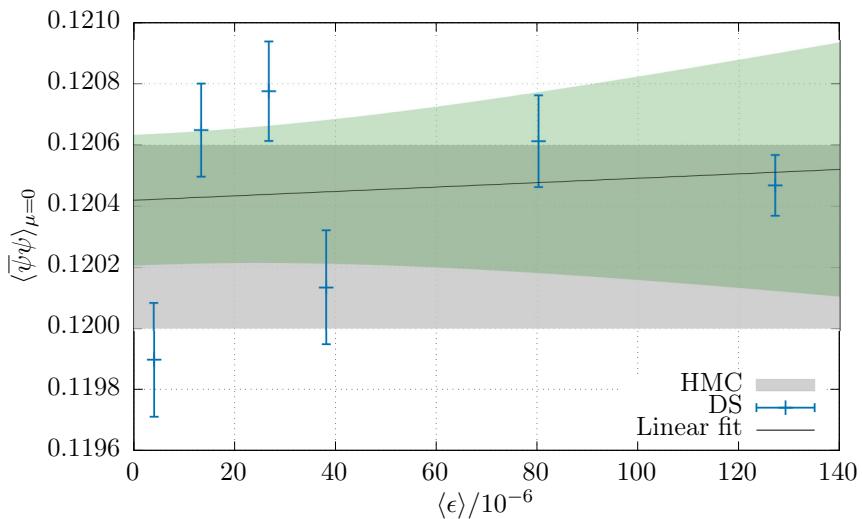
- Comparison @  $\mu = 0$  with HMC

# Naïve Staggered QCD



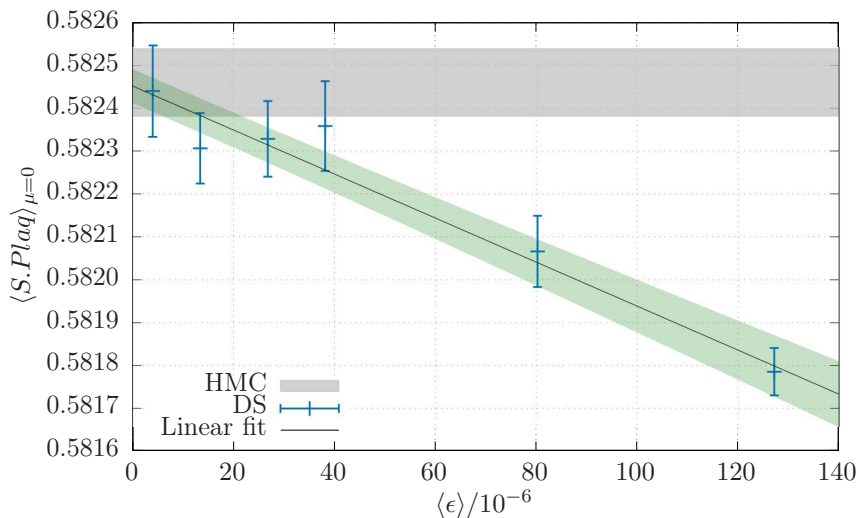
- Comparison @  $\mu = 0$  with HMC:  $6^4$

## Naïve Staggered QCD



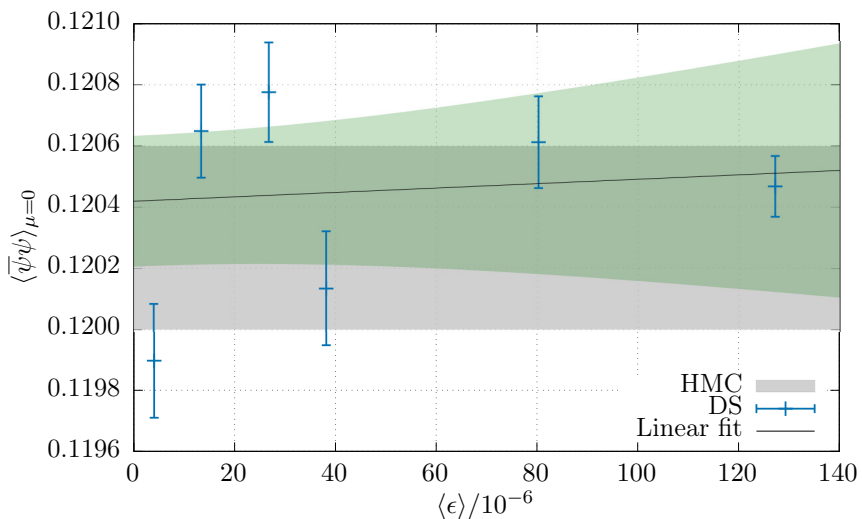
- Comparison @  $\mu = 0$  with HMC:  $6^4$  - Stepsize extrapolation

## Naïve Staggered QCD



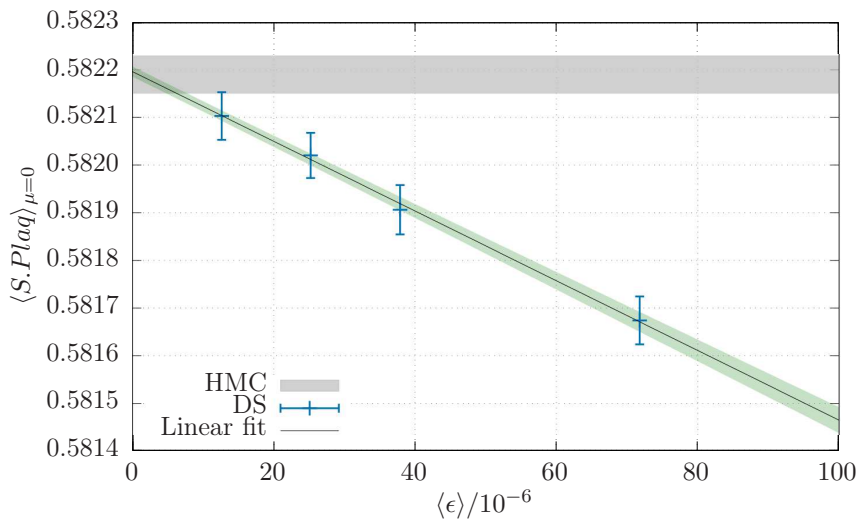
- Comparison @  $\mu = 0$  with HMC:  $6^4$  - Stepsize extrapolation

## Naïve Staggered QCD



- Comparison @  $\mu = 0$  with HMC:  $8^4$  - Stepsize extrapolation

# Naïve Staggered QCD



- Comparison @  $\mu = 0$  with HMC:  $8^4$  - Stepsize extrapolation

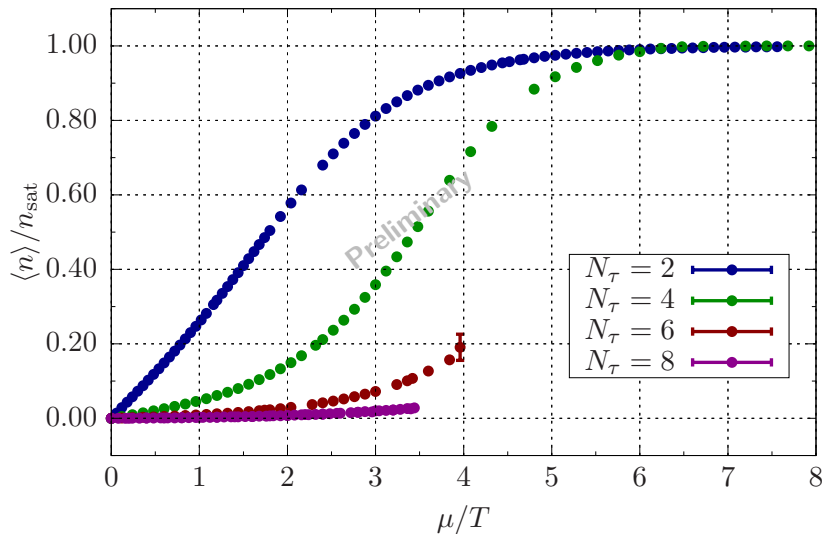


# Naïve Staggered QCD

- **Comparison @  $\mu = 0$  with HMC.**

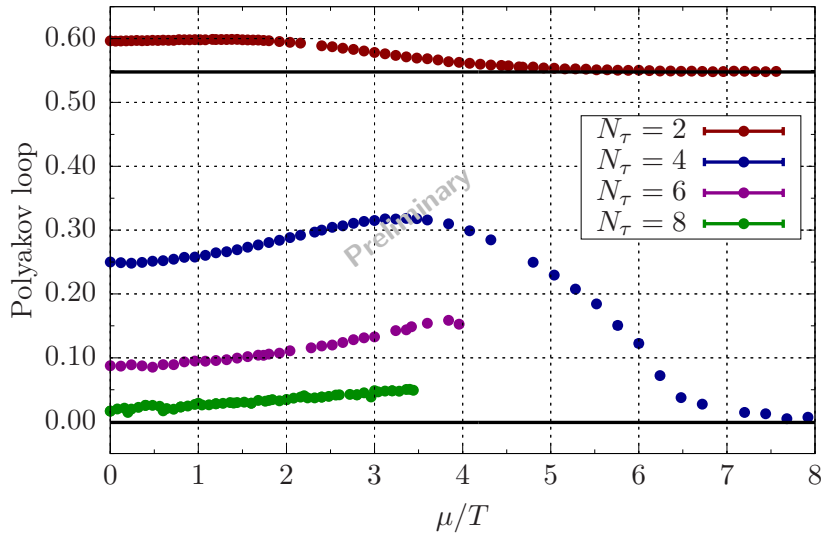
Volume	Plaquette		$\bar{\psi}\psi$	
	HMC	CL	HMC	CL
$6^4$	0.58246(8)	0.582452(4)	0.1203(3)	0.12042(2)
$8^4$	0.58219(4)	0.582196(1)	0.1316(3)	0.1319(2)
$10^4$			0.1372(3)	0.1372(2)
$12^4$			0.1414(4)	0.1411(1)

## Naïve Staggered QCD



- Fermion density

# Naïve Staggered QCD



- Polyakov loop

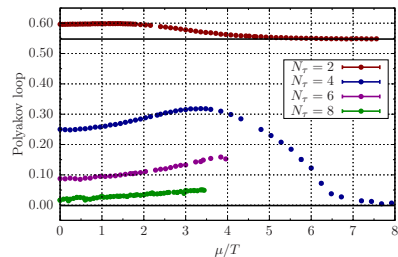
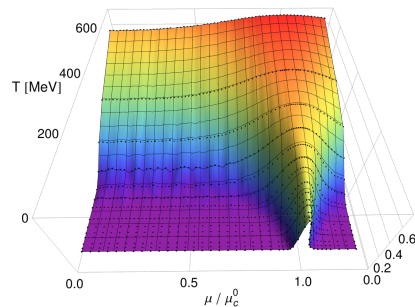
# Future work

## Conclusion

- Complex Langevin simulation can be used to study the QCD phase diagram
- Dynamical stabilisation improves convergence
- Work on the convergence, especially around  $\mu_c$ .

## Future work

- Start proper Full QCD simulations to identify phase structure of QCD.
- A lot of work to be done!



Thank you for your attention!