Cosmological Perturbation Theory beyond shell-crossing: Schrödinger equation approach

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Introduction

In this dissertation, I will show a novel approach to the study of large–scale structure formation in which the Cold Dark Matter is modelled by a complex scalar field whose dynamics are governed by coupled Schrödinger and Poisson equations.

This approach penetrates the weakly non-linear regime overcoming the weaknesses that the standard perturbation theories present:

- → they do not assure a density field that is positive everywhere;
- → they might totally break down when the particles' trajectories cross, a phenomenon known as *shell-crossing*, that leads the density field to generate a singularity.

The Standard Perturbation Theory

The Linearized Fluid approach

This Eulerian method works extremely well in the linear regime ($\delta << 1$). It consists to linearize the equations of motion of the CDM expressed in the *comoving coordinates*:

- The *Euler equation*

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v} \cdot \nabla_x)\vec{v} = -\frac{1}{a\rho}\nabla_x p - \frac{1}{a}\nabla_x \phi; \qquad (1)$$

- The *continuity equation*

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x(\rho\vec{v}) = 0; \qquad (2)$$

- The *Poisson equation*

$$\nabla_x^2 \phi - 4\pi G a^2 \delta\rho = 0. \qquad (3)$$

The Standard Perturbation Theory

The Linearized Fluid approach

Then we can proceed to linearize them and going to Fourier space we obtain the following set of equations:

$$\begin{cases} \dot{\vec{v}}_{\vec{k}} + H\vec{v}_{\vec{k}} = -\frac{ik}{a} c_s^2 \left(\delta_{\vec{k}} + \vec{\phi}_{\vec{k}} \right) \\ \dot{\delta}_{\vec{k}} + \frac{ik \cdot \vec{v}_{\vec{k}}}{a} = 0 \\ k^2 \phi_{\vec{k}} = -4\pi G a^2 \rho_b \delta_{\vec{k}} \end{cases} \tag{4}$$

From these one gets the differential equation:

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - 4\pi G \rho_b \delta_{\vec{k}} \approx 0 \tag{5}$$

Looking for solutions of the form $\delta \propto t^{\alpha}$ we obtain:

$$\alpha = 2/3 \Longrightarrow \delta_{\vec{k}} \propto t^{2/3} \quad \textit{"growing mode"}$$

$$\alpha = -1 \Longrightarrow \delta_{\vec{k}} \propto t^{-1} \quad \textit{"decaying mode"}$$

We consider two different formulations of continuity and Euler equation:

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot \left[(1+\delta)\vec{v}\right] = 0, \tag{6}$$

$$\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\phi \tag{7}$$

with $d\tau = dt/a$ is the *conformal time* and $\mathcal{H} = \frac{d \ln a}{d\tau} = Ha$ is the *conformal expansion rate*. Now we define $\theta(\vec{x},\tau) \equiv \nabla \cdot \vec{v}(\vec{x},\tau)$ and going to Fourier space (7) and (6) read

$$\frac{\partial \tilde{\delta}(\vec{k},\tau)}{\partial \tau} + \tilde{\theta}(\vec{k},\tau) = -\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\alpha(\vec{k}_1,\vec{k}_2)\tilde{\theta}(\vec{k}_1,\tau)\tilde{\delta}(\vec{k}_2,\tau) \tag{8}$$

$$\frac{\partial \tilde{\theta}(\vec{k},\tau)}{\partial \tau} + \mathcal{H}\tilde{\theta}(\vec{k},\tau) + \frac{3}{2}\mathcal{H}^2\tilde{\delta}(\vec{k},\tau) = -\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\beta(\vec{k}_1,\vec{k}_2)\tilde{\theta}(\vec{k}_2,\tau) \tag{9}$$

$\alpha(\vec{k}_1, \vec{k}_2)$ and $\beta(\vec{k}_1, \vec{k}_2)$ are the *mode coupling* functions:

$$\alpha(\vec{k}_1, \vec{k}_2) \equiv \frac{(\vec{k}_1 + \vec{k}_2) \cdot \vec{k}_1}{k_1^2}, \quad \beta(\vec{k}_1, \vec{k}_2) \equiv \frac{(\vec{k}_1 + \vec{k}_2)^2 (\vec{k}_1 \cdot \vec{k}_2)}{2 k_1^2 k_2^2}. \tag{10}$$

In order to find a solution, we make a perturbative expansion which lets formally solve equations (8) and (9):

$$\tilde{\delta}(\vec{k}, \tau) = \sum_{n=1}^{\infty} a^n(\tau) \delta^{(n)}(\vec{k}, \tau), \quad \tilde{\theta}(\vec{k}, \tau) = -\mathcal{H}(\tau) \sum_{n=1}^{\infty} a^n(\tau) \theta^{(n)}(\vec{k}, \tau). \tag{11}$$

The Standard Perturbation Theory

Eulerian non-linear Perturbation Theory

The equations (8) and (9) determine $\delta_n(\vec{k})$ and $\theta_n(\vec{k})$ in term of linear fluctuations:

$$\delta_n(\vec{k}) = \int d^3q_1 ... \int d^3q_n \delta_D(\vec{k} - \vec{q}_{1...n}) F_n(\vec{q}_1, ..., \vec{q}_n) \delta_1(\vec{q}_1) ... \delta_1(\vec{q}_n), \tag{12}$$

$$\theta_n(\vec{k}) = \int d^3q_1 ... \int d^3q_n \delta_D(\vec{k} - \vec{q}_{1...n}) G_n(\vec{q}_1, ..., \vec{q}_n) \delta_1(\vec{q}_1) ... \delta_1(\vec{q}_n) \tag{13}$$

where $F_n(\vec{q}_1, ..., \vec{q}_n)$ and $G_n(\vec{q}_1, ..., \vec{q}_n)$ are homogeneous functions, known as *kernels*, that are constructed from the fundamental mode coupling functions $\alpha(\vec{k}_1, \vec{k}_2)$ and $\beta(\vec{k}_1, \vec{k}_2)$. For instance, n=2:

$$F_2(\vec{q}_1, \vec{q}_2) = \frac{5}{7} + \frac{1}{2} \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}, \tag{14}$$

$$G_2(\vec{q}_1, \vec{q}_2) = \frac{3}{7} + \frac{1}{2} \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{4}{7} \frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}. \tag{15}$$

Therefore, once we obtained $\delta_n(\vec{k})$, we can calculate the lowest order cumulants of Eulerian Perturbation Theory that are:

$$S_3 = \frac{\langle\delta^3\rangle}{\langle\delta^2\rangle^2} = \frac{34}{7}, \quad S_4 = \frac{\langle\delta^4\rangle - 3\langle\delta^2\rangle^2}{\langle\delta^2\rangle^3} = \frac{60,712}{1323}. \tag{16}$$

The cumulants are obtained from the *cumulant generating function*:

$$\mathcal{C}(t) = \log[\mathcal{M}(t)]; \tag{17}$$

where $\mathcal{M}(t)$ is the *moment generating functions*:

$$\mathcal{M}(t) \equiv \sum_{p=0}^{\infty} \frac{\langle\delta^p\rangle}{p!} t^p. \tag{18}$$

The Schrödinger Perturbation Theory

Perturbation Theory with the Schrödinger equation

In order to find the Schrödinger equation we consider (1), (2) and (3) in these new variables:

$$\eta \equiv \frac{\rho}{\rho_b} = 1 + \delta, \quad \vec{u} \equiv \frac{\vec{v}}{a\dot{a}}, \quad \varphi \equiv \frac{3t_*^2}{2a_*^3}\phi, \quad a(t) = a_*(t/t_*)^{2/3}. \tag{19}$$

Then we have:

$$\begin{cases} \frac{\partial \vec{u}}{\partial a} + \vec{u} \cdot \nabla \vec{u} + \frac{3}{2a}\vec{u} = -\frac{3}{2a}\nabla\varphi \\ \frac{\partial \eta}{\partial a} + \vec{u} \cdot \nabla \eta + \eta \nabla \cdot \vec{u} = 0 \\ \nabla^2 \varphi = \frac{\delta}{a} \end{cases} \tag{20}$$

We assume an irrotational velocity field $\vec{u} = \nabla\Phi$ and rearrange the Euler equation in a more suitable form:

$$\frac{\partial \Phi}{\partial a} + \frac{1}{2}(\nabla\Phi)^2 + \frac{3}{2a}\Phi = -\frac{3}{2a}\varphi \tag{21}$$

$$\frac{\partial \eta}{\partial a} + \nabla \cdot (\eta \nabla \Phi) = 0 \tag{22}$$

We introduce a complex scalar field, that represents the CDM, of the following form:

$$\begin{cases} \psi(r,t) = e^{A(r,t)+\frac{i}{\hbar}B(r,t)} \\ \eta = \psi^*\psi = e^{2A(r,t)} \end{cases} \tag{23}$$

We fix $\Phi(r,t) = B(r,t)$ and with some calculations we find these quantities:

$$|\nabla B|^2 = \hbar^2\left(\nabla^2 A - \frac{2i}{\hbar}\frac{\partial A}{\partial a} + |\nabla A|^2 - \frac{\nabla^2\psi}{\psi}\right), \tag{24}$$

$$\frac{\partial B}{\partial a} = -i\hbar\left(\frac{1}{\psi}\frac{\partial \psi}{\partial a} - \frac{\partial A}{\partial a}\right), \tag{25}$$

$$B = \frac{\hbar}{2i}\ln\left(\frac{\psi}{\psi^*}\right). \tag{26}$$

Then substituing in (21), we obtain:

$$i\hbar\frac{\partial\psi}{\partial a} = -\frac{\hbar^2}{2}\nabla^2\psi + \left[V + \frac{\hbar^2}{2}\left(\nabla^2 A + |\nabla A|^2\right)\right]\psi \qquad (27)$$

where we introduced a *general potential* V defined as follow:

$$V = \frac{3}{2a}(B + \varphi) = \frac{3}{2a}\left(\frac{\hbar}{2i}\ln\left(\frac{\psi}{\psi^*}\right) + \varphi\right). \qquad (28)$$

The additive term in the rhs of the equation (27) is the *quantum pressure* term. This name is due to the fact that it resembles a pressure gradient and it can be ignored. Then we obtained the coupled Schrödinger and Poisson equations:

$$i\hbar\frac{\partial\psi}{\partial a} = -\frac{\hbar^2}{2}\nabla^2\psi + V\psi \qquad (29)$$

$$\nabla^2\varphi \equiv \nabla^2\left[\frac{2a}{3}\left(V + \frac{3i\hbar}{4a}\ln\left(\frac{\psi}{\psi^*}\right)\right)\right] = \frac{1}{a}(e^{2A} - 1) \qquad (30)$$

Substituting the wave function $\psi = e^{A(r,t)+\frac{i}{\hbar}B(r,t)}$ in the Schrödinger equation we obtain an equation that can be split in two coupled equations respectively for imaginary and real part.

Imaginary part.

$$\frac{\partial A}{\partial a} = -\frac{1}{2}\left(\nabla^2 B + 2\nabla A \cdot \nabla B\right). \tag{31}$$

Real part.

$$\frac{\partial B}{\partial a} = \frac{\hbar^2}{2}\left(\nabla^2 A + |\nabla A|^2\right) - \frac{1}{2}|\nabla B|^2 - V. \tag{32}$$

Now we go to Fourier space working in the correspondence limit, i.e. $\hbar \to 0$:

$$\frac{\partial A_k}{\partial a} = -\frac{1}{2}\left(k^2 B(\vec{k}) + 2\int d^3k_1 d^3k_2 \delta_D(\vec{k} - \vec{k}_1 - \vec{k}_2)\vec{k}_1 \cdot \vec{k}_2 A(\vec{k}_1)B(\vec{k}_2)\right) \quad (33)$$

$$\begin{aligned}\frac{\partial B_k}{\partial a} =& -\frac{1}{2}\int d^3k_1 d^3k_2 \delta_D(\vec{k} - \vec{k}_1 - \vec{k}_2)\vec{k}_1 \cdot \vec{k}_2 B(\vec{k}_1)B(\vec{k}_2) - \frac{3H^2a^2}{2k^2} \\ &\times \sum_{N\geq 1} \frac{2^N}{N!}\int d^3k_1 ... d^3k_N \delta_D(\vec{k} - \vec{k}_1 - ... - \vec{k}_N)A(\vec{k}_1)...A(\vec{k}_N)\end{aligned} \quad (34)$$

In order to render the equations (33) and (34) homogeneous in a and H we proceed to make a perturbative expansion of our scalar field using the *Ansätze* given by Szapudi and Kaiser (2003):

$$\begin{cases} A_k = \sum A_k^{(N)} a^N \\ B_k = -H \sum B_k^{(N)} a^{N+2} \end{cases} \tag{35}$$

where $A_k^{(N)}$ and $B_k^{(N)}$ are:

$$A_k^{(N)} = \int d^3k_1 ... \int d^3k_n \delta_D(\vec{k} - \vec{k}_{1...n}) F^{(N)}(\vec{k}_1, ..., \vec{k}_n) A_{k_1}^{(1)} ... A_{k_N}^{(1)}, \tag{36}$$

$$B_k^{(N)} = \frac{2}{k^2} \int d^3k_1 ... \int d^3k_n \delta_D(\vec{k} - \vec{k}_{1...n}) G^{(N)}(\vec{k}_1, ..., \vec{k}_n) A_{k_1}^{(1)} ... A_{k_N}^{(1)}, \tag{37}$$

where $F^{(N)}(\vec{k}_1, ..., \vec{k}_n)$ and $G^{(N)}(\vec{k}_1, ..., \vec{k}_n)$ are the kernels that are similar to Eulerian case.

The Schrödinger Perturbation Theory

Perturbation Theory with the Schrödinger equation

Here we give explicitly the N=2 case just for F:

$$F_2(\vec{k}_1, \vec{k}_2) = \frac{3}{7} + \frac{10}{7}\alpha(k_1, k_2) + \frac{2}{7}\beta(k_1, k_2) \quad \textit{with} \quad \begin{cases} \alpha(q_1, q_2) = \frac{(q_1 q_2)}{k_2^2} \\ \beta(q_1, q_2) = k^2 \frac{(q_1 q_2)}{(q_1^2 q_2^2)} \end{cases} \tag{38}$$

Once we obtained the expression of $A_k^{(N)}$ we can proceed to calculate its lowest significant cumulants:

$$S_3^A = \frac{26}{7}, \quad S_4^A = \frac{40,240}{1323}. \tag{39}$$

The cumulants of the density field are compute taking advantage of a recursive formula by Fry and Gatzañaga (1993), which link the cumulants of the scalar field $A(r,t)$ with the cumulants of $\delta = e^{2A} - 1$:

$$S_3 = b^{-1}(S_3^A + 3c_2) = \frac{34}{7}, \tag{40}$$

$$S_4 = b^{-2}(S_4^A + 12c_2 S_3^A + 4c_3 + 12c_2^2) = \frac{60,712}{1323}, \tag{41}$$

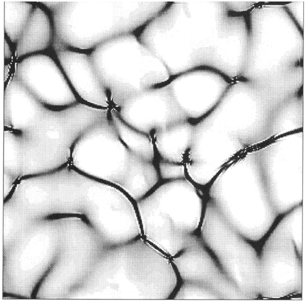
where the coefficients are: $b = 2$ and $c_N = b_N / b = 2^{N-1}$.

It's evident how this novel approach totally overcomes the limitations of the standard perturbation theories:

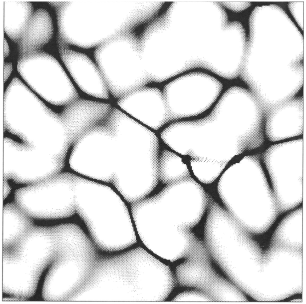
- the wave function ψ causes the density $\delta = \eta - 1 = e^{2A} - 1$ to assume only positive values;
- Szapudi and Kaiser (2003) do not consider trajectories of single particles but a complex scalar field, hence this approach doesn't break down at shell-crossing. Moreover in the wave function no singularities occur at any time.

Conclusions

Hence, in this work, we introduced the Schrödinger Perturbation Theory showing that the lowest order cumulants predicted by Eulerian Perturbation Theory for the dark matter field δ are exactly recovered.
Below we compare the two-dimensional CDM universe simulated using the Schrödinger method and N-body technique (Widrow and Kaiser 1993).



(a) Schrödinger method



(b) N-Body

Thanks for the attention!