

# Ideal hydrodynamics limit extensions explored: Polarization and gauge theory



Based on [1810.12468](#), [1807.02796](#), [1701.08263](#), [1604.05291](#)  
with D. Montenegro, L. Tinti

What is this talk about

**The necessity** of a field theory perspective

Hydrodynamics is neither transport nor string theory!

**Introduction** to the field theory of hydrodynamics

Our knowledge of hydrodynamics rewritten as symmetries

Perhaps not ideal for solving problems, but worth thinking about!

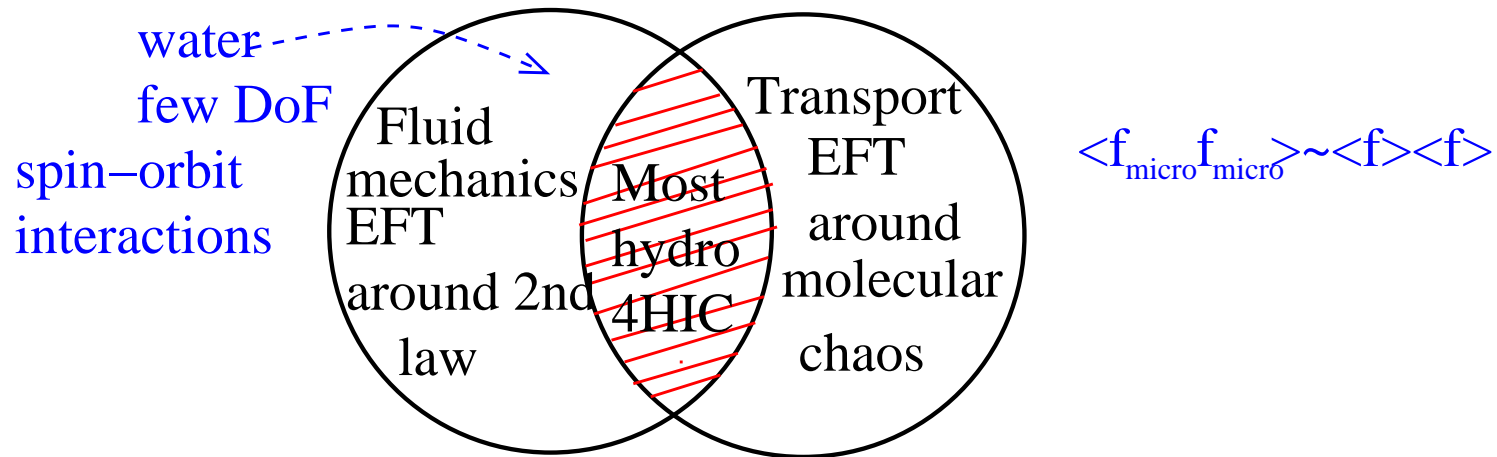
**Extending hydrodynamics I** Polarization

**Extending hydrodynamics II** Gauge symmetries

PS: So far, theory only!

## Hydro is not (just) transport!

Its constituents are usually neither billiard balls not black holes!

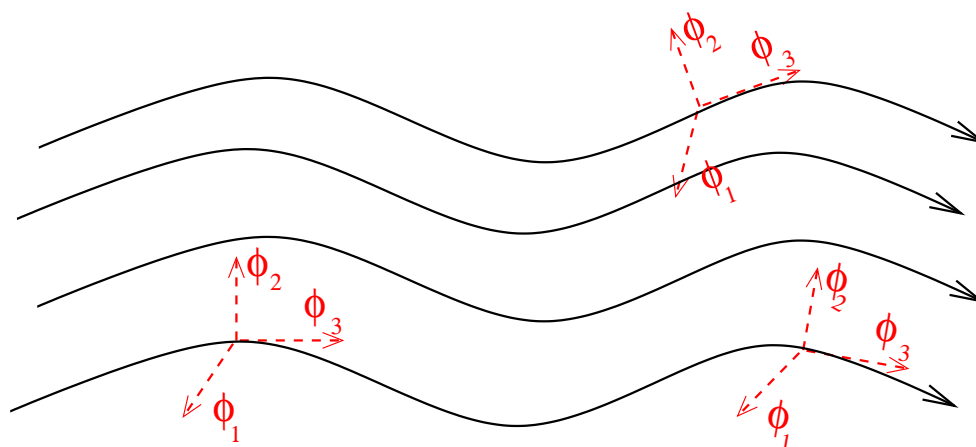


Take water: Boltzmann equation breaks down as particles tightly correlated, hydro works well!

Worth defining it explicitly in terms of approximate local **entropy maximization only!** One can do this with Lagrangian techniques!

What might work: EFT techniques (Nicolis et al,1011.6396 (JHEP))

Continuous mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates  $\phi_I(x^\mu), I = 1...3$  of the position of a fluid cell originally at  $\phi_I(t = 0, x^i), I = 1...3$ . (Lagrangian hydro . NB: no conserved charges)



The system is a **Fluid** if it's Lagrangian obeys some symmetries (Ideal hydrodynamics  $\leftrightarrow$  Isotropy in comoving frame) Solutions generally break these, Excitations (Sound waves, vortices etc) can be thought of as "Goldstone bosons".

**Translation invariance** at Lagrangian level  $\leftrightarrow$  Lagrangian can only be a function of  $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$  Now we have a “continuous material”!

**Homogeneity/Isotropy** means the Lagrangian can only be a function of  $B = \det B^{IJ}, \text{diag} B^{IJ}$   
The comoving fluid cell must not see a “preferred” direction  $\Leftarrow SO(3)$  invariance

**Invariance under Volume-preserving diffeomorphisms** means the Lagrangian can only be a function of  $B$  (actually  $b = \sqrt{B}$ )  
In all fluids a cell can be infinitesimally deformed  
(with this, we have a fluid. If this last requirement is not met, Nicolis et al all call this a “Jelly”)

A few exercises for the bored public Check that  $L = -F(B)$  leads to

$$T_{\mu\nu} = (P + \rho)u_\mu u_\nu - P g_{\mu\nu}$$

provided that

$$\rho = F(B) , \quad p = F(B) - 2F'(B)B , \quad u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K$$

(A useful formula is  $\frac{db}{d\partial_\mu \phi_I} \partial_\nu \phi_I = u^\mu u^\nu - g^{\mu\nu}$  )

Equation of state chosen by specifying  $F(b)$  . “Ideal”:  $\Leftrightarrow F(B) \propto b^{2/3}$

$b$  is identified with the entropy and  $b \frac{dF(B)}{dB}$  with the microscopic temperature.

$u^\mu$  fixed by  $u^\mu \partial_\mu \phi^{\forall I} = 0$  . Vortices become **Noether currents of diffeomorphisms!**

This is all really smart, but why?

Hydro is not (just) transport: Hydrodynamics is based on three scales

$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

$l_{micro}$  stochastic,  $l_{mfp}$  dissipative. If  $l_{micro} \sim l_{mfp}$  soundwaves

**Of amplitude** so that momentum  $P_{sound} \sim (area)\lambda (\delta\rho) c_s \gg T$

**And wavenumber**  $k_{sound} \sim P_{sound}$

**Survive** (ie their amplitude does not decay to  $E_{sound} \sim T$ )  $\tau_{sound} \gg 1/T$

**Transport:** Beyond Molecular chaos **AdS/CFT:** Beyond large  $N_c$   
It turns out Polarization, gauge symmetries mess this  $l_{micro}$  hierarchy!

## Ideal hydrodynamics and the microscopic scale

The most general Lagrangian is

$$L = T_0^4 F \left( \frac{B}{T_0^4} \right) \quad , \quad B = T_0^4 \det B^{IJ} \quad , \quad B^{IJ} = |\partial_\mu \phi^I \partial^\mu \phi^J|$$

Where  $\phi^{I=1,2,3}$  is the comoving coordinate of a volume element of fluid.

**NB:**  $T_0 \sim \Lambda g$  microscopic scale, includes thermal wavelength and  $g \sim N_c^2$  (or  $\mu/\Lambda$  for dense systems ).  $T_0 \rightarrow \infty \Rightarrow$  classical limit

It is therefore natural to identify  $T_0$  with the microscopic scale!

*Kn* behaves as a gradient,  $T_0$  as a Planck constant!!!



At  $T_0 < \infty$  quantum and thermal fluctuations can produce sound waves and vortices, “weighted” by the usual path integral prescription!

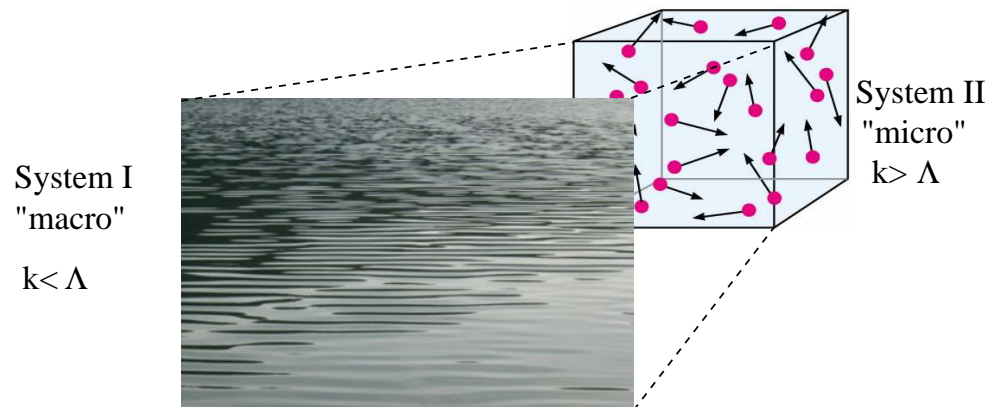
$$L \rightarrow \ln \mathcal{Z} \quad \mathcal{Z} = \int \mathcal{D}\phi_i \exp \left[ -T_0^4 \int F(B) d^4x \right], \langle \mathcal{O} \rangle \sim \frac{\partial \ln \mathcal{Z}}{\partial \dots}$$

$$\left( \text{eg. } \langle T_{\mu\nu}^x T_{\mu\nu}^{x'} \rangle = \frac{\partial^2 \ln \mathcal{Z}}{\partial g_{\mu\nu}(x) \partial g_{\mu\nu}(x')} \right)$$

$T_0 \sim n^{-1/3}$ , unlike Knudsen number, behaves as a “Planck constant”. EFT expansion and lattice techniques should give all allowed terms and correlators. **Coarse-graining will be handled here!**

For analytical calculations fluid can be perturbed around a hydrostatic ( $\phi_I = \vec{x}$ ) background

$$\phi_I = \vec{x} + \underbrace{(\vec{\pi}_L)}_{\text{sound}} + \underbrace{(\vec{\pi}_T)}_{\text{vortex}}$$



And we discover a fundamental problem: Vortices carry arbitrary small energies but stay put! No S-matrix in hydrostatic solution!

$$L_{linear} = \underbrace{\dot{\vec{\pi}}_L^2 - c_s^2 (\nabla \cdot \vec{\pi}_L)^2}_{\text{sound wave}} + \underbrace{\dot{\pi}_T^2}_{\text{vortex}} + \text{Interactions}(\mathcal{O}(\pi^3, \partial\pi^3, \dots))$$

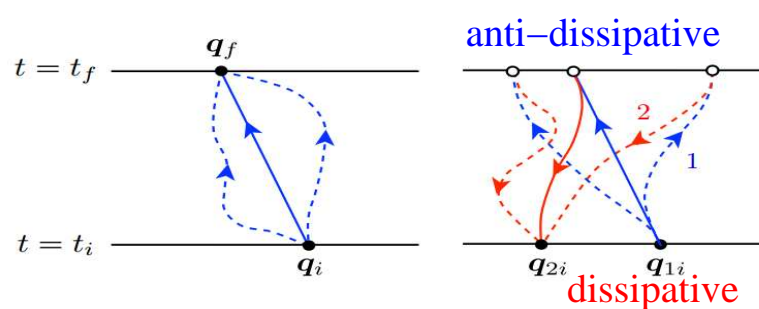
Unlike sound waves, **Vortices** can not give you a theory of free particles, since they do not propagate: They carry energy and momentum but stay in the same place! Can not expand such a quantum theory in terms of free particles.

Physically: “quantum vortices” can live for an arbitrary long time, and dominate any vacuum solution with their interactions. **This does not mean the theory is ill-defined, just that it is strongly non-perturbative!**

Polarization might help here!

The big problem with Lagrangians... usually only non-dissipative terms  
 But there are a few ways to fix it. We focus on coordinate doubling  
 (Galley, but before Morse+Feschbach)

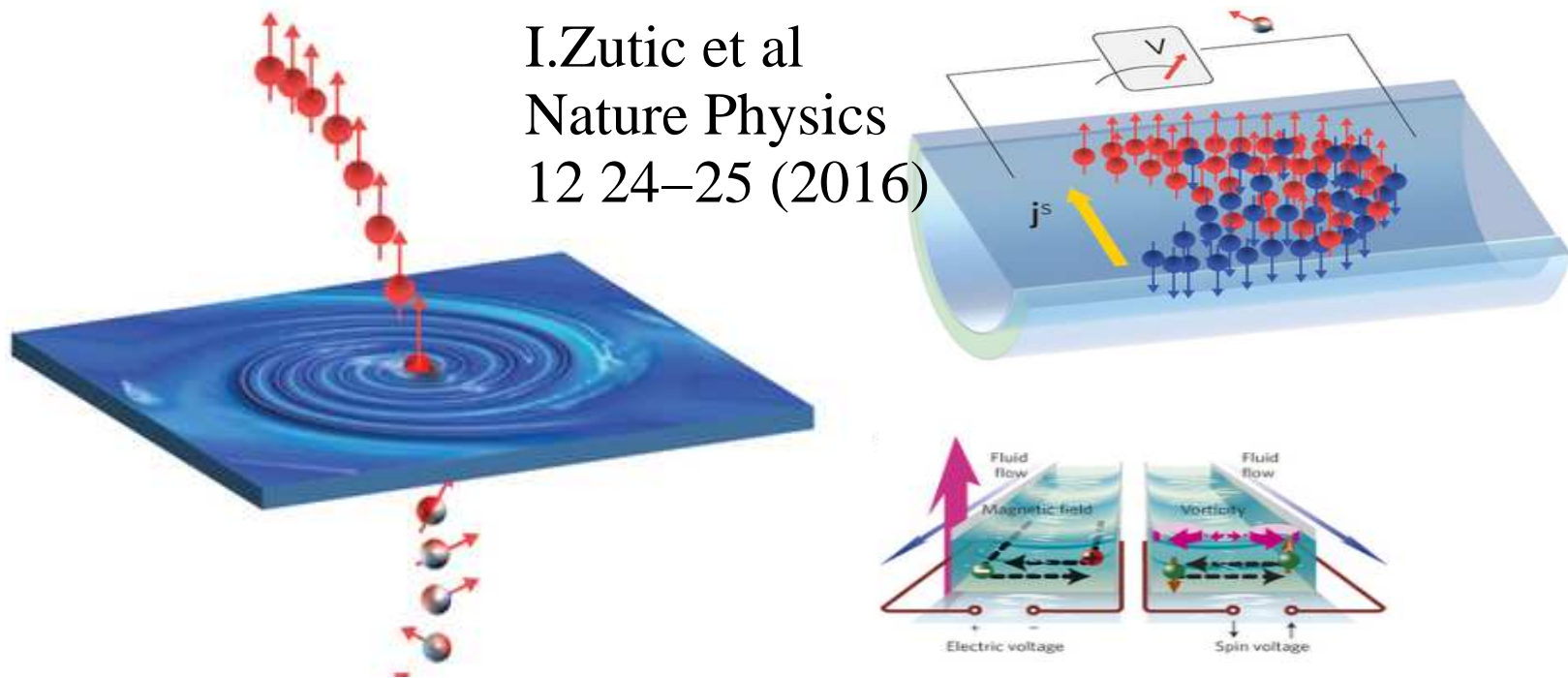
Dissipative  
 extension  
 of Hamilton's  
 principle



$$L = \frac{1}{2} \left( \underbrace{m\dot{x}^2 - wx^2}_{SHO} \right) \rightarrow \underbrace{(m\dot{x}_+^2 - wx_+^2)}_{\mathcal{L}_1} - \underbrace{(m\dot{x}_-^2 - wx_-^2)}_{\mathcal{L}_2} + \underbrace{\alpha(x_+x_- - \dot{x}_-x_+)}_{\mathcal{K}}$$

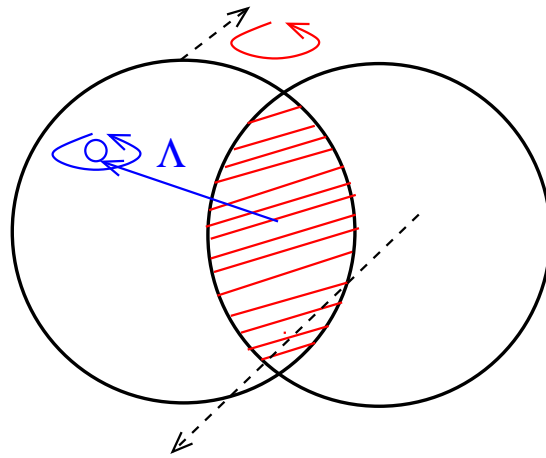
Standard techniques give you two sets of equations, one with a damped harmonic oscillator, the other “anti-damped”. Navier-Stokes and Israel-Steward (GT, D. Montenegro, PRD, in press)

## Application 1: Hydro with polarization



Ultracold atoms: Zutic, Matos-Abiague, "Spin Hydrodynamics", Nature Physics **12** 24-25 Takahashi et al", Nature Physics **12** 52-56 (2016)

In the context of heavy ion physics... STAR, Nature, 1701.06657



Initial rapidity gradient and initial transparency could generate initial angular momentum. Cooper-Fry like ansatz assuming equilibrium between spin and vorticity at freeze-out work well (Y.Karpenko's talk) but theory propagating vorticity and polarization throughout evolution of fluid is lacking. Vorticity-polarization coupling is a violation of molecular chaos, cannot be described with kinetic theory consistently!

What is ideal hydro? A conceptual difficulty!

**Entropy conserved** always at maximum at each point in spacetime

**Local isotropy** in the comoving frame

**Vorticity is conserved** (Kelvins theorem)

**Continuum limit** when you break up cells, intensive results stay the same

With polarization, only the first has a chance of being realized even in the ideal limit

Combining polarization with the ideal hydrodynamic limit, defined as

- (i) The dynamics within each cell is faster than macroscopic dynamics, and it is expressible only in terms of local variables and with no explicit reference to four-velocity  $u^\mu$  (gradients of flow are however permissible, in fact required to describe local vorticity).
  - (ii) Dynamics is dictated by local entropy maximization, within each cell, subject to constraints of that cell alone. Macroscopic quantities are assumed to be in local equilibrium inside each macroscopic cell
  - (iii) Only excitations around a hydrostatic medium are sound waves, vortices
- (i-iii) ,with symmetries and EFT define the theory



**Polarization described by** antisymmetric rank-2 tensor  $y_{\mu\nu}$

$$L = F(b, y_{\mu\nu}) = F(b(1 \pm y^2 + \dots))$$

$y_{\mu\nu}$  not conserved, dictated by entropy maximization

**In equilibrium** polarization proportional to vorticity to avoid non-hydrodynamic modes and maintain maximum of entropy

$$d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial V} dV + \frac{\partial \mathcal{F}}{\partial e} de + \underbrace{\frac{\partial \mathcal{F}}{\partial [\Omega_{\mu\nu}]}}_{=y_{\mu\nu}} d[\Omega_{\mu\nu}] = 0$$

**Equations of motion** from Lagrangian, not conservation laws  
Not enough conservation laws to close equations

Linearization leads to violation of causality because of Ostrogradski's theorem

Dependence of lagrangian on acceleration (vorticity) gives acausal linearized modes (Vortices and sound-waves mix , dispersion relation  $w = \sum_1^4 A_i k^i$  )

To fix this, we need polarization to relax to vorticity, a la Israel-Stewart

$$\tau_{\Omega} u_{\alpha} \partial^{\alpha} y_{\mu\nu} + y_{\mu\nu} = \chi(T, y) \Omega_{\mu\nu}$$

As a consequence ( G Torrieri, D Montenegro, 1807.02796 ) lower limit to viscosity in anti-ferromagnetic fluids

$$\tau_Y^2 \geq \frac{8c\chi^2(b_o, 0)}{(1 - c_s^2)b_o F'(b_o)} \quad , \quad \frac{\eta}{s} \geq T\tau_Y$$

A bottom-up viscosity limit . Ferromagnetic fluid retains IR acausal mode, Banks-Casher relation for ferromagnetism!

## Gauge theory and local thermalization

**The formalism we introduced earlier** is ok for quark polarization but problematic for gluon polarization: Gauge symmetry means one can exchange locally angular momentum states for transversely polarized spin states. So vorticity vs polarization is ambiguous

**Using the energy-momentum tensor for dynamics is even more problematic** for spin  $T_{\mu\nu}$  acquires a "pseudo-gauge" transformation

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \frac{1}{2} \partial_\lambda (\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda})$$

where  $\Phi$  is fully antisymmetric.  $\delta S / \delta g_{\mu\nu}$  and canonical tensors are limits of choice of  $\Phi$ . But in a gauge theory, pseudo-Gauge transformations **are** gauge transformations! Affects EFTs based on  $T_{\mu\nu}$  (Hong Liu, Florkowski and collaborators)

## From global to gauge conserved currents

A reminder: Within Lagrangian field theory a scalar chemical potential is added by adding a  $U(1)$  symmetry to system.

$$\phi_I \rightarrow \phi_I e^{i\alpha} \quad , \quad L(\phi_I, \alpha) = L(\phi_I, \alpha + y) \quad , \quad J^\mu = \frac{dL}{d\partial_\mu \alpha}$$

generally flow of  $b$  and of  $J$  not in same direction. Can impose a well-defined  $u^\mu$  by adding chemical shift symmetry

$$L(\phi_I, \alpha) = L(\phi_I, \alpha + y(\phi_I)) \rightarrow L = L(b, y = u_\mu \partial^\mu \alpha)$$

A comparison with the usual thermodynamics gives us

$$\mu = y \quad , \quad n = dF/dy$$

Generalization from  $U(1)$  to generic group easy

$$\alpha \rightarrow \{\alpha_i\} \quad , \quad \exp(i\alpha) \rightarrow \exp\left(i \sum_i \alpha_i \hat{T}_i\right)$$

One subtlety: Currents stay parallel to  $u_\mu$  but chemical potentials become adjoint, since rotations in current space still conserved

$$y = J^\mu \partial_\mu \alpha_i \rightarrow y_{ab} = J_a^\mu \partial_\mu \alpha_b$$

Lagrangian still a function of  $dF(b, \{\mu\})/dy_{ab}$  , “**flavor chemical potentials**”

From global to gauge invariance! Lagrangian invariant under

$$\{y_{ab}\} \rightarrow y'_{ab} = U_{ac}^{-1}(x)y_{cd}U_{db}(x) \quad , \quad U_{ab}(x) = \exp\left(i \sum_i \alpha_i(x)\hat{T}_i\right)$$

However, gradients of  $x$  obviously change  $y$  .

$$\begin{aligned} y_{ab} \rightarrow U_{ac}^{-1}(x)y_{cd}U_{bd}(x) &= U^{-1}(x)_{ac}J_f^\mu U_{cf}U_{fg}^{-1}\partial_\mu\alpha_gU_{bg} = \\ &= U^{-1}(x)_{ac}J_f^\mu U_{cf}\partial_\mu\left(U_{fg}^{-1}\alpha_dU_{bd}(x)\right) - J_a^\mu(U\partial_\mu U)_{fb}\alpha_f \end{aligned}$$

Only way to make lagrangian gauge invariant is

$$F(b, J_j^\mu \partial_\mu \alpha_i) \rightarrow F(b, J_j^\mu (\partial_\mu - U(x)\partial_\mu U(x)) \alpha_i)$$

Which is totally unexpected, profound and crazy

## The swimming ghost!

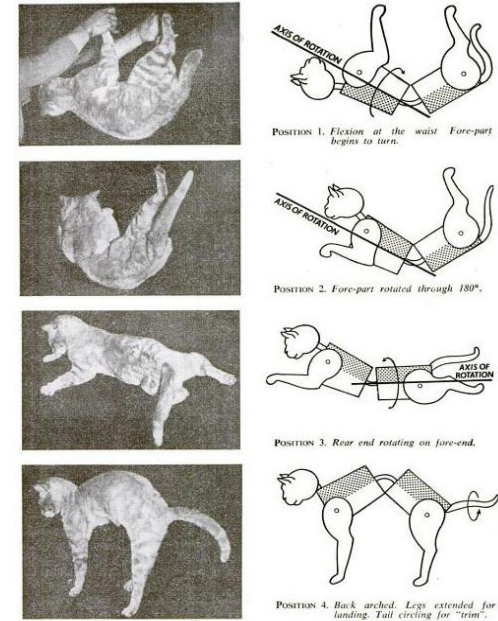
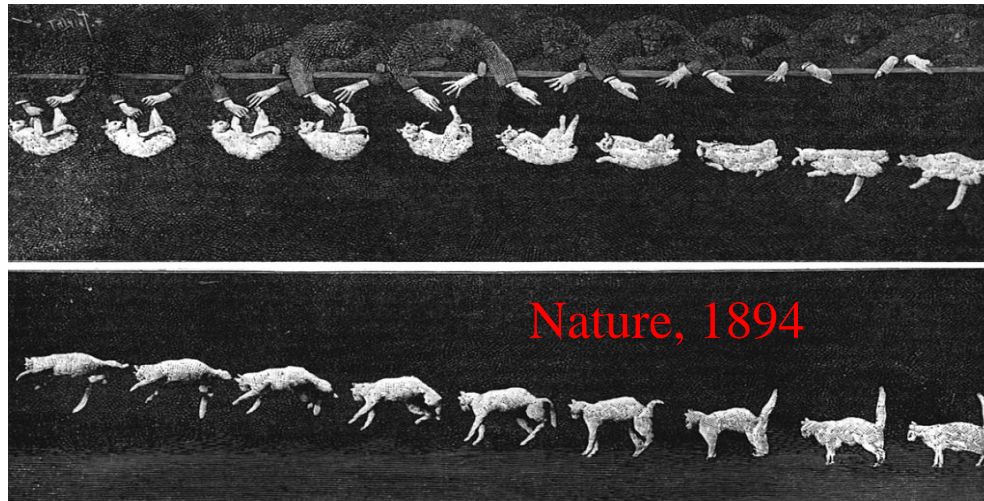
$$F(b, J_j^\mu \partial_\mu \alpha_i) \rightarrow F(b, J_j^\mu (\partial_\mu - U(x) \partial_\mu U(x)) \alpha_i)$$

Means the ideal fluid lagrangian depends on velocity!. no real ideal fluid limit possible  
the system “knows it is flowing” at local equilibrium! **NB:** For U(1)

$$\hat{T}_i \rightarrow 1 \quad , \quad y_{ab} \rightarrow \mu_Q \quad , \quad u_\mu \partial^\mu \alpha_i \rightarrow A_\tau$$

So second term can be gauged to a redefinition of the chemical potential  
(the electrodynamic potentials effect on the chemical potential).

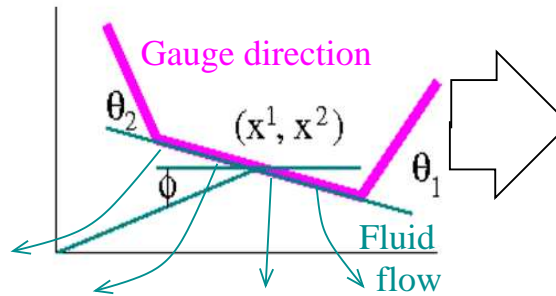
Cannot do it for Non-Abelian gauge theory, “twisting direction” in color space  
It turns out this has an old analogue...



S. Montgomery (2003): How does a cat always fall on its feet without anything to push themselves against? The shape of spaces a cat can deform themselves into defines a “set of gauges” a cat can choose without change of angular momentum.



Purcell, Shapere+Wilczek, Avron+Raz : A similar process enables swimmers to move through viscous liquids with no applied force



Now imagine each fluid cell filled with a “swimmer”, with arms and legs outstretched in “gauge” directions...



In ideal limit all currents proportional to  $u_\mu$  . But gauge symmetry requires “ghost” excitations, proportional to gradients of currents, to not be physical. So free energy HAS to depend on flow.

Classic on this, B. Bistrovic, R. Jackiw, H. Li, V. P. Nair and S. Y. Pi, *Phys. Rev. D* **67**, 025013 (2003) , “NonAbelian fluid dynamics in Lagrangian formulation,” missed this subtlety as no local equilibrium defined!

## Whats going on? A more statistical mechanics perspective

We perturb the hydrostatic limit, where  $\phi_I = X_I$ , and isolate a transverse mode (vortex) and a longitudinal mode (sound wave)

$$\phi_I = X_I + \vec{\pi}_I^{sound} + \vec{\pi}_I^{vortex} \quad , \quad \nabla \cdot \vec{\pi}_I^{vortex} = \nabla \times \vec{\pi}_I^{sound} = 0$$

Since the derivative of the free energy w.r.t.  $b$  is positive, sound waves and vortices do “work”. Let us now assume the system has a “color chemical potential” in some direction. Let us change the color chemical potential in space according to

$$\Delta\mu(x) = \sum_i (\mu_i(x)^{swim} + \mu_i(x)^{swirl}) \hat{T}_i \quad , \quad \nabla_i \cdot \mu_i^{swim} = \nabla_i \times \mu_i^{swirl} = 0$$

Because of gauge redundancy, the derivatives of the free energy with respect to color (“color susceptibility”) will typically be negative. So the two can balance!!!!

But this breaks the "hierarchy" of statistical mechanics

It mixes micro and macro perturbations!

In statistical mechanics, what normally distinguishes "work" from "heat" is coarse-graining, the separation between micro and macro states. Quantitatively, probability of thermal fluctuations is normalized by  $1/(c_V T)$  and microscopic correlations due to viscosity are  $\sim \eta/(Ts)$ . Since for a usual fluid, there is a hierarchy between microscopic scale, Knudsen number and gradient

$$\frac{1}{c_V T} \ll \frac{\eta}{(Ts)} \ll \partial u_\mu$$

Gauge symmetry breaks it, since it equalizes perturbations at both ends of this!

Is there a Gauge-independent way of seeing this? Perhaps!

One can write the effective Lagrangian in a Gauge-invariant way using **Wilson-Loops** . But the effective Lagrangian written this way will have an infinite number of terms, in a series weighted by the characteristic Wilson loop size. For a locally equilibrated system, this series does not commute with the gradient. Just like with Polymers, the system should have **multiple anisotropic non-local minima** which mess up any Knudsen number expansion. **Some materials are inhomogeneous and anisotropic at equilibrium, YM could be like this!**

**Lattice would not see it** , as there are no gradients there. There is an entropy maximum, and it is the one the lattice sees. The problems arise if you "coarse-grain" this maximum into each microscopic cell and try to do a gradient expansion around this equilibrium, unless you have color neutrality.

## Gauge theory and polarization

Since  $u^\mu \partial_\mu$  is in the Lagrangian, let us compare vorticity and Wilson loops!

$$\text{Vorticity : } \oint J_\mu dx^\mu \neq 0 \quad , \quad \text{Wilson loop : } \oint dx_\mu \partial^\mu U_{ab} \equiv \int_\Sigma d\Sigma_{\mu\nu} F_{ab}^{\mu\nu}$$

Lagrangian will in general have gauge-invariant terms proportional to  $Tr_a \omega_{\mu\nu a} F_a^{\mu\nu}$

$$F(b, J_j^\mu (\partial_\mu - U(x) \partial_\mu U(x)), Tr_a \omega_{\mu\nu a} F_a^{\mu\nu}) \quad , \quad \omega_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \partial^\alpha J_a^\beta$$

where as usual

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c$$

$$F(b, J_j^\mu (\partial_\mu - U(x)\partial_\mu)U(x), Tr_a \omega_{\mu\nu a} F_a^{\mu\nu}), \quad \omega_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \partial^\alpha J_a^\beta$$

Unlike in Jackiw et al ,  $F_{\mu\nu}$  is not field strength but just a polarization tensor, whose value is set by entropy maximization.

$\chi(y, b)$  and  $\tau_Y$  defineable as in hydrodynamics with polarization, but  $\chi(y, b) = dF(\dots)/d\omega_{\mu\nu a}$  must be gauge invariant, and also depend on  $u_\mu$  .

In analogy with hydrodynamics with polarization, most likely  $F_i^{\mu\nu}$  relaxes to  $\omega_i^{\mu\nu}$ , with some time-scale  $\tau$  from causality, with a "non-abelian"  $f_{ijk} \in \omega\omega$  correction.

But polarization and vorticity relaxation only up to gauge-dependence

$$\tau u^\mu \partial_\mu \text{Tr}_i [G_{\mu\nu}^i]^2 + \text{Tr}_i [G_{\mu\nu}^i]^2 = \chi \text{Tr}_i [\omega_{\mu\nu}^i]^2 + \mathcal{O}(f_{ijk} \omega_j \omega_k)$$

There should be "swirling modes" rotating in color space where vorticity and relaxation never relaxes.



Development of EoMs, linearization, etc. of this theory in progress!

**A crazy guess, speculation** Remember that all flow dependence through  $\mu_{ab}$  color chemical potentials. What if local equilibrium happens when they go to zero, i.e. color density is neutral.

**Could colored-swimming ghosts quickly be produced, and then locally thermalize and color-neutralize the QGP?**



What about gauge-gravity duality?

**Large  $N$**  non-hydrodynamic modes go away in the planar limit

There are  $N$  ghost modes and  $N^2$  degrees of freedom

**Conformal fixed point** most likely means ghosts non-dynamical

Not yet sure of this, but conformal invariance reduces pseudo-Gauge transformations to

$$\Phi_{\lambda,\mu\nu} \xrightarrow[\text{conformal}]{} g_{\sigma\mu}\partial_\nu\phi - g_{\sigma\nu}\partial_\mu\phi$$

where  $\phi$  is a scalar function. Irrelevant for dynamics.

As shown in Capri et al ( [1404.7163](#) ) Gribov copies for a Yang-Mills theory non-dynamical there. It would be a huge job to do this for hydrodynamics.

## Conclusions

**Hydrodynamics is not** a limit of transport, AdS/CFT or any other microscopic theory

**Hydrodynamics is** an EFT built around symmetries and entropy maximization and should be treated as such

**Once you realize this** , generalizing it to theories with extra DoFs, symmetries etc. becomes straight-forward.

**Lots of things to do** Gauge symmetry looks particularly interesting!

**Can we do better?** Put theory on the lattice, work with Thiago Nunes