Hydrodynamics with spin

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recent works with **F. Becattini, B. Friman, A. Jaiswal, A. Kumar, R. Ryblewski, and E. Speranza**
PRC97 (2018) 041901, PRD97 (2018) 116017, PRC98 (2018) 044906
PRC99 (2019) 011901, PLB789 (2019) 419

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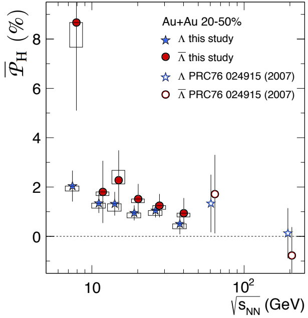
- **Non-central heavy-ion collisions create fireballs with large global angular momenta** which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
- **Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions**, both from the experimental and theoretical point of view
  L. Adamczyk et al. (**STAR**), (2017), **Nature 548 (2017) 62-65**
  Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid
  www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever



PERFECT-FLUID HYDRODYNAMICS = LOCAL EQUILIBRIUM + CONSERVATION LAWS
one usually includes energy, linear momentum, baryon number, ...

FOR PARTICLES WITH SPIN, THE CONSERVATION OF ANGULAR MOMENTUM IS NOT TRIVIAL
new hydrodynamic variables should be introduced

Canonical (quantization) version

$$e^{-(E-\mu)/T} \longrightarrow e^{-p\cdot\beta(x)+\xi(x)} \longrightarrow \widehat{\rho}_{\rm LEQ} = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_{\rm can}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{S}^{\mu,\lambda\nu}_{\rm can} - \xi\widehat{j}^\mu\right)\right] \qquad (1)$$

Σ is a space-like hypersurface, for example, corresponding to a constant LAB time t, in this case $\widehat{\rho}_{\rm LEQ} = \widehat{\rho}_{\rm LEQ}(t)$

β_ν and ξ are the Lagrange multiplier functions, whose meaning is the ratio between the local four-velocity u^μ and temperature T (a four-temperature vector) and the ratio between local chemical potential μ and T

one has to introduce an antisymmetric tensor field $\omega_{\lambda\nu}$, it is dubbed the **spin chemical potential** or the **spin polarization tensor**

Canonical tensors obtained by the Noether Theorem:

$$\partial_\mu \widehat{T}^{\mu\nu}_{\rm can} = 0, \quad \partial_\mu \widehat{J}^{\mu,\lambda\nu}_{\rm can} = 0. \tag{2}$$

Angular momentum has orbital and spin parts

$$\widehat{J}^{\mu,\lambda\nu}_{\rm can} = x^\lambda \widehat{T}^{\mu\nu}_{\rm can} - x^\nu \widehat{T}^{\mu\lambda}_{\rm can} + \widehat{S}^{\mu,\lambda\nu}_{\rm can} \equiv \widehat{L}^{\mu,\lambda\nu}_{\rm can} + \widehat{S}^{\mu,\lambda\nu}_{\rm can}, \tag{3}$$

$$\partial_\mu \widehat{J}^{\mu,\lambda\nu}_{\rm can} = \widehat{T}^{\lambda\nu}_{\rm can} - \widehat{T}^{\nu\lambda}_{\rm can} + \partial_\mu \widehat{S}^{\mu,\lambda\nu}_{\rm can} = 0, \qquad \partial_\mu \widehat{S}^{\mu,\lambda\nu}_{\rm can} = \widehat{T}^{\nu\lambda}_{\rm can} - \widehat{T}^{\lambda\nu}_{\rm can}. \tag{4}$$

Pseudo-gauge transformation

(different localization of energy density and angular momentum, global charges not changed)

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu} \right), \tag{5}$$

$$\widehat{S}'^{\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}. \tag{6}$$

Belinfante's construction: superpotential defined as $\widehat{\Phi} = \widehat{S}^{\lambda,\mu\nu}_{\rm can}$

$$\widehat{T}^{\mu\nu}_{\rm Bel} = \widehat{T}^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda \left(\widehat{S}^{\lambda,\mu\nu}_{\rm can} - \widehat{S}^{\mu,\lambda\nu}_{\rm can} - \widehat{S}^{\nu,\lambda\mu}_{\rm can} \right), \quad \widehat{S}^{\lambda,\mu\nu}_{\rm Bel} = 0. \tag{7}$$

Canonical version

$$\widehat{\rho}_{\rm LEQ} = \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_{\rm can}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{S}^{\mu,\lambda\nu}_{\rm can} - \xi\widehat{j}^\mu \right) \right] \tag{8}$$

Global equilibrium

in global equilibrium we require that $\widehat{\rho}_{\rm LEQ}$ = const.
this implies

β satisfies the Killing equation, spin polarization is given by thermal vorticity $\varpi_{\lambda\nu}$

$$\partial_\lambda\beta_\nu + \partial_\nu\beta_\lambda = 0, \qquad \omega_{\lambda\nu} = \varpi_{\lambda\nu} \equiv -\frac{1}{2}\left(\partial_\lambda\beta_\nu - \partial_\nu\beta_\lambda\right) = \text{const.} \tag{9}$$

Local equilibrium, two options:
1) $\omega_{\lambda\nu}(x) = \varpi_{\lambda\nu}(x) = -\frac{1}{2}\left(\partial_\lambda\beta_\nu(x) - \partial_\nu\beta_\lambda(x)\right)$
2) $\omega_{\lambda\nu}(x)$ and $\varpi_{\lambda\nu}(x)$ are independent,
β and $\omega_{\lambda\nu}(x)$ "control" different densities

based on the works by F. Becattini and collaborators
also in local equilibrium $\omega_{\lambda\nu}(x) = \varpi_{\lambda\nu}(x)$

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin

2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface
(defined often by the condition T=const)

3) Calculate thermal vorticity $\varpi_{\alpha\beta}(x) \neq$ const

4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$

5) Make predictions about spin polarization

SUCH A METHOD WORKS WELL,
DESCRIBES MOST OF THE DATA, BUT...CAN WE TAKE IT FOR GRANTED?

in local equilibrium $\widehat{\rho}_{\rm LEQ}$ is approximately constant (with dissipation effects neglected)

$$T^{\mu\nu} = {\rm tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{T}^{\mu\nu}\right), \quad S^{\mu,\lambda\nu} = {\rm tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{S}^{\mu,\lambda\nu}\right), \quad j^{\mu} = {\rm tr}\left(\widehat{\rho}_{\rm LEQ}\,\widehat{j}^{\mu}\right). \tag{10}$$

these tensors are all functions of the hydrodynamic variables β_μ, $\omega_{\mu\nu}$, and ξ

$$T^{\mu\nu} = T^{\mu\nu}[\beta,\omega,\xi], \quad S^{\mu,\lambda\nu} = S^{\mu,\lambda\nu}[\beta,\omega,\xi], \quad j^{\mu} = j^{\mu}[\beta,\omega,\xi], \tag{11}$$

and satisfy the conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^{\mu} = 0 \tag{12}$$

standard scalar functions $f(x,p)$ are generalized to 2×2 Hermitean matrices in spin space for each value of the space-time position x and four-momentum p

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32

$$\left[f^+(x,p)\right]_{rs} \equiv f^+_{rs}(x,p) = \bar{u}_r(p)X^+u_s(p), \tag{13}$$

$$[f^-(x,p)]_{rs} \equiv f^-_{rs}(x,p) = -\bar{v}_s(p)X^-v_r(p). \tag{14}$$

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right]M^\pm, \quad M^\pm = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]. \tag{15}$$

here $\Sigma^{\mu\nu}$ is the Dirac spin operator, electric- and magnetic-like three-vectors

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}. \tag{16}$$

special case in this talk

$$M^\pm = 1 \pm \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}. \tag{17}$$

The spin observables are represented by the Pauli matrices $\boldsymbol{\sigma}$ and the expectation values of $\boldsymbol{\sigma}$ provide information on the polarization of spin-1/2 particles in their rest frame

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[1 - \frac{1}{2}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right], \tag{18}$$

average polarization per particle

$$\boldsymbol{P} \quad = \quad \frac{1}{m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right] = \boldsymbol{b}_* \quad (\boldsymbol{b}\text{ field in the particle rest frame}) \tag{19}$$

$$\langle\boldsymbol{P}(x,p)\rangle = \frac{1}{2}\,\frac{\mathrm{tr}_2\left[(f^+ + f^-)\boldsymbol{\sigma}\right]}{\mathrm{tr}_2\left[f^+ + f^-\right]} = -\frac{1}{4}\boldsymbol{P}. \tag{20}$$

De Groot, van Leeuwen, van Weert: *Relativistic Kinetic Theory. Principles and Applications*

GLW framework

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\, \delta^{(4)}(k-p) u^r(p)\bar{u}^s(p) f^+_{rs}(x,p), \tag{21}$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\, \delta^{(4)}(k+p) v^s(p)\bar{v}^r(p) f^-_{rs}(x,p). \tag{22}$$

Clifford-algebra expansion

(used in many early works on QED and QGP plasma, e.g., H.T. Elze, M. Gyulassy, D. Vasak, Phys.Lett. B177 (1986) 402)

for the equilibrium

$$\mathcal{W}^{\pm}_{\rm eq}(x,k) \quad = \quad \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5 \mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^\mu \mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) + \gamma_5\gamma^\mu \mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].$$

and any other Wigner function

$$\mathcal{W}^{\pm}(x,k) \quad = \quad \frac{1}{4}\left[\mathcal{F}^{\pm}(x,k) + i\gamma_5 \mathcal{P}^{\pm}(x,k) + \gamma^\mu \mathcal{V}^{\pm}_{\mu}(x,k) + \gamma_5\gamma^\mu \mathcal{A}^{\pm}_{\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{\mu\nu}(x,k)\right].$$

WF, A. Kumar, R. Ryblewski, Phys. Rev. C98 (2018) 044906

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = C[\mathcal{W}(x,k)]. \tag{23}$$

Here K^μ is the operator defined by the expression

$$K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu, \tag{24}$$

In the case of global equilibrium, with the vanishing collision term, the Wigner function $\mathcal{W}(x,k)$ exactly satisfies the equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = 0. \tag{25}$$

the leading order terms in $\hbar$ can be taken from $\mathcal{W}^{\pm}_{\rm eq}(x,k)$

$$\mathcal{F}^{(0)} = \mathcal{F}_{\rm eq}, \tag{26}$$
$$\mathcal{P}^{(0)} = 0, \tag{27}$$
$$\mathcal{V}^{(0)}_\mu = \mathcal{V}_{{\rm eq},\mu}, \tag{28}$$
$$\mathcal{A}^{(0)}_\mu = \mathcal{A}_{{\rm eq},\mu}, \tag{29}$$
$$\mathcal{S}^{(0)}_{\mu\nu} = \mathcal{S}_{{\rm eq},\mu\nu}. \tag{30}$$

From global to local equilibrium

in the NLO in $\hbar$ we get the kinetic equation (well known in the literature)

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0, \tag{31}$$

$$k^\mu \partial_\mu \, \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \, \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \tag{32}$$

Global equilibrium — Eq. (31) and Eq. (32) are exactly fulfilled, what about local equilibrium

Local equilibrium — only moments of Eq. (31) and Eq. (32) are satisfied

$$\partial_\alpha N^\alpha_{\rm eq}(x) = 0, \quad \partial_\alpha T^{\alpha\beta}_{\rm GLW}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}_{\rm GLW}(x) = 0. \tag{33}$$

GLW — forms proposed in the textbook on the kinetic theory by de Groot - van Leeuwen - van Weert charge current

$$N^{\alpha}_{\rm GLW} = n u^{\alpha}, \qquad n = 4 \sinh(\xi)\, n_{(0)}(T) \tag{34}$$

energy-momentum tensor (with a perfect-fluid form)

$$T^{\mu\nu}_{\rm GLW}(x) \quad = \quad (\varepsilon + P) u^{\mu} u^{\nu} - P g^{\mu\nu}, \tag{35}$$

$$\varepsilon = 4 \cosh(\xi)\, \varepsilon_{(0)}(T)\,, \quad P = 4 \cosh(\xi)\, P_{(0)}(T), \tag{36}$$

$n_{(0)}(T)$, $\varepsilon_{(0)}(T)$, $P_{(0)}(T)$ — particle density, energy density, and pressure of classical particles at the temperature T

spin tensor

$$S^{\lambda,\mu\nu}_{\rm GLW} \quad = \quad \frac{\hbar \cosh(\xi)}{m^2} \int dP\, e^{-\beta \cdot p} p^{\lambda} \left(m^2 \omega^{\mu\nu} + 2 p^{\alpha} p^{[\mu} \omega^{\nu]}{}_{\alpha} \right) = S^{\lambda,\mu\nu}_{\rm ph} + S^{\lambda,\mu\nu}_{\Delta}. \tag{37}$$

only $S^{\lambda,\mu\nu}_{\rm ph}$ was used in WF, B. Friman, A. Jaiswal, E.Speranza, Phys.Rev. C97 (2018) 041901

LO generates corrections in the NLO

$$\mathcal{P}^{(1)} = -\frac{1}{2m}\,\partial^\mu \mathcal{A}_{\mathrm{eq},\mu}, \tag{38}$$

$$\mathcal{V}^{(1)}_\mu \quad = \quad -\frac{1}{2m}\partial^\nu \mathcal{S}_{\mathrm{eq},\nu\mu}, \tag{39}$$

$$\mathcal{S}^{(1)}_{\mu\nu} = \frac{1}{2m}\left(\partial_\mu \mathcal{V}_{\mathrm{eq},\nu} - \partial_\nu \mathcal{V}_{\mathrm{eq},\mu}\right). \tag{40}$$

IMPORTANT IF the canonical formalism is used

$$T^{\mu\nu}_{\mathrm{GLW}}(x) = \frac{1}{m}\mathrm{tr}_4 \int d^4k\, k^\mu\, k^\nu \mathcal{W}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu \mathcal{F}(x,k). \tag{41}$$

$$T^{\mu\nu}_{\mathrm{can}}(x) = \int d^4k\, k^\nu \mathcal{V}^\mu(x,k) \tag{42}$$

quantum corrections induce asymmetry $T^{\mu\nu}_{\mathrm{can}}(x) \neq T^{\nu\mu}_{\mathrm{can}}(x)$

From canonical to GLW case 1

Including the components of $\mathcal{V}^\mu(x,k)$ up to the first order in the equilibrium case we obtain

$$T^{\mu\nu}_{\rm can}(x) = T^{\mu\nu}_{\rm GLW}(x) + \delta T^{\mu\nu}_{\rm can}(x) \tag{43}$$

where

$$\delta T^{\mu\nu}_{\rm can}(x) = -\frac{\hbar}{2m}\int d^4k k^\nu \partial_\lambda \mathcal{S}^{\lambda\mu}_{\rm eq}(x,k) = -\partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x). \tag{44}$$

The canonical energy-momentum tensor is conserved

$$\partial_\alpha T^{\alpha\beta}_{\rm can}(x) = 0. \tag{45}$$

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm can} &= \frac{\hbar \sinh(\zeta)\cosh(\xi)}{\zeta}\int dP\, e^{-\beta\cdot p}\left(\omega^{\mu\nu}p^\lambda + \omega^{\nu\lambda}p^\mu + \omega^{\lambda\mu}p^\nu\right) \\ &\equiv \frac{\hbar w}{4\zeta}\left(u^\lambda\omega^{\mu\nu} + u^\mu\omega^{\nu\lambda} + u^\nu\omega^{\lambda\mu}\right) = S^{\lambda,\mu\nu}_{\rm GLW} + S^{\mu,\nu\lambda}_{\rm GLW} + S^{\nu,\lambda\mu}_{\rm GLW}. \end{aligned} \tag{46}$$

The canonical spin tensor is not conserved

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x) + \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x). \tag{47}$$

if we introduce the tensor $\Phi_{\rm can}^{\lambda,\mu\nu}$ defined by the relation

$$\Phi_{\rm can}^{\lambda,\mu\nu} \equiv S_{\rm GLW}^{\mu,\lambda\nu} - S_{\rm GLW}^{\nu,\lambda\mu}, \tag{48}$$

we can write

$$S_{\rm can}^{\lambda,\mu\nu} = S_{\rm GLW}^{\lambda,\mu\nu} - \Phi_{\rm can}^{\lambda,\mu\nu} \tag{49}$$

and

$$T_{\rm can}^{\mu\nu} = T_{\rm GLW}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi_{\rm can}^{\lambda,\mu\nu} + \Phi_{\rm can}^{\mu,\nu\lambda} + \Phi_{\rm can}^{\nu,\mu\lambda} \right). \tag{50}$$

The canonical and GLW frameworks are connected by a pseudo-gauge transformation. Similarly to Belinfante, it leads to a symmetric energy-momentum tensor, however, does not eliminate the spin degrees of freedom.

we introduce the phase-space density $\Delta\Pi_\mu$ of the **Pauli-Lubański vector**

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p\frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}. \tag{51}$$

only the spin-part contributes here, the results are the same for the canonical and GLW versions

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}E_p\frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\hbar\cosh(\xi)}{(2\pi)^3}\,e^{-p\cdot\beta}p^\lambda\tilde{\omega}_{\mu\beta}. \tag{52}$$

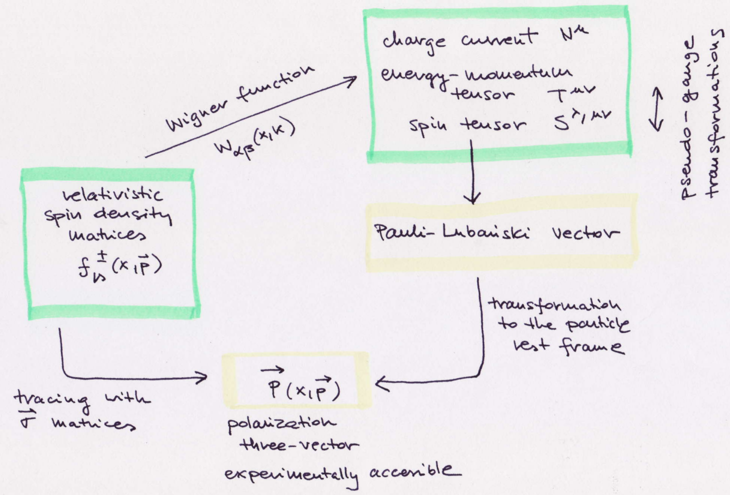
dividing by the total density of particles and antiparticles, we find

$$\pi_\mu(x,p) \equiv \frac{\Delta\Pi_\mu(x,p)}{\Delta\mathcal{N}(x,p)} = -\frac{\hbar}{4m}\,\tilde{\omega}_{\mu\beta}\,p^\beta. \tag{53}$$

in PRF

$$\pi^0_* = 0, \qquad \boldsymbol{\pi}_* = -\frac{\hbar}{4}\boldsymbol{P}. \tag{54}$$

This is an important result showing that the space part of the PL vector in PRF agrees with the mean polarization three-vector


charge current N^μ
energy-momentum tensor $T^{\mu\nu}$
spin tensor $S^{\lambda,\mu\nu}$
Wigner function
$W_{\alpha\beta}(x,k)$
relativistic spin density matrices
$f^{\pm}_{rs}(x,\vec{p})$
Pauli-Lubanski vector
pseudo-gauge transformations
transformation to the particle rest frame
tracing with $\vec{\sigma}$ matrices
$\vec{P}(x,\vec{p})$
polarization three-vector
experimentally accessible

internal angular momentum tensor $s^{\alpha\beta}$

M. Matthison, Neue mechanik materieller systemes, Acta Phys. Polon. 6 (1937) 163.

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta. \tag{55}$$

$$s \cdot p = 0, \quad s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta} \tag{56}$$

A straightforward generalization of the phase-space distribution function $f(x, \boldsymbol{p})$ is a spin dependent distribution $f(x, \boldsymbol{p}, s)$

$$\int dS \ldots = \frac{m}{\pi \mathfrak{s}} \int d^4 s \, \delta(s \cdot s + \mathfrak{s}^2) \, \delta(p \cdot s) \ldots \tag{57}$$

$$\mathfrak{s}^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} \tag{58}$$

$$\int dS = \frac{m}{\pi \mathfrak{s}} \int d^4 s \, \delta(s \cdot s + \mathfrak{s}^2) \, \delta(p \cdot s) = 2 \tag{59}$$

equilibrium distribution functions for particles and antiparticles in the form

$$f_{\rm eq}^{\pm}(x,p,s) = \exp\left(-p\cdot\beta(x) \pm \xi(x) + \frac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right). \qquad (60)$$

conserved "currents"

$$N_{\rm eq}^{\mu} = \int dP \int dS\, p^{\mu}\left[f_{\rm eq}^{+}(x,p,s) - f_{\rm eq}^{-}(x,p,s)\right], \qquad (61)$$

$$T_{\rm eq}^{\mu\nu} = \int dP \int dS\, p^{\mu}p^{\nu}\left[f_{\rm eq}^{+}(x,p,s) + f_{\rm eq}^{-}(x,p,s)\right] \qquad (62)$$

$$S_{\rm eq}^{\lambda,\mu\nu} = \int dP \int dS\, p^{\lambda}\, s^{\mu\nu}\left[f_{\rm eq}^{+}(x,p,s) + f_{\rm eq}^{-}(x,p,s)\right] \qquad (63)$$

For $|\omega_{\mu\nu}| < 1$ we obtain the formalism that agrees with that based on the quantum description of spin (in the GLW version)

PL vector can be expressed by the simple expression

$$\pi_\mu = -\mathfrak{s} \frac{\tilde{\omega}_{\mu\beta}}{P} \frac{p^\beta}{m} L(P \mathfrak{s}), \tag{64}$$

where $L(x)$ is the Langevin function defined by the formula

$$L(x) = \coth(x) - \frac{1}{x}. \tag{65}$$

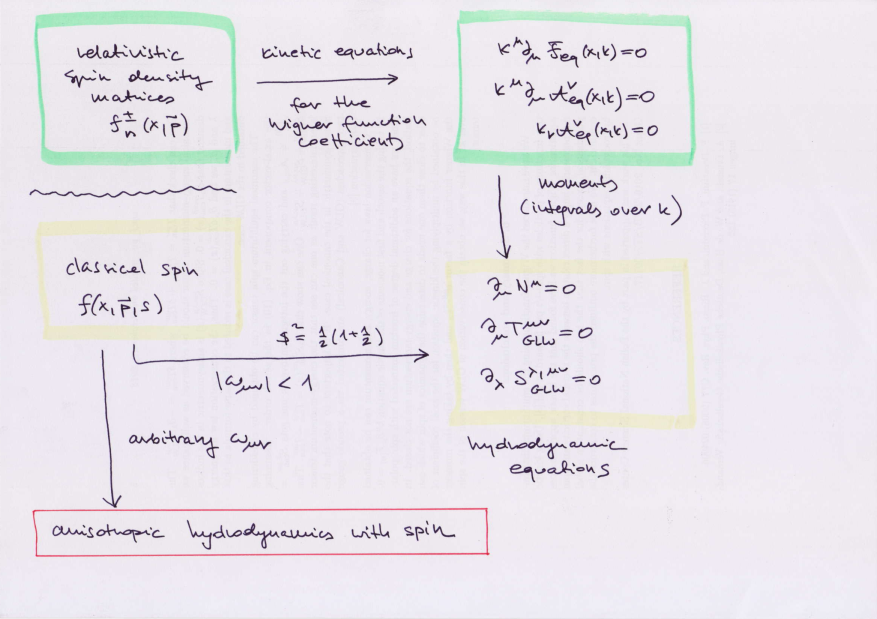
in PRF the direction of the PL vector agrees with that of the polarization vector $\boldsymbol{P}$. For small and large P we obtain two important results:

$$\boldsymbol{\pi}_* = -\mathfrak{s} \frac{\boldsymbol{P}}{P}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s} = \sqrt{\frac{3}{4}}, \quad \text{if} \quad P \gg 1 \tag{66}$$

and

$$\boldsymbol{\pi}_* = -\mathfrak{s}^2 \frac{\boldsymbol{P}}{3}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s}^2 \frac{P}{3} = \frac{P}{4}, \quad \text{if} \quad P \ll 1. \tag{67}$$

Summary 2


relativistic spin density matrices $f_n^{\pm}(x, \vec{p})$
kinetic equations
for the Wigner function coefficients
$k^\mu \partial_\mu \mathcal{F}_{eq}(x,k) = 0$
$k^\mu \partial_\mu \mathcal{A}^\nu_{eq}(x,k) = 0$
$k_\nu \mathcal{A}^\nu_{eq}(x,k) = 0$
moments (integrals over k)
classical spin $f(x, \vec{p}, s)$
$s^2 = \frac{1}{2}(1+\frac{1}{2})$
$|\omega_{\mu\nu}| < 1$
$\partial_\mu N^\mu = 0$
$\partial_\mu T^{\mu\nu}_{GLW} = 0$
$\partial_\lambda S^{\lambda,\mu\nu}_{GLW} = 0$
arbitrary $\omega_{\mu\nu}$
hydrodynamic equations
anisotropic hydrodynamics with spin

Conclusions

1. The arguments collected in our works suggest using **the de Groot - van Leeuwen - van Weert (GLW) forms of the energy-momentum and spin tensors**, together with their conservation laws, as the building blocks for construction of hydrodynamics with spin.
2. **The GLW framework can be connected with the canonical one (obtained with the help of the Noether theorem from the underlying Lagrangian) through a pseudo-gauge transformation** that has been explicitly constructed. Both, the GLW and canonical frameworks include spin degress of freedom, hence can be used at the same footing to describe spin polarization phenomena.
3. **The pseudo-gauge transformation from the canonical to the Belinfante forms neglects the spin degrees of freedom and leads to a formalism that is not satisfactory for description of the polarization phenomena** — the total angular momentum in the Belinfante approach has the form of the orbital angular momentum which can be always set equal to zero by a Lorentz transformation.
4. **Using the classical concept of spin one can formulate a consistent framework of hydrodynamics with spin**, which for small values of the polarization agrees with the approach using relativistic spin-density matrices. The classical-spin approach is free from the problems connected with normalization of the polarization three-vector and indicates that the hydrodynamic system becomes anisotropic if the spin densities are large. The classical approach allows also for the explicit definition of a conserved entropy current.

more in arXiv:1811.04409

Quark Matter 2021 in Kraków, Oct. 4-9, 2021, ICE Congress Center