Electromagnetic (and Hadronic) Air Showers

Paolo Lipari (INFN, Roma La Sapienza)

ISAPP school 2018. LHC meets Cosmic Rays
CERN october 28th 2018

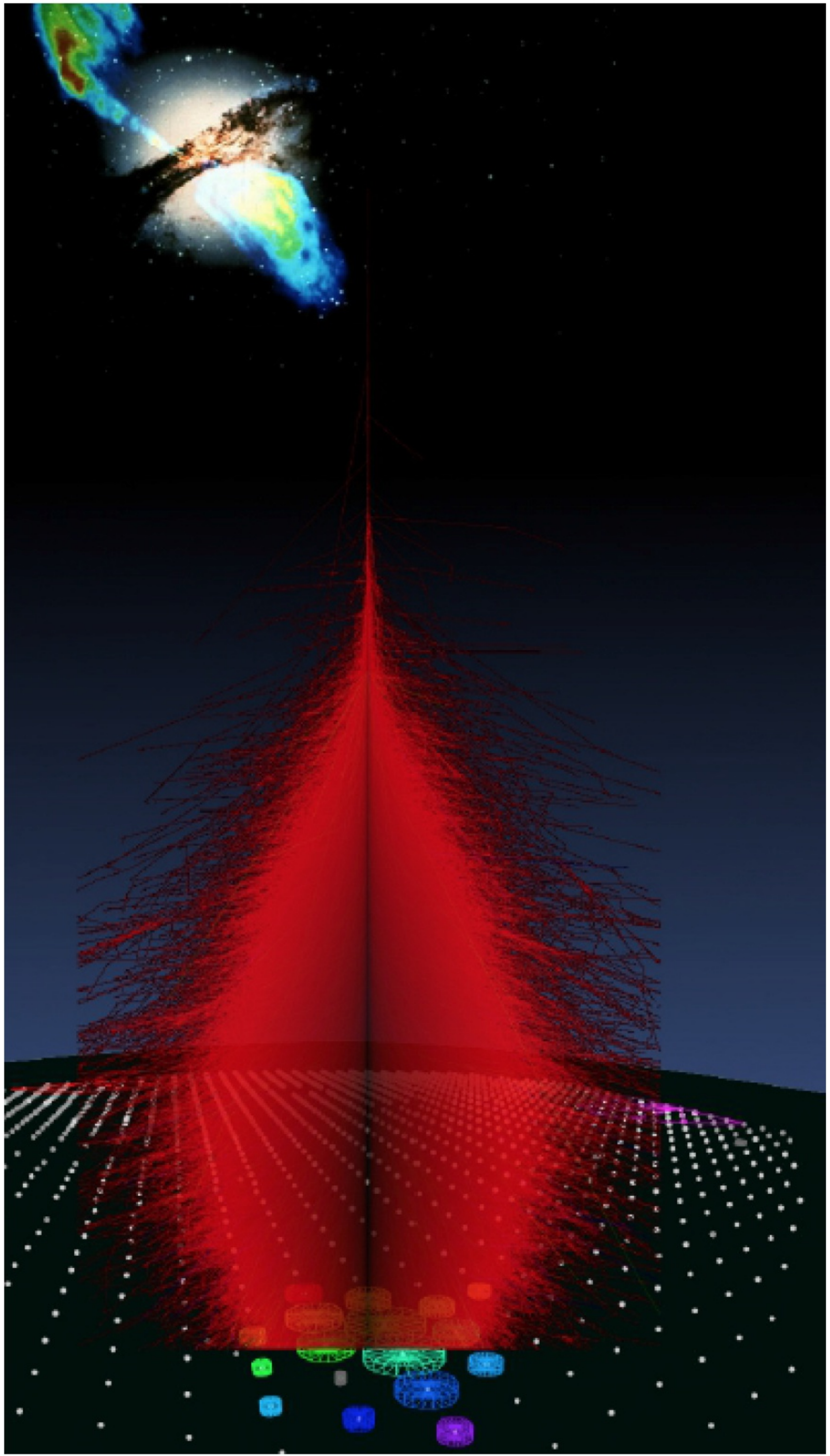


image by Ralf Ulrich

the Source

the Shower

the Data

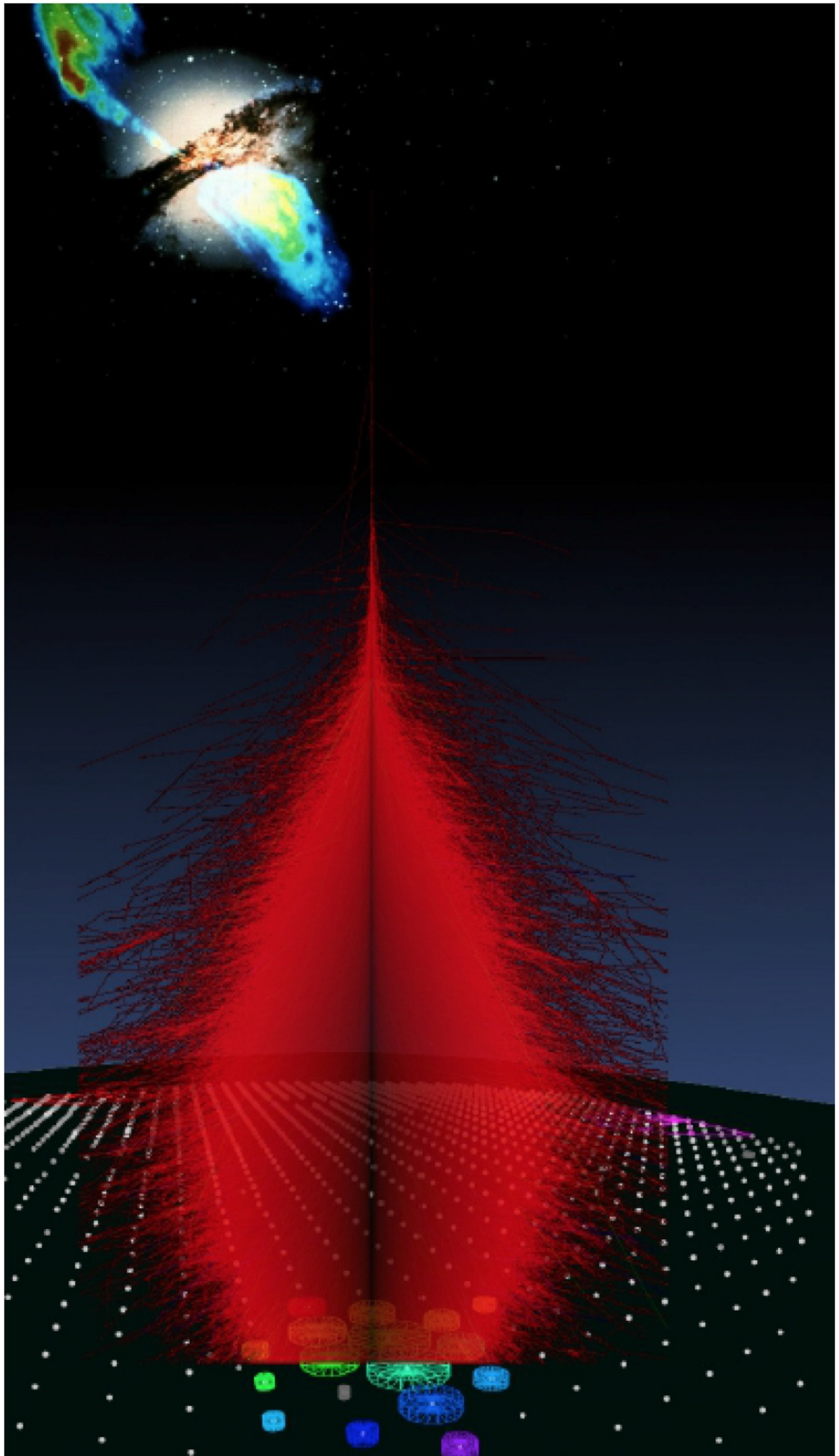


image by Ralf Ulrich

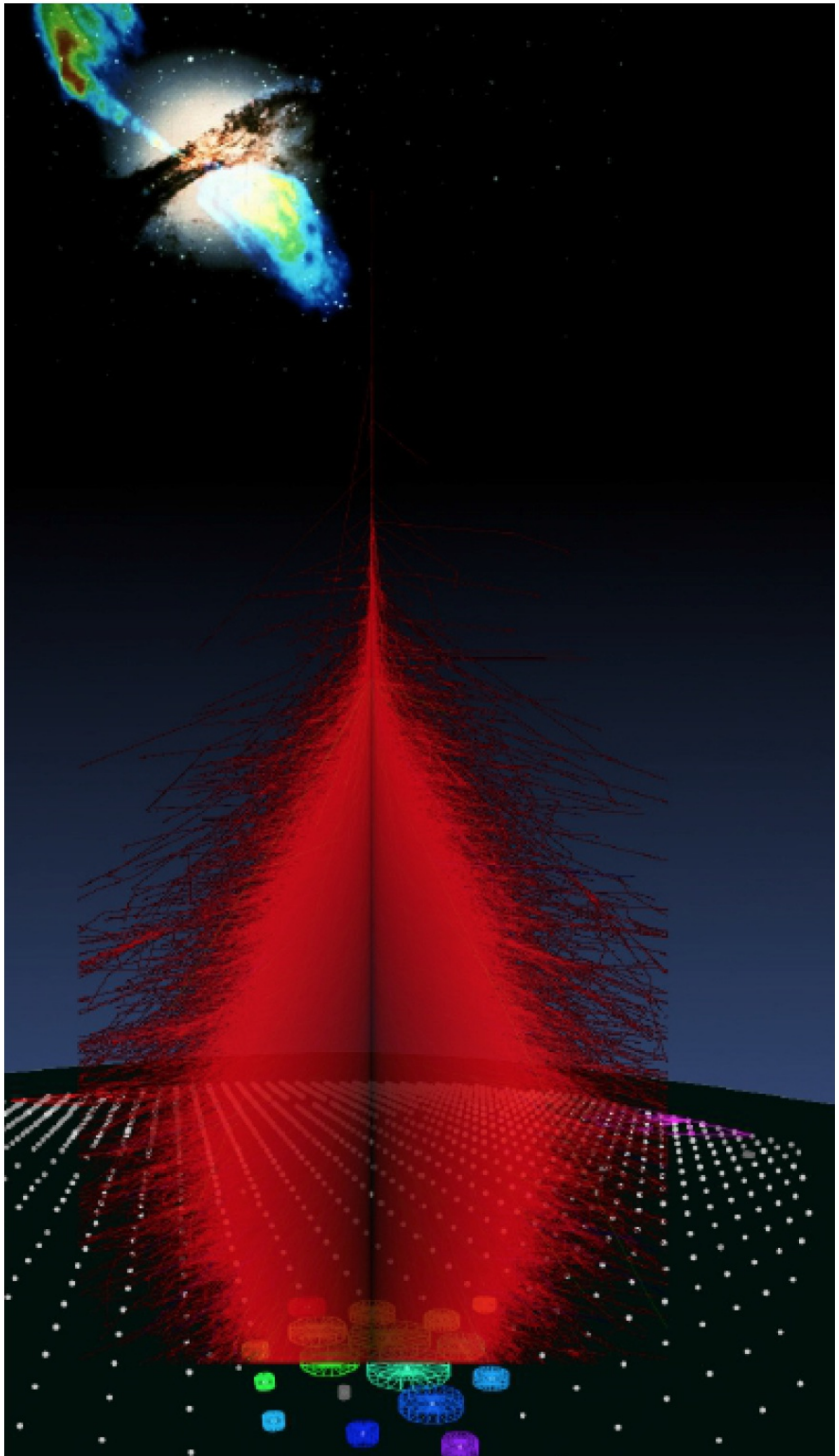
the Source

From spectra and composition of observed particles infer properties of the astrophysical sources

the Shower

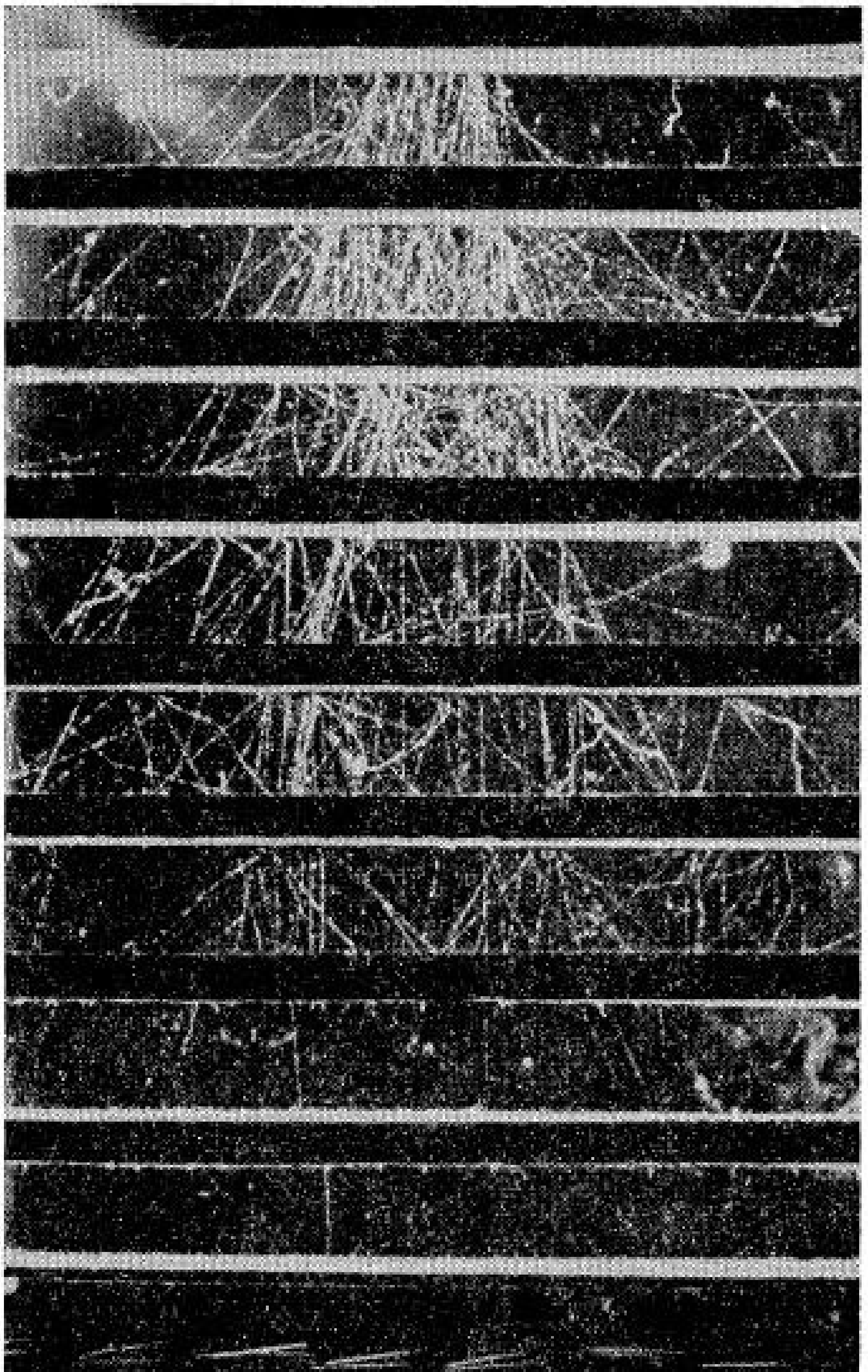
To reconstruct the energy (and identity) of the primary particle, one needs a sufficiently good understanding of the shower development.

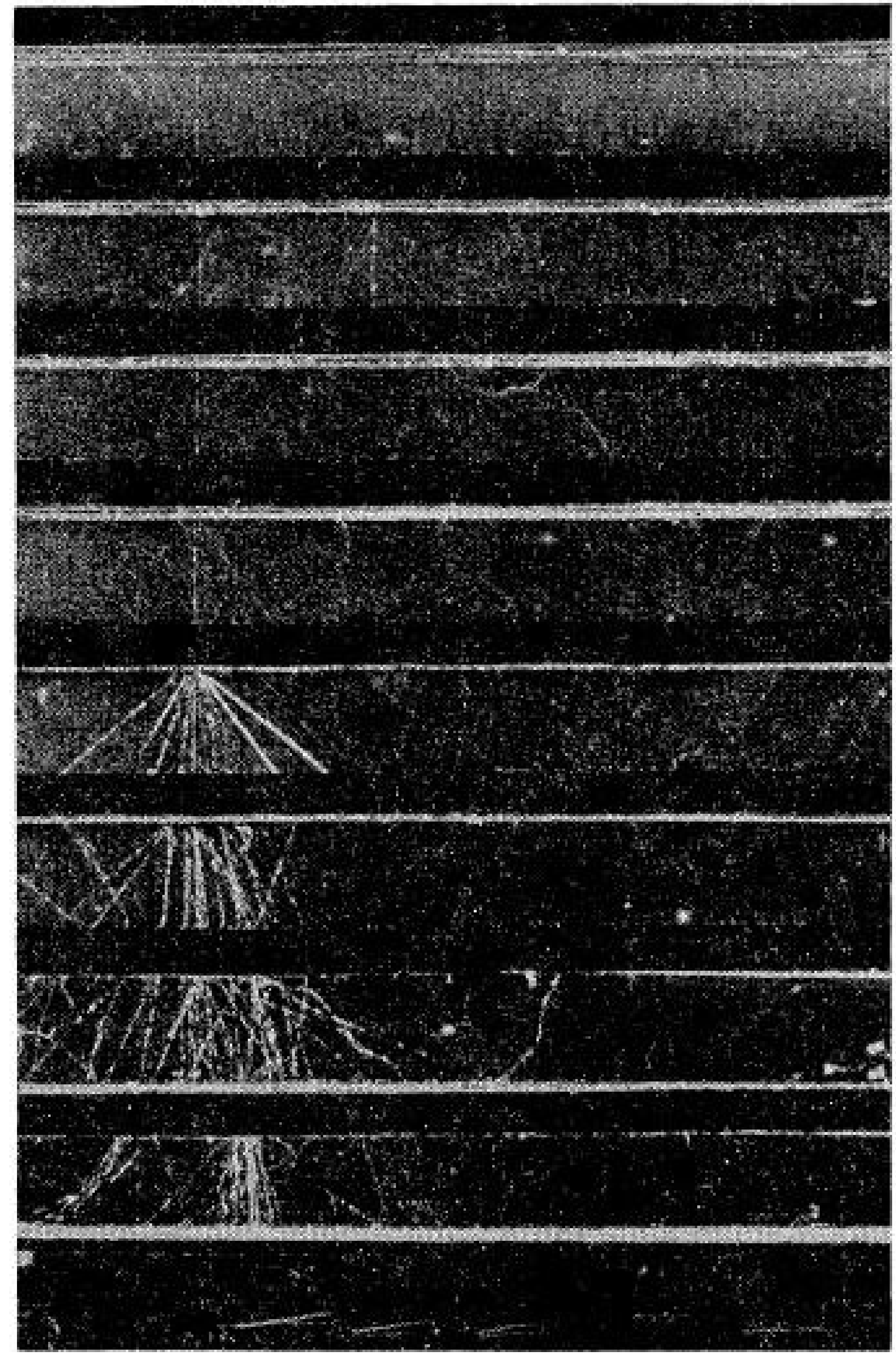
the Data

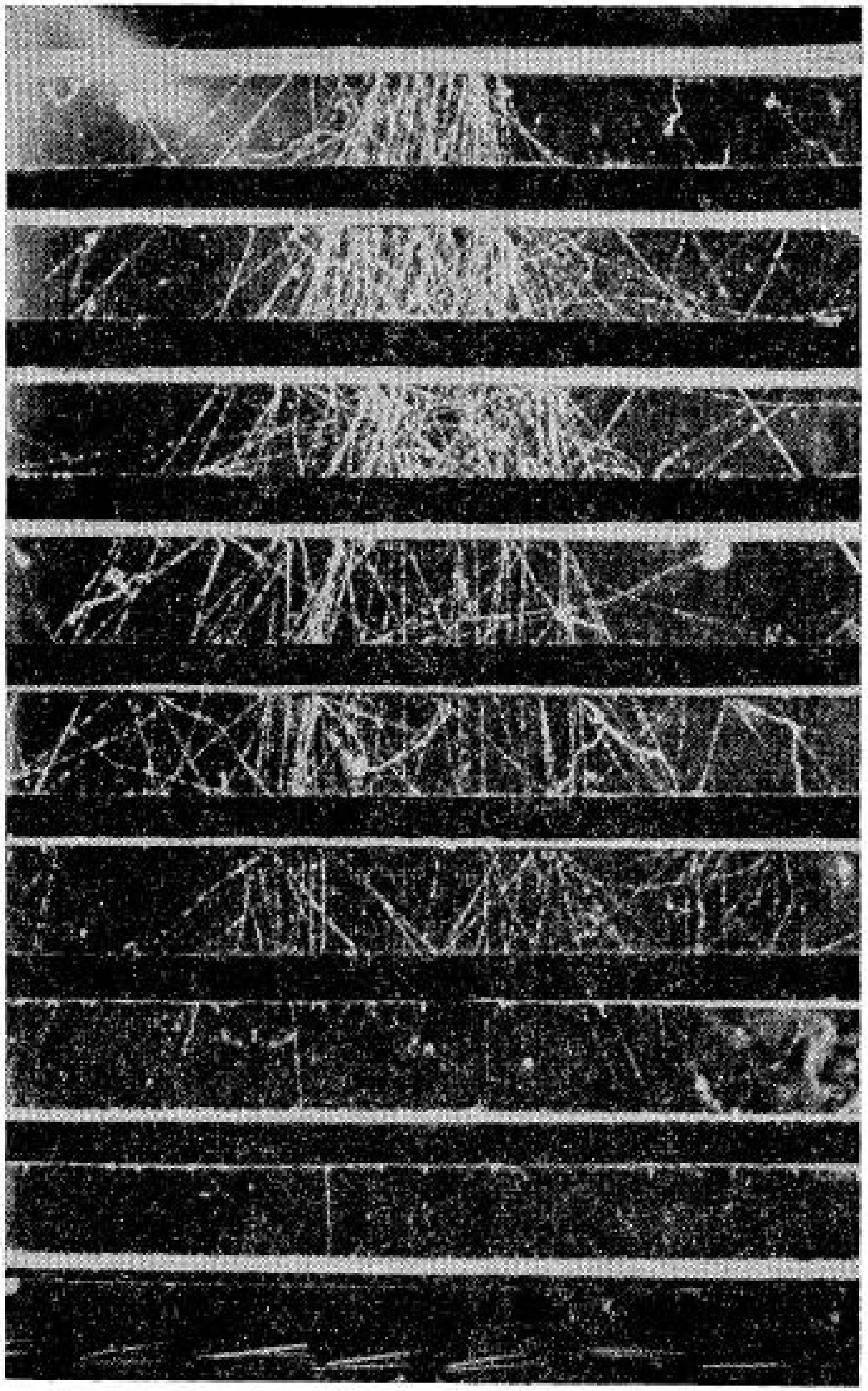


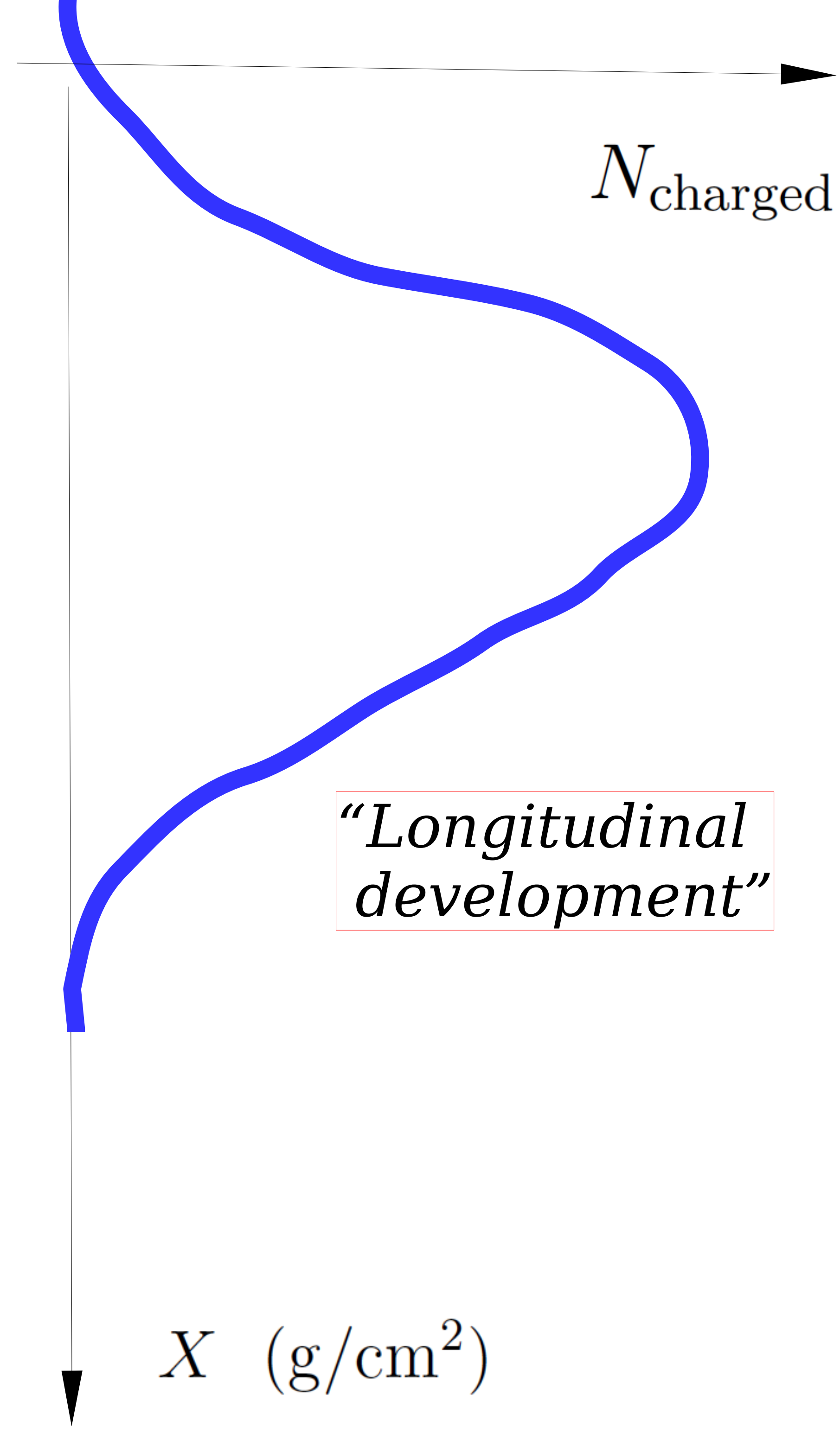
Is it possible to infer from the observations information about the properties of hadronic interactions at energies not accessible to accelerators

?

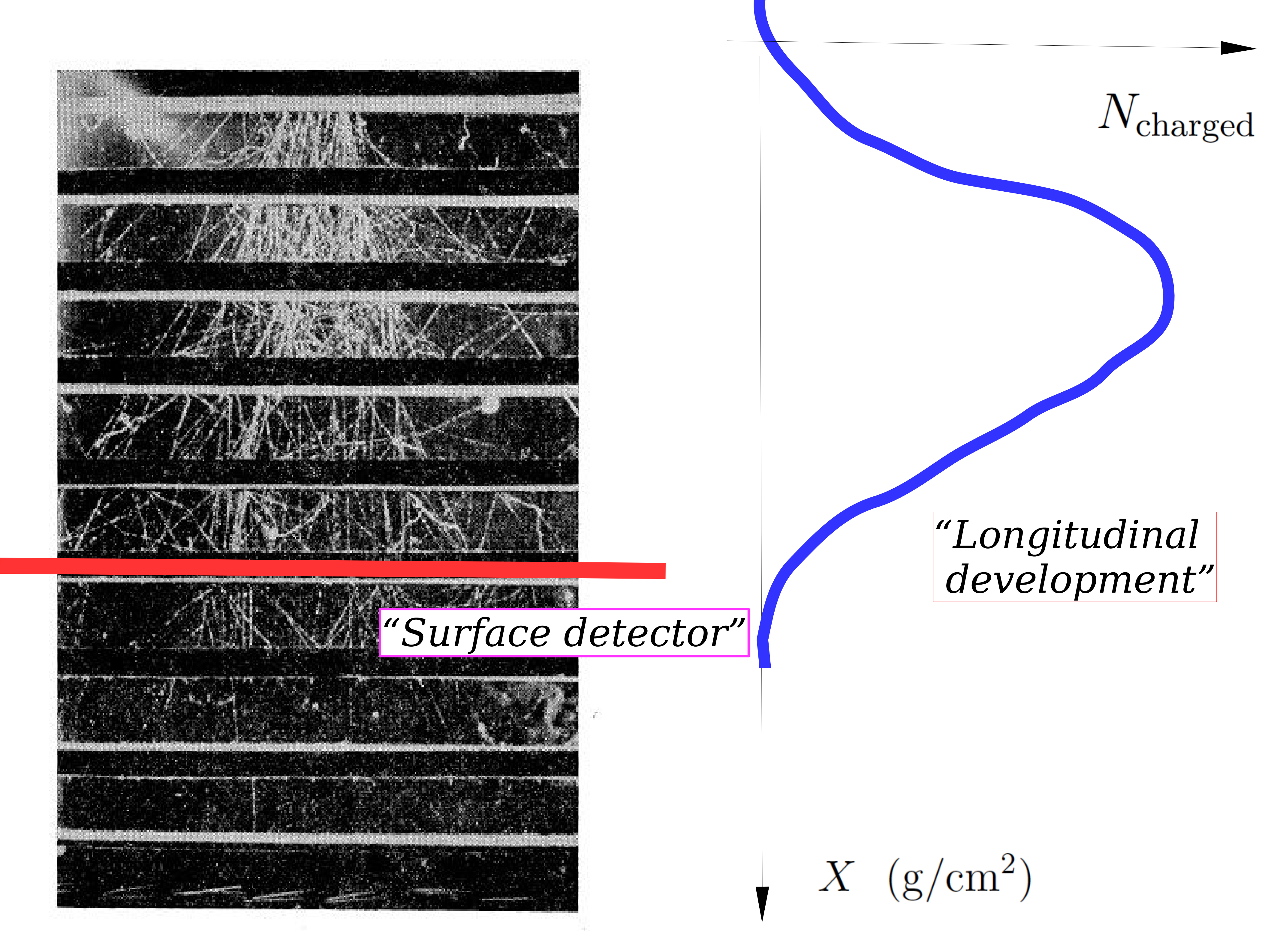




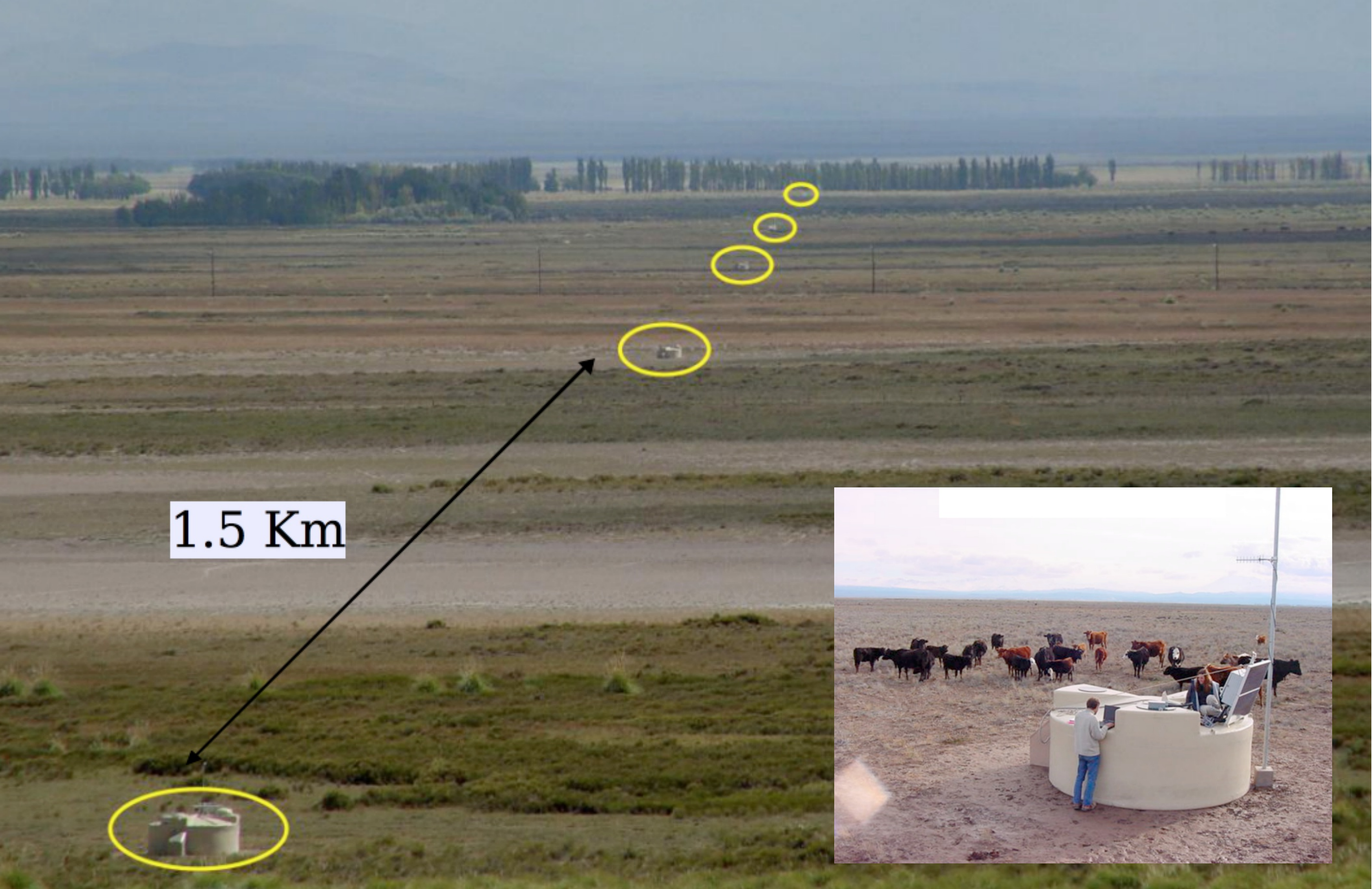




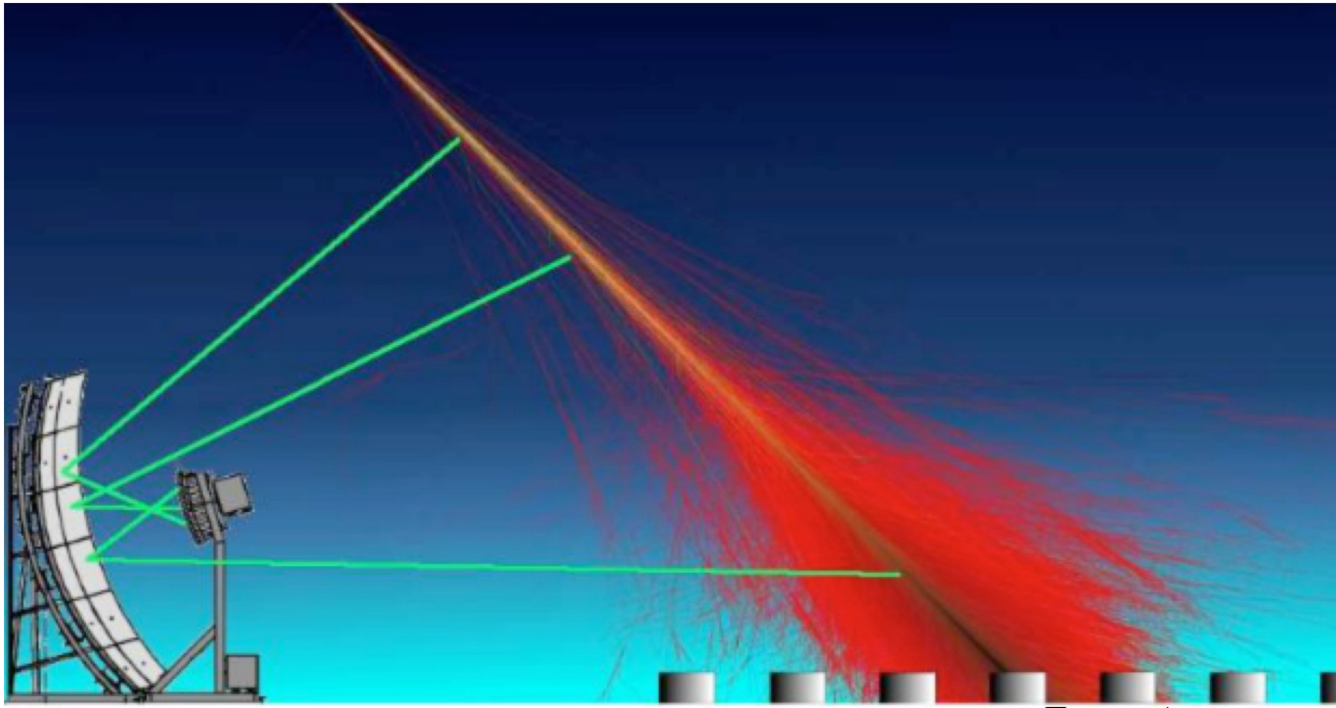
N_{charged}
"Longitudinal development"
X (g/cm^2)

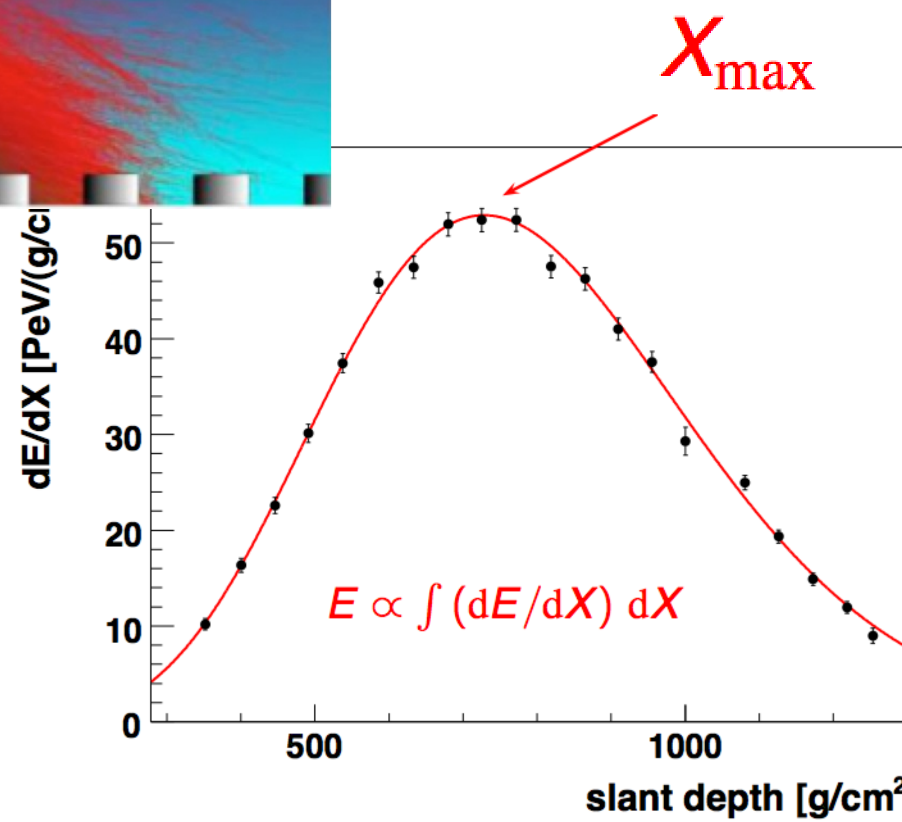

N_{charged}
“Longitudinal development”
“Surface detector”
X (g/cm^2)

AUGER detector in ARGENTINA



Longitudinal development





Start discussing a

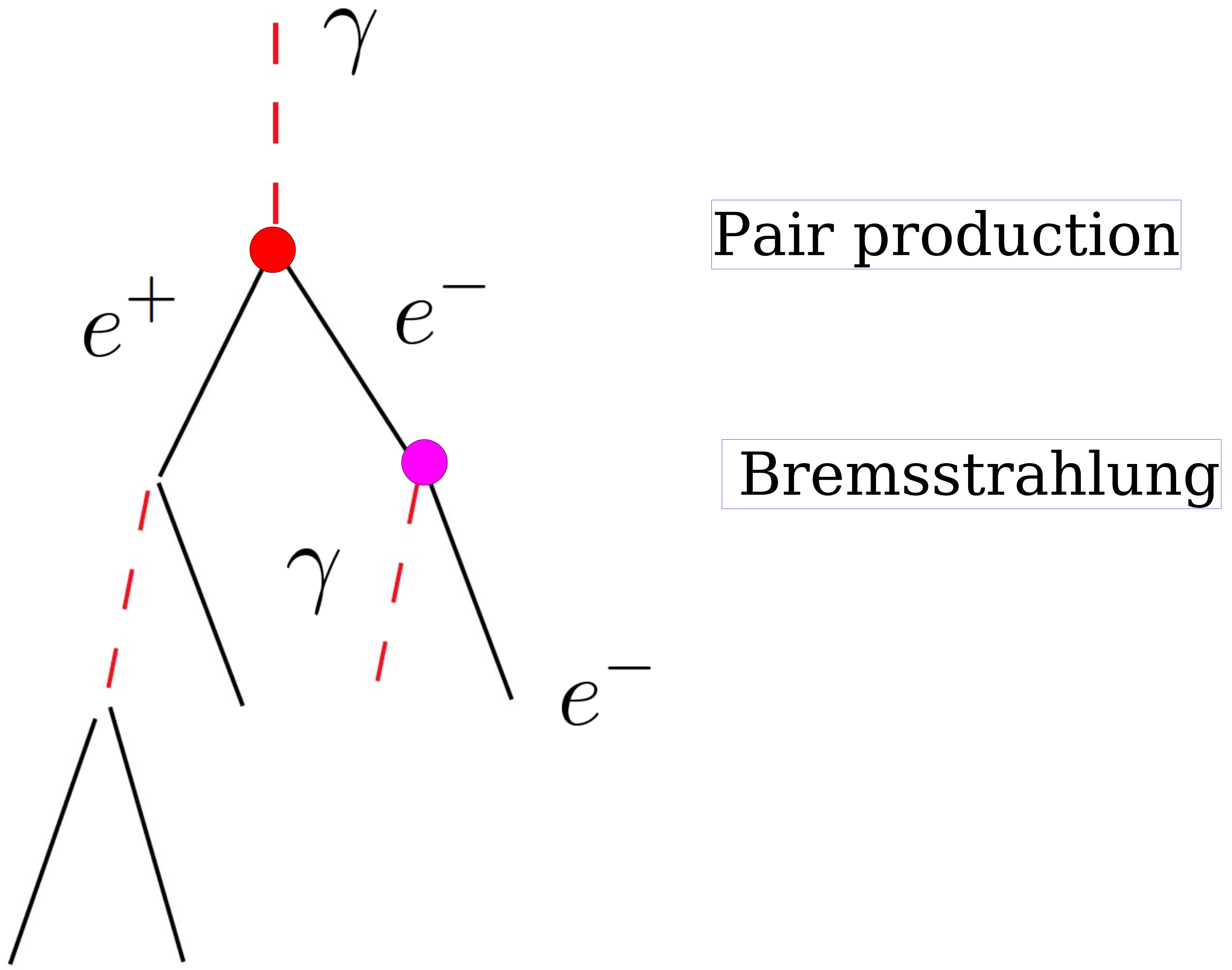
purely **electromagnetic shower**

formed by 3 types of particles $e^- \; e^+ \; \gamma$

(effectively 2 types) $e^{\mp} \quad \gamma$

Determined by 3 fundamental processes;

[a] Bremsstrahlung
[b] Pair Production
[c] Ionization losses



γ
e^+
e^-
γ
e^-
Pair production
Bremsstrahlung

FUNDAMENTAL PROCESSES in QUANTUM ELECTRODYNAMICS

COMPTON SCATTERING

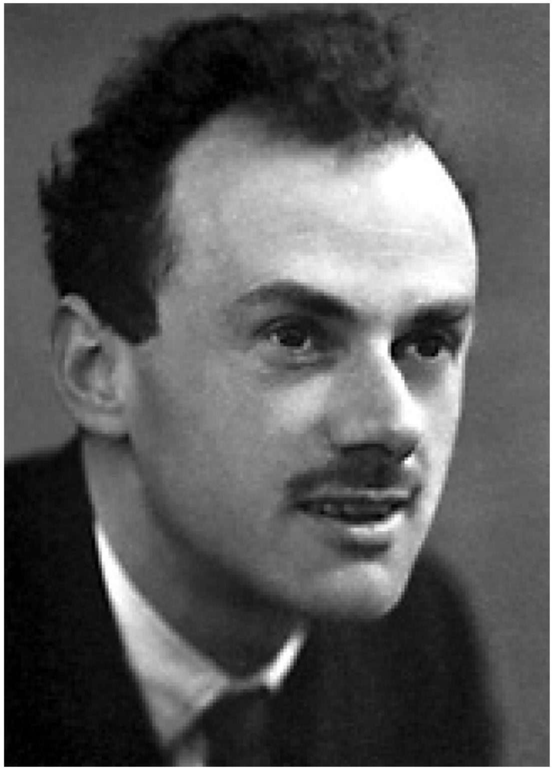
$$e + \gamma \longrightarrow e + \gamma$$

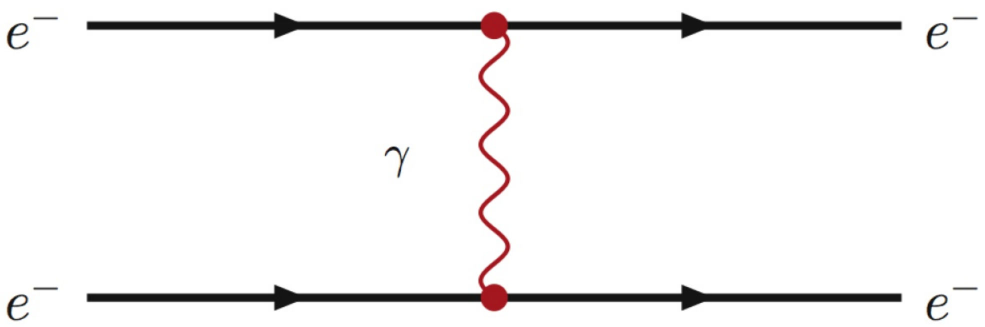
ELECTRON-POSITRON ANNIHILATION

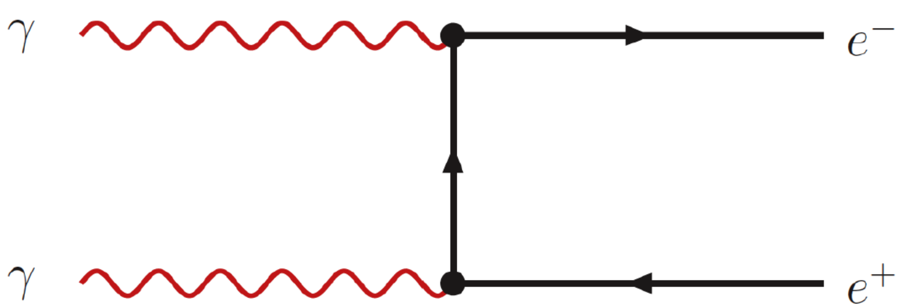
$$e^+ + e^- \longrightarrow \gamma + \gamma$$

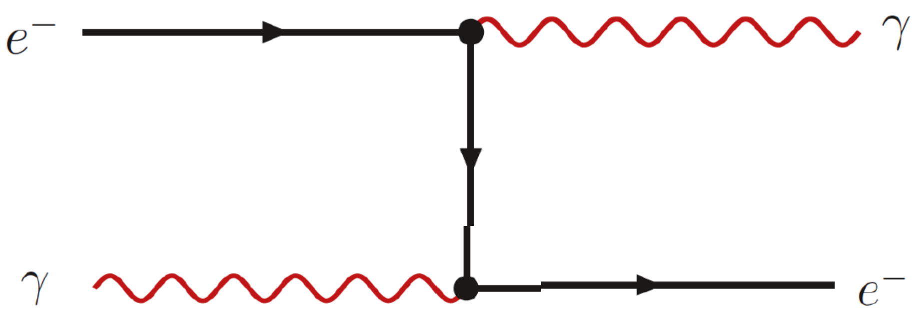
ELECTRON-POSITRON CREATION

$$\gamma + \gamma \longrightarrow e^+ + e^-$$









Dirac (1930)
Prediction of the existence of the *positron*

Compton scattering

$$e^{\pm} + \gamma \to e^{\pm} + \gamma$$

Elastic scattering between an electron/positron and a photon

More in general the cross section for
the scattering between a charged particle and a photon.
scales with the square the particle mass:

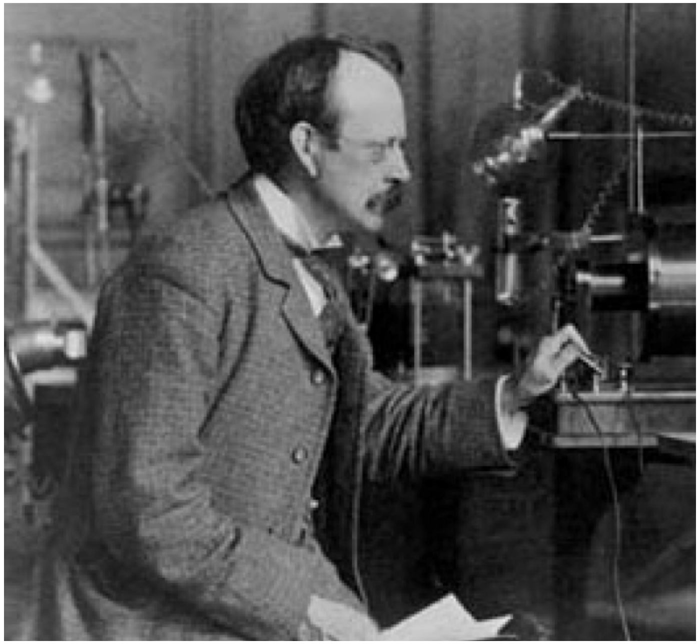
$$\sigma[a\gamma \to a\gamma] \propto \frac{q_a^4}{m_a^2}$$

Thomson cross section

for the electron-photon scattering
(limit for small photon energy)

$$e + \gamma \longrightarrow e + \gamma$$

Can be obtained as
a classical physics result



[J.J. Thomson (1897
discovery of the electron]

Plane wave of light incident
on an electron.
The electron accelerates because of the electric field,
and radiates energy:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/unit time/unit solid angle}}{\text{Incident energy flux in energy/unit area/unit time}}}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{m\,c^2}\right)^2 \sin^2\Theta$$

Linearly Polarized Light

Θ (angle with respect to the polarization vector)

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{m\,c^2}\right)^2 \frac{1}{2}(1+\cos^2\theta)$$

Unpolarized light

θ (angle with respect to photon direction)

$$\sigma_{\text{Thomson}} = \frac{8\pi}{3} \frac{e^2}{(m_e c^2)^2} = \frac{8\pi}{3} r_0^2$$

$$\sigma_{\text{Thomson}} \simeq 6.65 \times 10^{-25} \text{ cm}^2$$

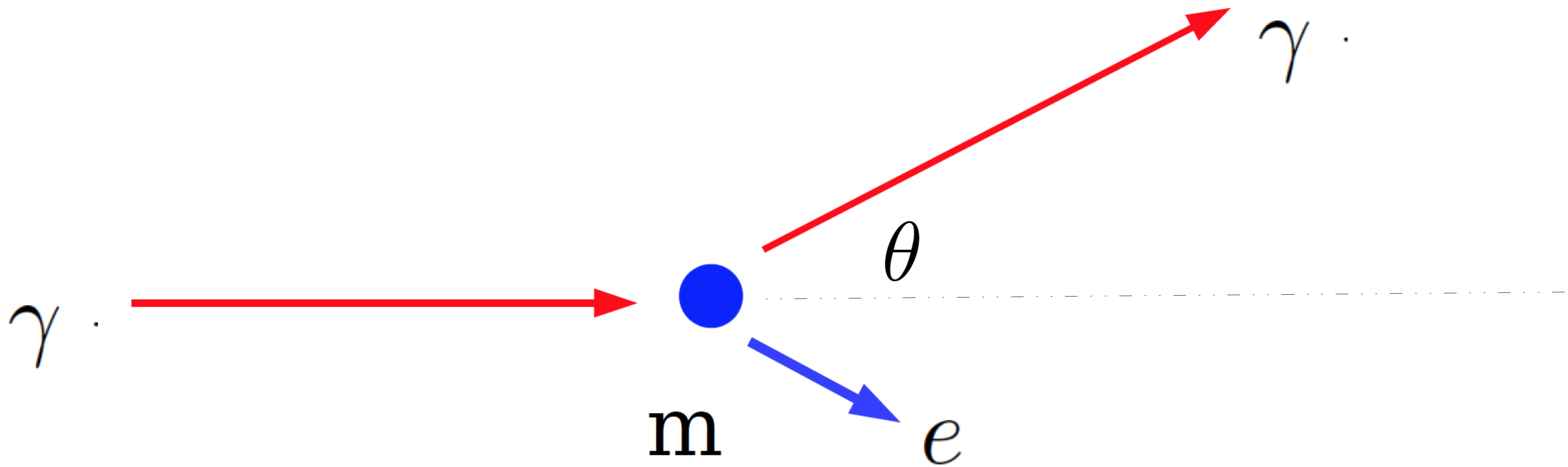
$$r_0 = \frac{e^2}{(m_e c^2)^2}$$

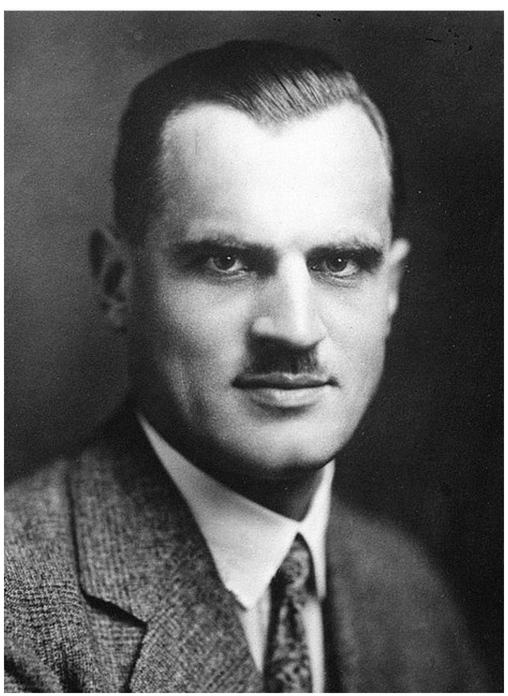
Electron "Classical Radius"

$$r_0 = \frac{\alpha}{m_e^2} \left(\frac{\hbar c}{c^4} \right)$$

$$r_0 = 2.81 \times 10^{-13} \text{ cm}$$

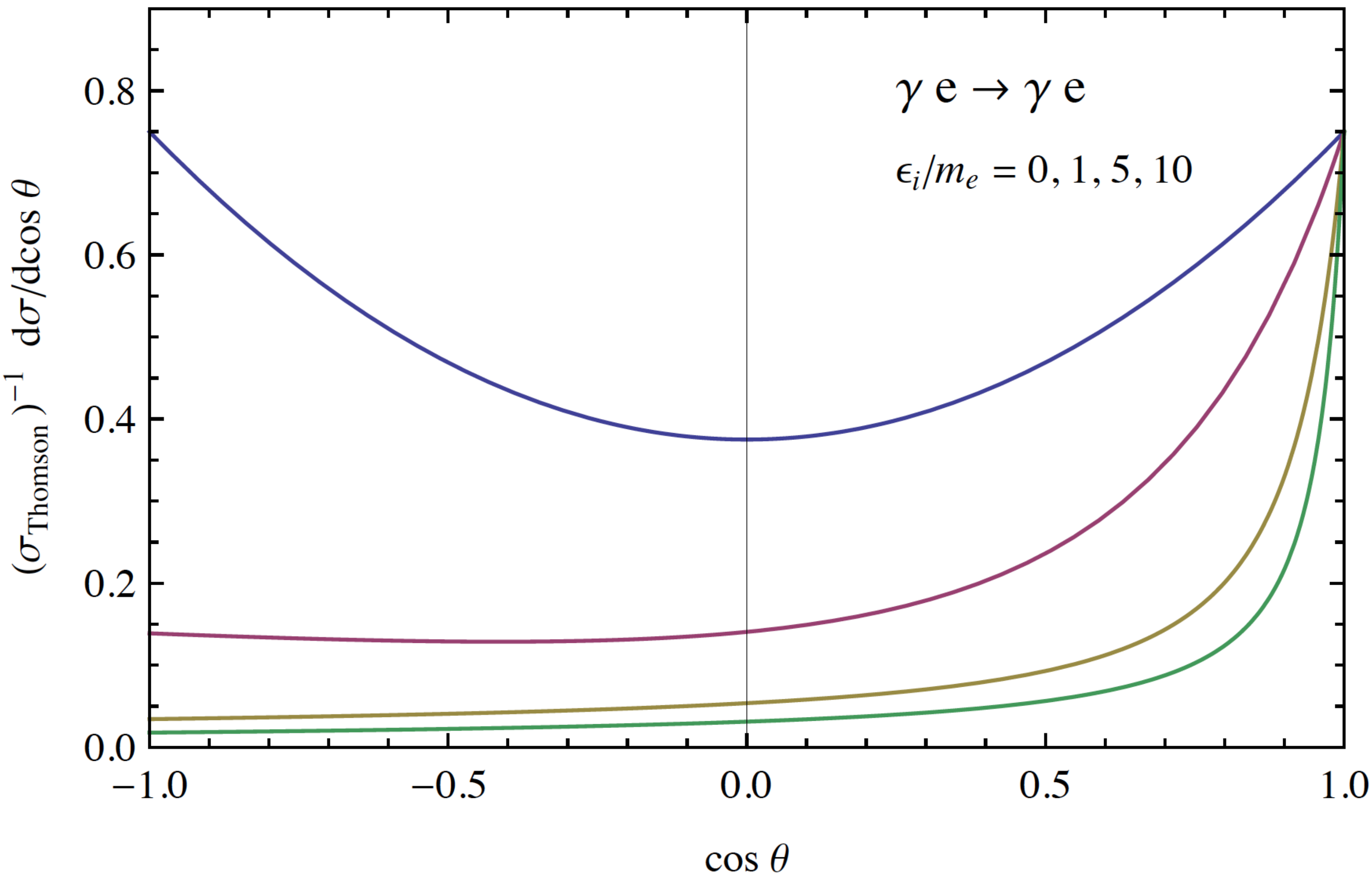
scattering in
the electron rest frame





Arthur Compton experiment (1923)
measurement of the change in wavelength
of the scattered light.

$$k_f = \frac{k_i}{1 + \frac{k_i}{m_e}\,(1 - \cos\theta)}$$

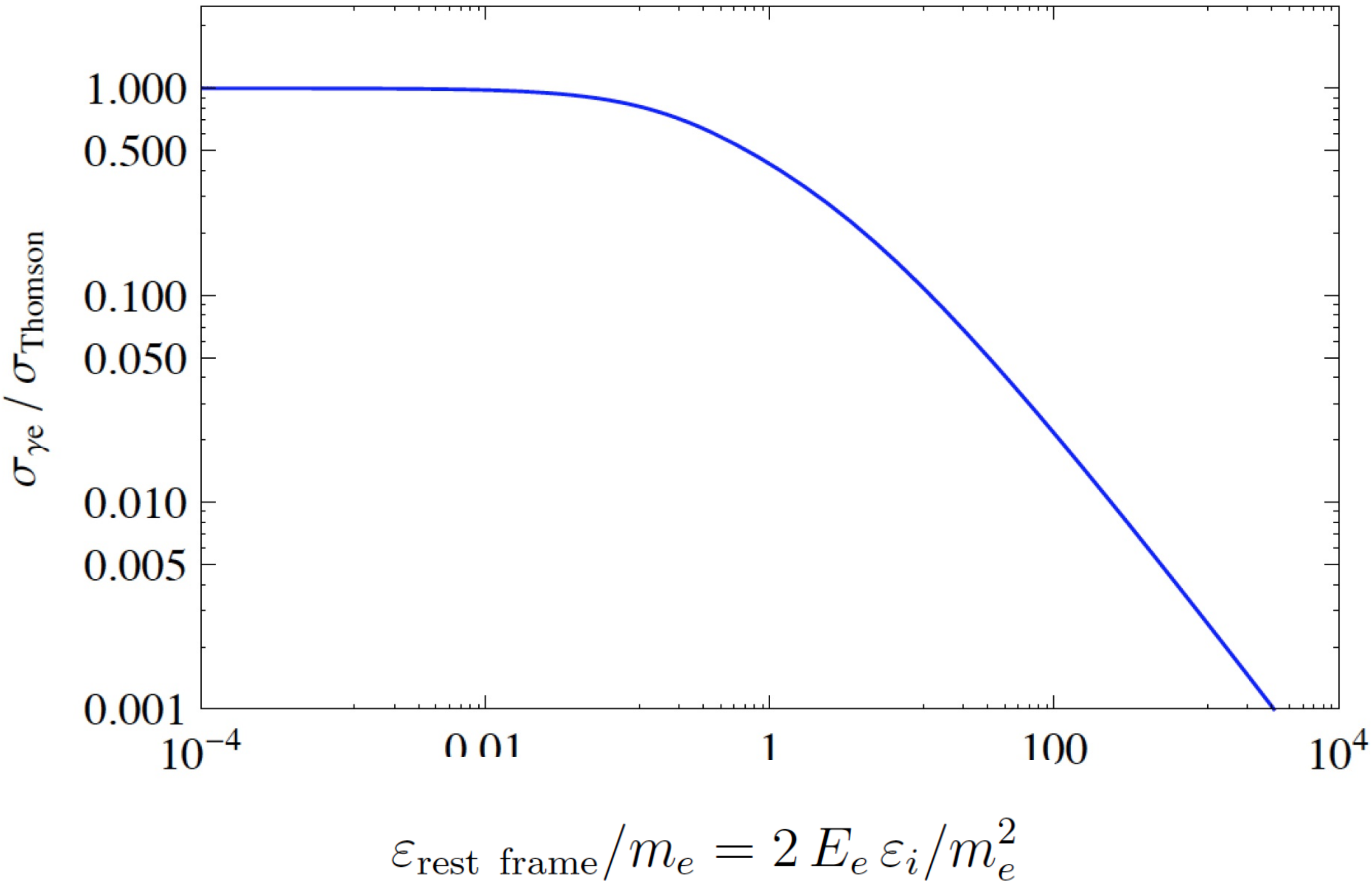


Differential cross section

$$\frac{d\sigma}{d\cos\theta}(\cos\theta;\varepsilon_i)$$

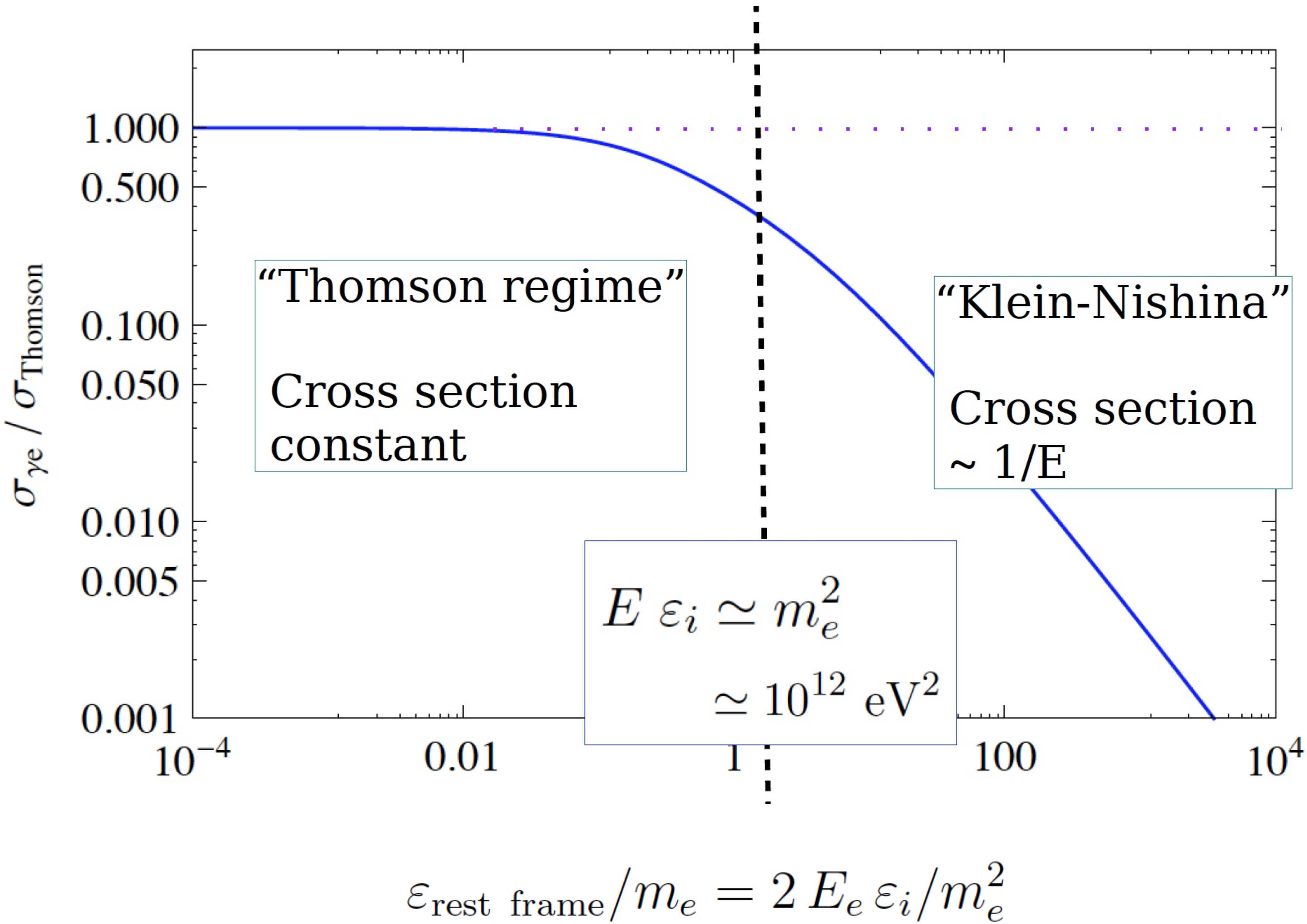
Compton Scattering

Total cross section



Compton Scattering

Total cross section



Process of pair creation

$$\boxed{\gamma + \gamma \to e^{+} + e^{-}}$$

Kinematical threshold for the reaction

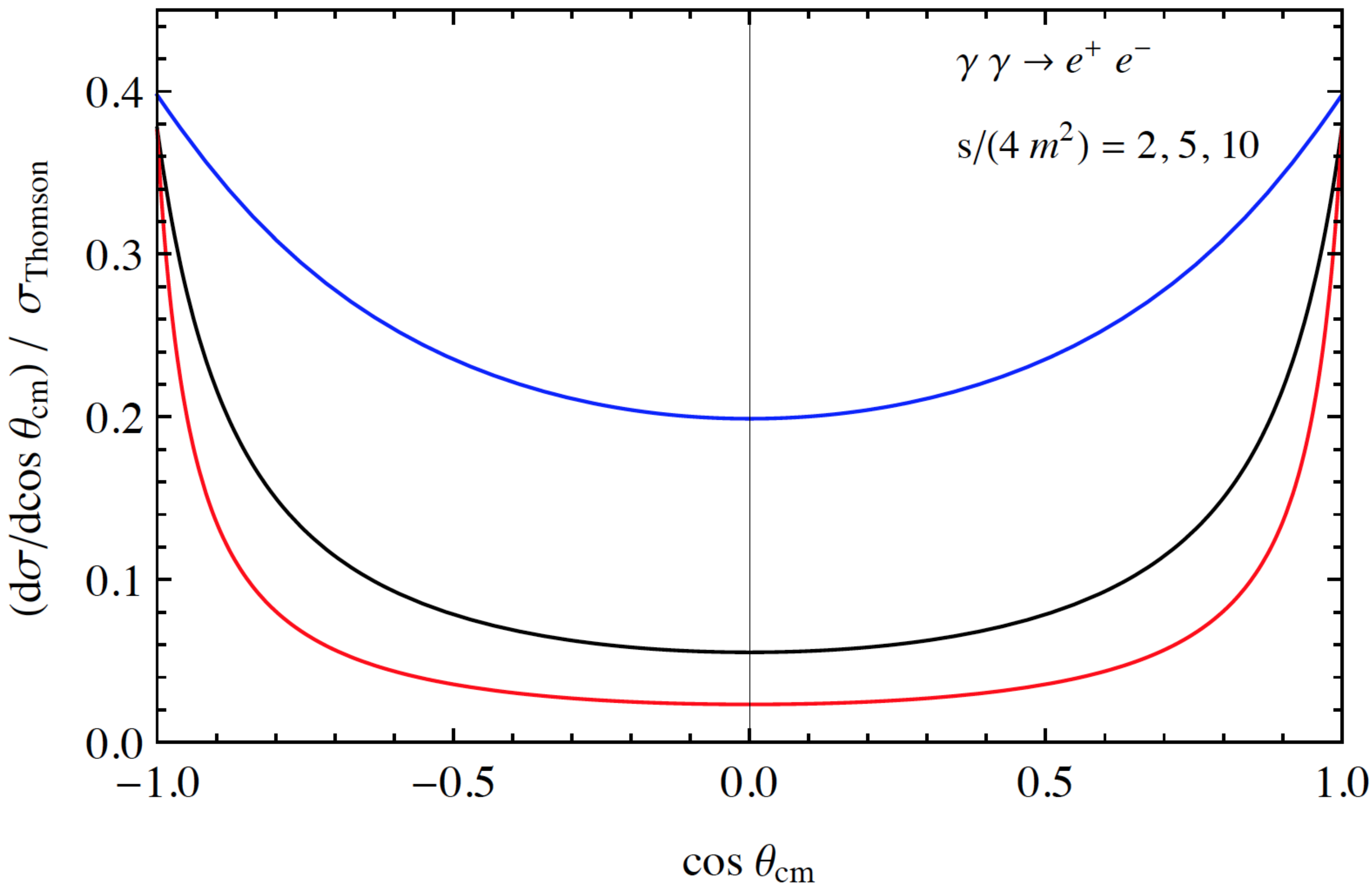
$$s = (\text{center of mass energy})^2 \geq (2m_e)^2$$

$$(k_1 + k_2)^2 \geq (2\, m_e)^2$$

$$(E_1 + E_2)^2 - |\vec{p}_1 + \vec{p}_2|^2 \geq 4m_e^2$$

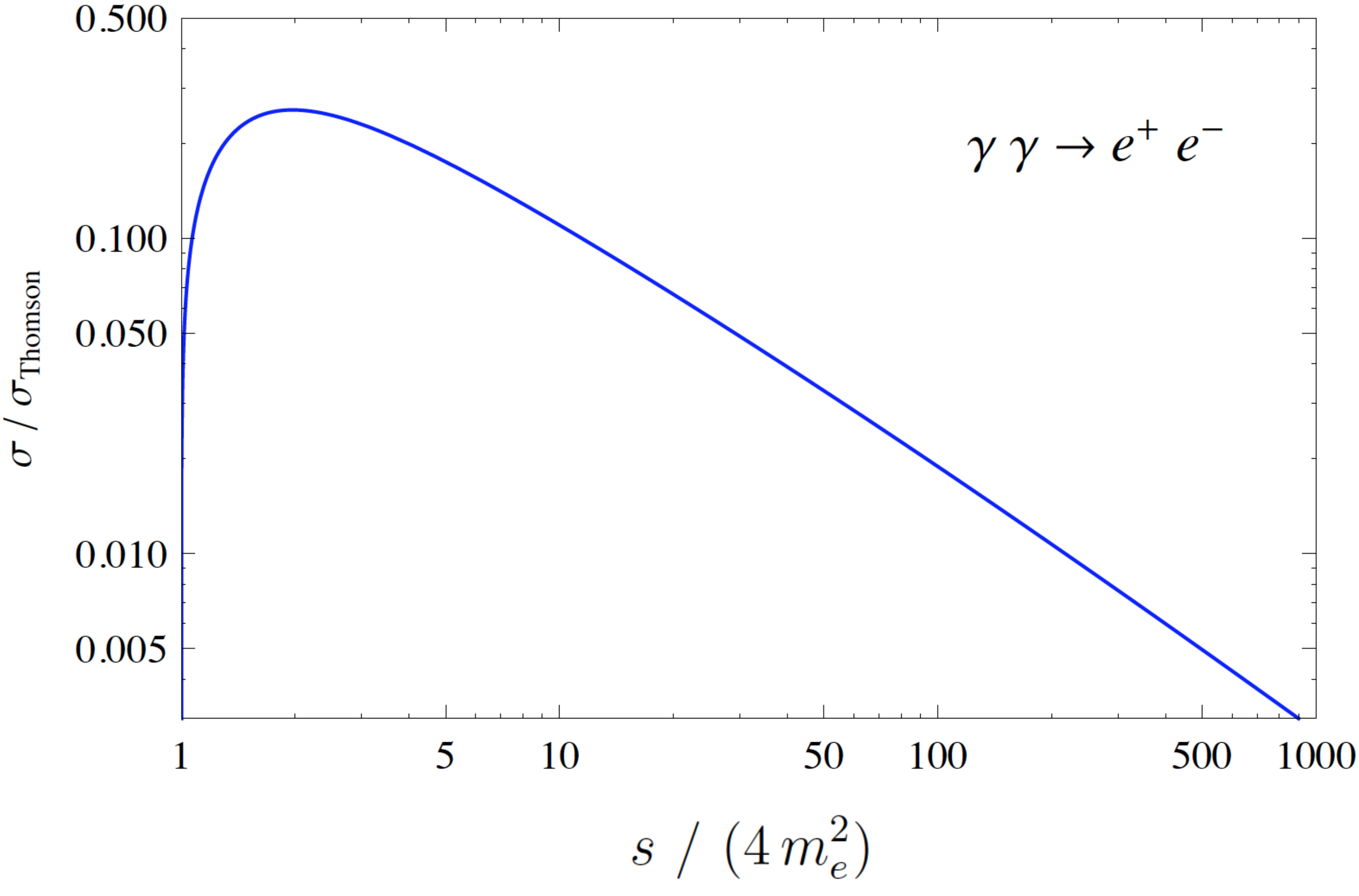
$$2\, E_1\, E_2\, (1 - \cos\theta_{12}) \geq 4\, m_e^2$$

Angular distribution in the c.m. Frame $\dfrac{d\sigma}{d\cos\theta^*}(\cos\theta^*, s)$

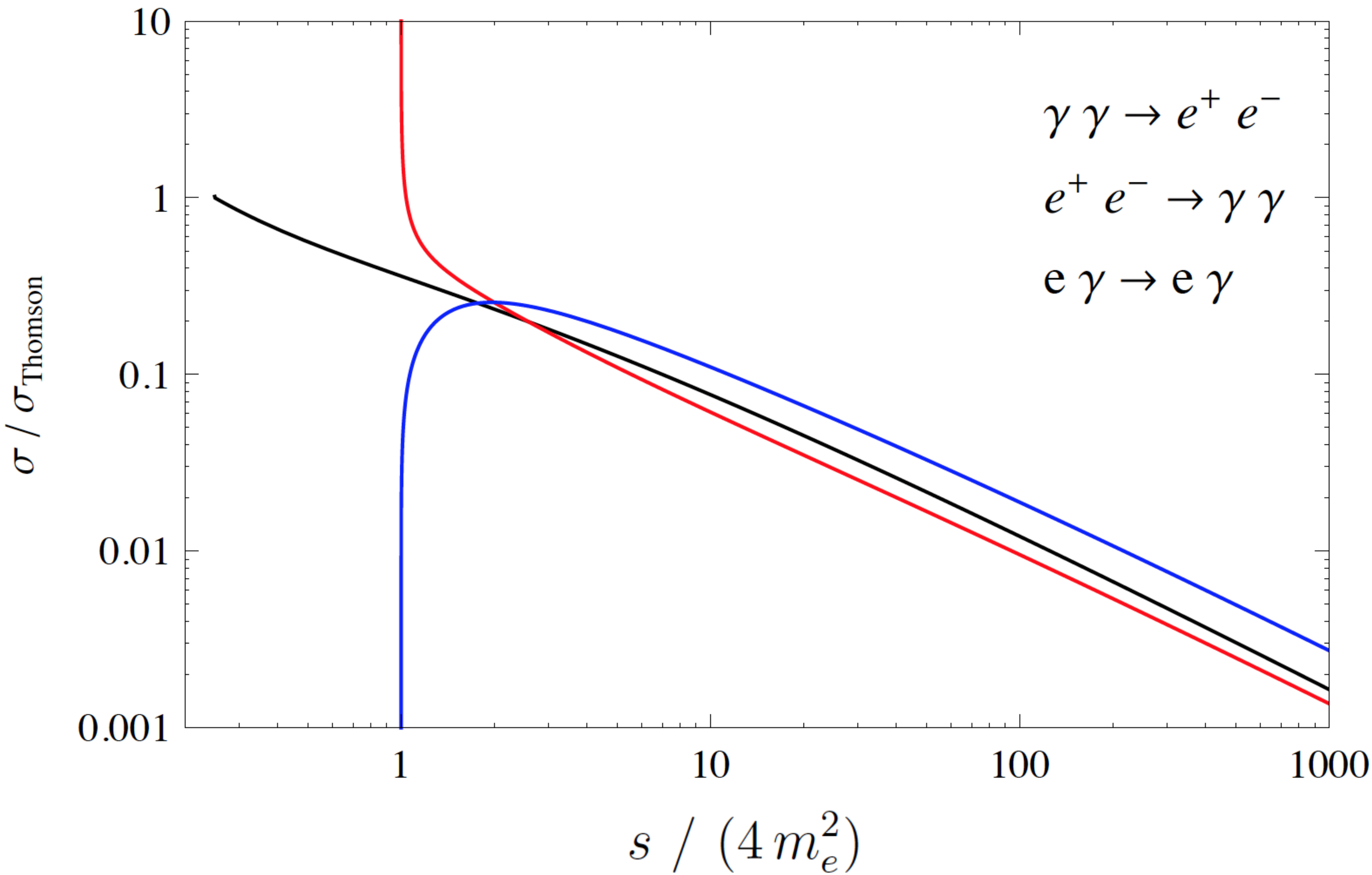


$$\boxed{\gamma + \gamma \rightarrow e^+ + e^-}$$

Total cross section



Cross section for three processes



$\boxed{\text{Bremsstrahlung}}$

$$e + Z \rightarrow e + \gamma + Z$$

$\boxed{\text{Pair Creation}}$

$$\gamma + Z \rightarrow e^{+} + e^{-} + Z$$

$\boxed{\text{Bremsstrahlung}}$

$$e + Z \rightarrow e + \gamma + Z$$

$$e + \gamma_{\text{virtual}} \rightarrow e + \gamma$$

$\boxed{\text{Pair Creation}}$

$$\gamma + Z \rightarrow e^{+} + e^{-} + Z$$

$$\gamma + \gamma_{\text{virtual}} \rightarrow e^{+} + e^{-}$$

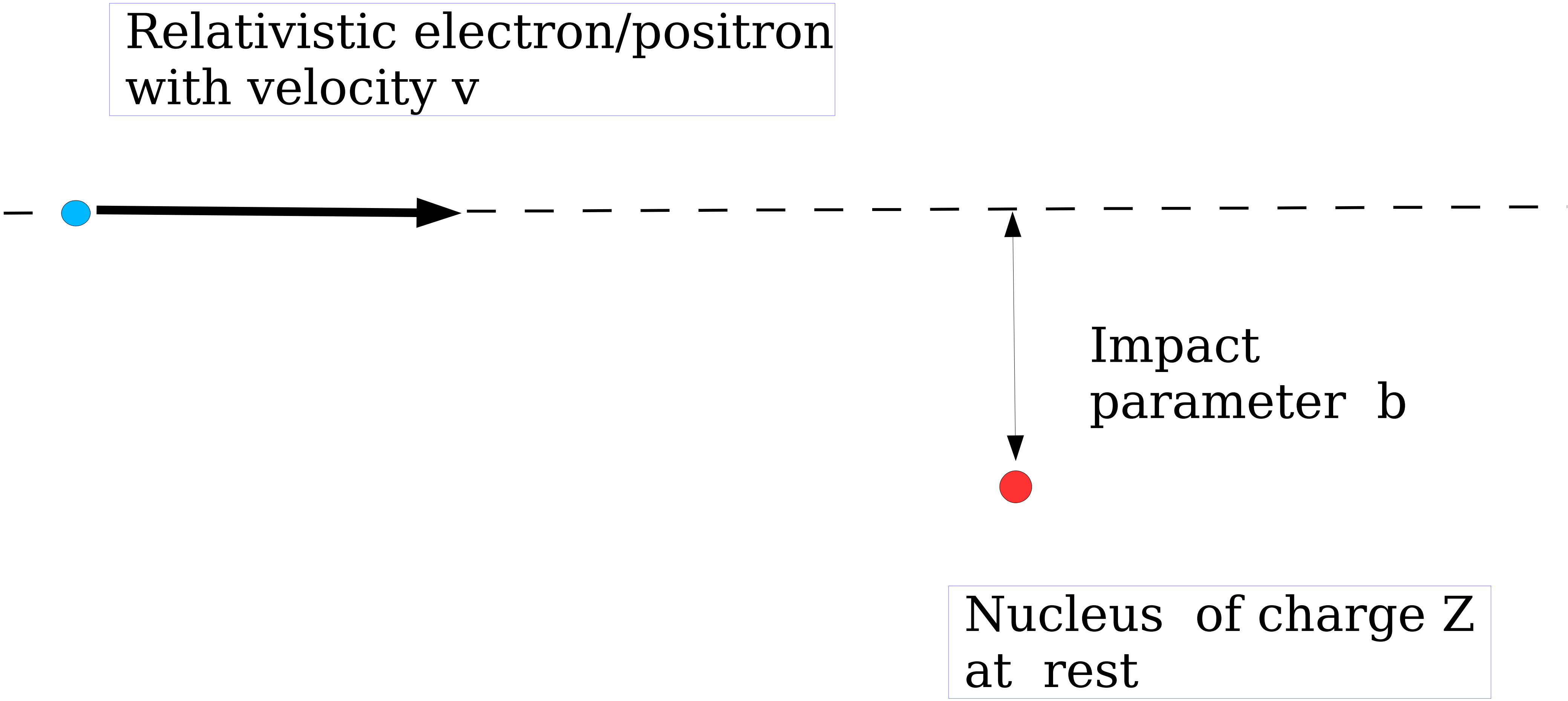
Bremsstrahlung

$$e + Z \to e + \gamma + Z$$

$$e + \gamma_{\text{virtual}} \to e + \gamma$$

Interaction of the electron with a "virtual photon"
that form the electromagnetic field of
the target nucleus.

[Chapter 15 of Jackson, "Classical electrodynamics"
Bremsstrahlung, the method of virtual quanta.]

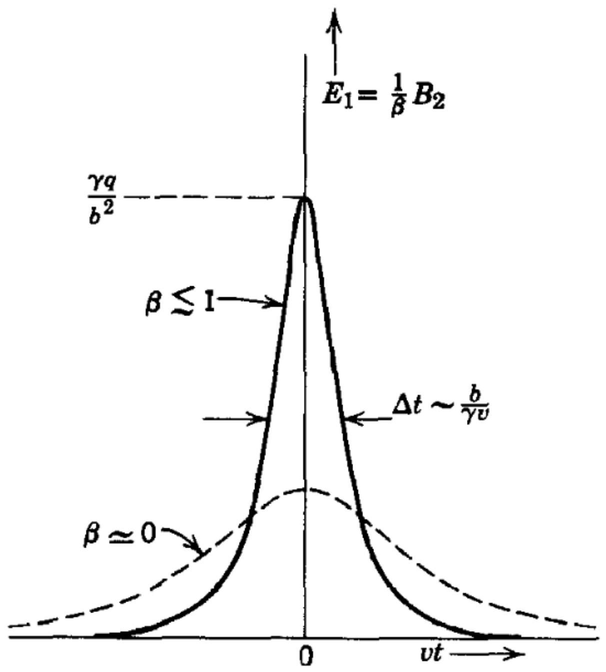


The field of the nucleus can be described as an ensemble of photons.

Obtain the varying electrical and magnetic fields in the rest frame of the electron

$$\boxed{\vec{E}(t) \quad \vec{B}(t)}$$

The observer at P, that sees a relativistic particle "zipping by" at relativistic speeed ($\beta \simeq 1, \quad \gamma \gg 1$) sees an electromagnetic field that is undistinguishable from an electromagnetic plane wave propagating along the z direction. $\vec{E}$, $\vec{B}$, $\hat{v}$ mutually perpendicular.



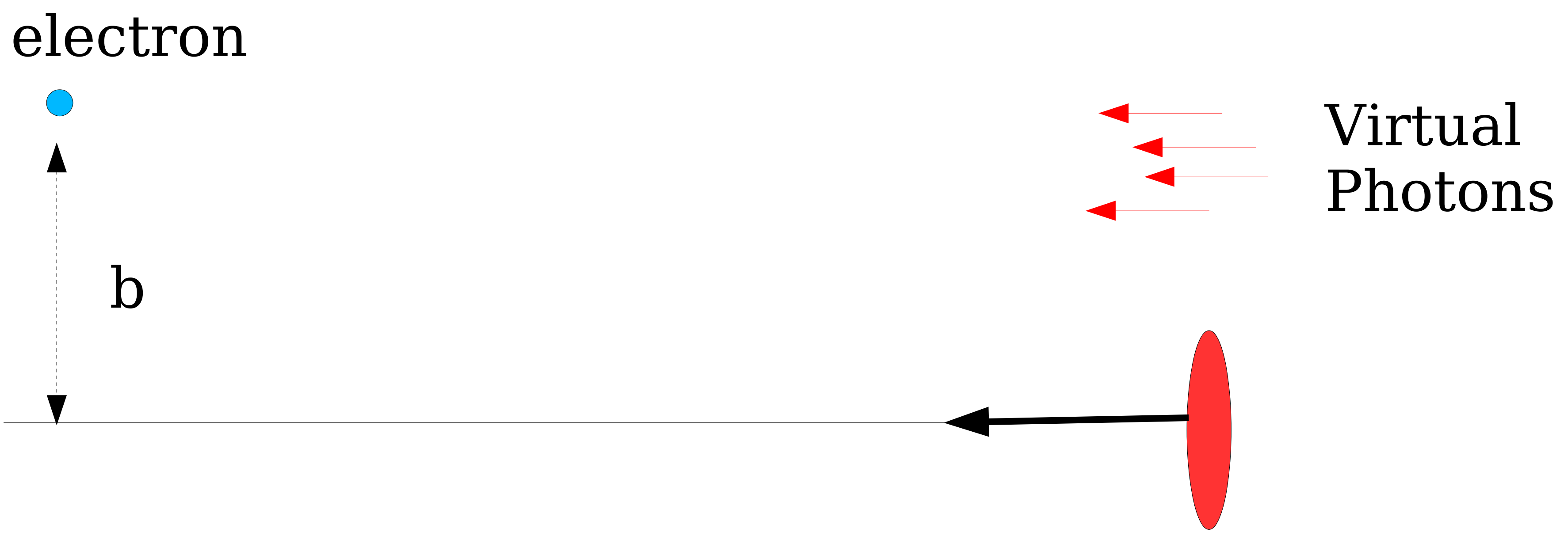
$$E_x(t) = \frac{q\, b\, \gamma}{[b^2 + (\beta\, \gamma\, t)^2]^{3/2}}$$

$$B_y(t) = \frac{q\, \beta\, \gamma\, t}{[b^2 + (\beta\, \gamma\, t)^2]^{3/2}}$$

electron/photon interaction with a nucleus.

can be seen as an interaction with
an ensemble of virtual photons

$$\varepsilon = \hbar\,\omega$$



Energy Fluence : Energy/(unit Area)

$$\int d\omega \, |E(\omega)|^2 = \int dt \, |E(t)|^2$$

$$|E(\omega)|^2 = \frac{d\mathcal{E}}{d\omega \; dA} = \frac{dN_\gamma^{\rm virtual}}{d\omega \; dA} \; (\hbar \; \omega) \quad \omega$$

$$\boxed{\frac{dN_\gamma^{\rm virtual}}{d\omega \; dA} = \frac{q^2}{\pi^2} \frac{1}{b^2} \; \frac{1}{\omega}}$$

Spectrum
extends only
up a maximum value

$$\frac{dN_\gamma^{\rm virtual}}{d\omega \; dA} \simeq 0$$

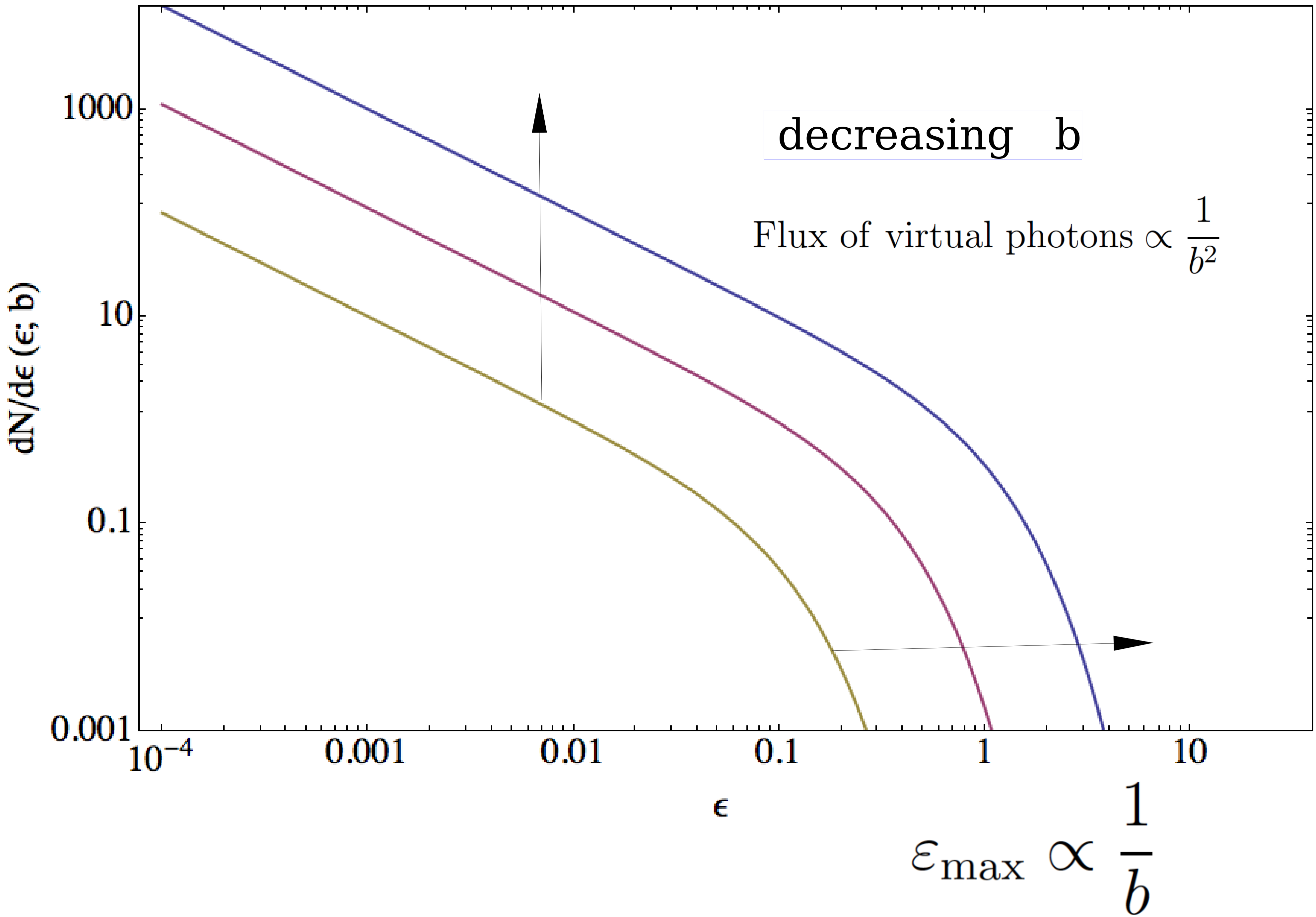
Flux of virtual photons associated with the electromagnetic field of a nucleus
(virtual photons)/(unit-area unit-energy)

$$\boxed{\frac{dN_{\gamma}^{\text{virtual}}}{dA\ d\varepsilon} = \frac{q^2}{\pi^2}\,\frac{1}{b^2}\,\frac{1}{\varepsilon}}$$

b = impact parameter

ε = energy of photon (in laboratory frame)

$$\frac{\varepsilon\, b}{\hbar\, c} \lesssim 1$$



decreasing b
Flux of virtual photons $\propto \frac{1}{b^2}$
dN/dϵ (ϵ; b)
1000
10
0.1
0.001
10^{-4}
0.001
0.01
0.1
1
10
ϵ
$\varepsilon_{\max} \propto \frac{1}{b}$

Compute the cross section for pair production:

Photon creating an electron-positron pair
in the field of a nucleus

$$\gamma + Z \to e^+ + e^- + Z$$

The process can be seen as the pair creation
of the incident photon with a virtual photon (of the nucleus field).

$$\gamma + \gamma_{\text{virtual}} \to e^+ + e^-$$

Calculation of the cross section:
integrate over impact parameter, and the virtual photon energy

$$\sigma_{\gamma\to e^+e^-}(E_\gamma) = \int d^2b \int d\varepsilon \left[\frac{dN_\gamma^{\text{virtual}}}{d^2b\ d\varepsilon}\right] \sigma_{\gamma\gamma\to e^+e^-}(\hat{s})$$

Calculation of the cross section:
integrate over impact parameter, and the virtual photon energy

$$\sigma_{\gamma\to e^+e^-}(E_\gamma) = \int d^2b \int d\varepsilon \left[\frac{dN_\gamma^{\text{virtual}}}{d^2b\ d\varepsilon}\right] \sigma_{\gamma\gamma\to e^+e^-}(\hat{s})$$

$\hat{s} =$ c.m energy of the photon-photon interaction

$$= (p_\gamma + p_{\gamma^*})^2 = 4\,E_\gamma\ \varepsilon$$

$$\int d^2b \, \frac{1}{b^2} \times [\ldots]$$

Integration
over impact parameter

$$(2\pi) \int_{b_{\min}}^{b_{\max}} db \;\; b \, \frac{1}{b^2} \times [\ldots]$$

$$(2\pi) \quad \log\left(\frac{b_{\max}}{b_{\min}}\right) \times [\ldots]$$

What are $b_{\max}$ and $b_{\min}$?

$b_{\min}$ is determined by Quantum mechanics and the indetermination principle:

$$\Delta x \, \Delta p \lesssim \hbar$$

$$b \, (m_e \, c) \lesssim \hbar$$

$$\boxed{b_{\min} = \frac{\hbar}{m_e \, c}}$$

What are $b_{\max}$ and $b_{\min}$?

$b_{\max}$

For a fully ionized nucleus depends on the energy of the virtual photon ε

$$\frac{\varepsilon\, b}{\hbar\, c} \leq 1 \qquad\qquad b_{\max} \simeq \frac{\hbar\, c}{\varepsilon}$$

For a nucleus in an atom must also be smaller that the size of the atom

$$b_{\max} \leq R_{\mathrm{atom}}$$

$$R_{\mathrm{hydrogen}} \simeq a_{\mathrm{Bohr}} = \frac{\hbar^2}{m_e\, e^2}$$

$$R_{\mathrm{atom}} \approx \frac{\hbar}{m_e\, c}\; 183\; Z^{-1/3}$$ Atom of Thomas-Fermi

$$b_{\max} = \min\left[\frac{\hbar c}{\varepsilon}, R_{\mathrm{atom}}\right]$$

"Full screening approximation"

$$\ln\left(\frac{b_{\max}}{b_{\min}}\right) \simeq \ln 183\, Z^{-1/3}$$

Total Pair Production Cross section

$$\sigma_{\gamma \to e^+ e^-}(K) = \int d^2 b \int d\varepsilon \; \boxed{\frac{dN_\gamma^{\text{virtual}}}{dA\; d\omega}} \; \sigma_{\gamma\gamma \to e^+ e^-}(\hat{s})$$

$$\boxed{\hat{s} = 4\, K\, \varepsilon}$$

$$\boxed{\frac{dN_\gamma^{\text{virtual}}}{dA\; d\varepsilon} = \frac{q^2}{\pi^2}\, \frac{1}{b^2}\, \frac{1}{\varepsilon}}$$

$$\sigma_{\gamma \to e^+ e^-}(K) = \frac{\alpha\; Z^2}{\pi^2} \left[(2\,\pi) \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \right]$$

$$\int_{4\, m^2/K}^{\infty} \frac{d\varepsilon}{\varepsilon}\; \sigma_{\gamma\gamma}(4\, K\, \varepsilon)$$

$$\frac{2\,\alpha\;Z^2}{\pi}\;\log\left[\frac{b_{\max}}{b_{\min}}\right]\;\int_{4\,m^2}^{\infty}\frac{ds}{s}\;\sigma_{\gamma\gamma}(s)$$

$$\boxed{\int_{4\,m^2}^{\infty}\frac{ds}{s}\sigma_{\gamma\gamma\to e^+e^-}(s)=(2\pi\;r_0^2)\;\frac{7}{9}}$$

$$\sigma_{\gamma\to e^+e^-}(K)=\frac{2\,\alpha\;Z^2}{\pi}\;\log\left[\frac{b_{\max}}{b_{\min}}\right]\quad(2\pi\;r_0^2)\;\frac{7}{9}$$

Cross section is *constant*

Cross section for pair production for an average nucleus in air

$$\sigma^{\mathrm{air}}_{\gamma \to e^+ e^-} \simeq 5.0 \times 10^{-25}\ \mathrm{cm}^2$$

Interaction length of a photon in air

$$\lambda_{\mathrm{pair}} \simeq \frac{\langle m \rangle}{\sigma^{\mathrm{air}}_{\gamma \to e^+ e^-}} \simeq 47\ \frac{\mathrm{g}}{\mathrm{cm}^{-2}}$$

$$N_\gamma(X) = N(0)\ e^{-X/\lambda_{\mathrm{pair}}}$$

Attenuation of
a photon beam
(X = column density)

$$\sigma[e \to e\gamma] \to \infty$$

Bremsstrahlung cross section

$$\left.\frac{d\sigma}{dE_\gamma}\right|_{e\to e\gamma} (E_\gamma, E_e) = \text{finite}$$

diverges only for $E_\gamma \to 0$

$$\propto E_\gamma^{-1}$$

$$\left.\frac{d\sigma}{dv}\right|_{e\to e\gamma} (v) = \text{finite} \qquad v = \frac{E_\gamma}{E_v}$$

Independent from the electron energy !

$$\frac{d\sigma_{\text{brems}}}{dv}(v, E_e) = \int_{\varepsilon_{\min}(v,E_e)}^{\infty} d\varepsilon\ [\ldots] \qquad v = \frac{E_\gamma}{E_e}$$

$$\frac{d\sigma_{\text{pair}}}{du}(u, E_\gamma) = \int_{\varepsilon_{\min}(u,E_e)}^{\infty} d\varepsilon\ [\ldots] \qquad u = \frac{E_{e^+}}{E_\gamma}$$

$$\boxed{\begin{aligned} \varepsilon_{\min}(v, E_e) &= \frac{m^2}{4\,E_e}\,\frac{v}{1-v} \\ \varepsilon_{\min}(u, E_\gamma) &= \frac{m^2}{4\,E_e\,u(1-u)} \end{aligned}}$$

$$\left.\frac{dE}{dX}\right|_{\mathrm{brems}} = \frac{N_A}{A} \int dE_\gamma \; E_\gamma \; \frac{d\sigma}{dE_\gamma}(E_\gamma, E_e)$$

$$\frac{dE}{dX} = \left\{ \frac{N_A}{A} \int_0^1 dv \; v \; \frac{d\sigma}{dv}(v) \right\} E = \frac{E}{\lambda_{\mathrm{rad}}}$$

$$\lambda_{\mathrm{rad}} = \frac{N_A}{A} \, 4\,\alpha\, r_0^2 \; \left\{ \ln[183 \; Z^{-1/3}] + \frac{1}{18} \right\}$$

$$\boxed{\lambda_{\mathrm{rad}} = \frac{N_A}{A} \, 4\,\alpha\, r_0^2 \; \ln[183 \; Z^{-1/3}]}$$

Radiation Length

Physical Meaning of the "Radiation Length"

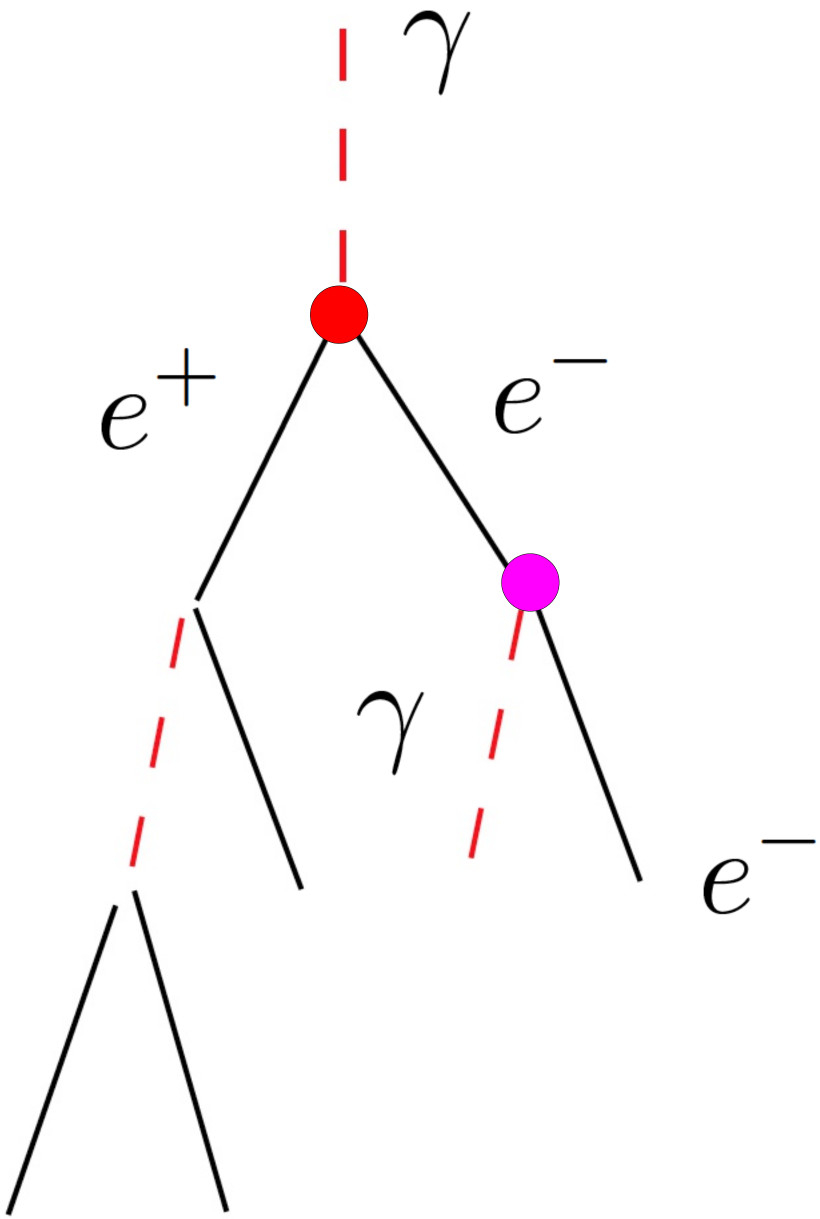
$$\langle E_e(X) \rangle = E_e(0)\; e^{-X/\lambda_{\rm rad}}$$

Physical Meaning of the "Pair Length"

$$N_\gamma(X) = N(0)\; e^{-X/\lambda_{\rm pair}}$$

$$\lambda_{\rm rad}^{\rm air} \simeq 37\; \frac{\rm g}{\rm cm^2}$$

$$\lambda_{\rm pair} \simeq \frac{9}{7}\; \lambda_{\rm rad}$$



$\psi(u)$

Pair production

$\varphi(v)$

Bremsstrahlung

BREMSSTRAHLUNG

$$v = \frac{E_\gamma}{E_e}$$

Fully ionized free nucleus (approximation of target infinite mass)

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{e\to e+\gamma}(v;E) = 4\,Z^2\,\alpha\,r0^2$$

$$\frac{1}{v}\;\left[1+(1-v^2)-\frac{2}{3}(1-v)\right]\;\left[\ln\left(\frac{2\,E}{m}\,\frac{v}{1-v}\right)-\frac{1}{2}\right]$$

High Energy Limit (Full screening)

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{e\to e+\gamma}(v;E) = 4\,Z^2\,\alpha\,r0^2$$

$$\frac{1}{v}\;\left\{\left[1+(1-v^2)-\frac{2}{3}(1-v)\right]\;\ln\left(183\,Z^{-1/3}\right)+\frac{1}{9}(1-v)\right\}$$

PAIR PRODUCTION

$$u = \frac{E_{e^+}}{E_\gamma}$$

Fully ionized free nucleus (approximation of target infinite mass)

$$\left.\frac{d\sigma}{du}\right|_{\gamma\to e^+e^-}(u;K) = 4\ Z^2\ \alpha\ r0^2$$

$$\left[u^2 + (1-u)^2 + \frac{2}{3}u\,(1-v)\right]\ \left[\ln\left(\frac{2\ K}{m}\,u(1-u)\right) - \frac{1}{2}\right]$$

High Energy Limit (Full screening)

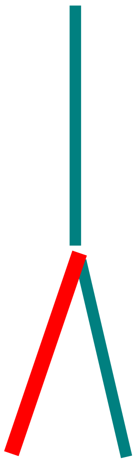
$$\left.\frac{d\sigma}{du}\right|_{\gamma\to e^+e^-}(u;K) = 4\ Z^2\ \alpha\ r0^2$$

$$\left\{\ \left[u^2 + (1-u)^2 + \frac{2}{3}u\,(1-v)\right]\ \ln\left(183\,Z^{-1/3}\right) - \frac{1}{9}u(1-u)\right\}$$

The “SPLITTING FUNCTIONS”

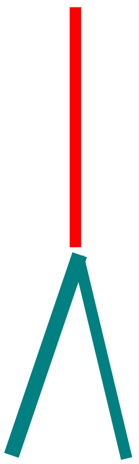
$$\varphi(v) = \left[\frac{d\sigma}{dv}(v)\right]_{\mathrm{brems}} \left(\frac{N_A}{A}\,\lambda_{\mathrm{rad}}\right)$$

$$\varphi(v) = \frac{1}{v}\,\left[1 - \left(\frac{2}{3} - 2\,b\right)\,(1-v) + (1-v)^2\right]$$



$$\psi(u) = \left[\frac{d\sigma}{du}(u)\right]_{\mathrm{pair}} \left(\frac{N_A}{A}\,\lambda_{\mathrm{rad}}\right)$$

$$\psi(u) = (1-u)^2 + \left(\frac{2}{3} - 2\,b\right)\,(1-u)\,u + u^2$$



Introduce a correction parameter b
[nothing to do with the impact parameter]

$$b \simeq \frac{1}{18 \log(183\, Z^{-1/3})}$$

$$b \simeq 0.0135$$

(for air)

$$\sigma_0 = \int_0^1 du\ \psi(u) = \frac{7}{9} - \frac{b}{3}$$

$$\int_0^1 dv\ v\ \varphi(v) = 1 + b$$

$$\varphi(v) \quad dv\, dt$$

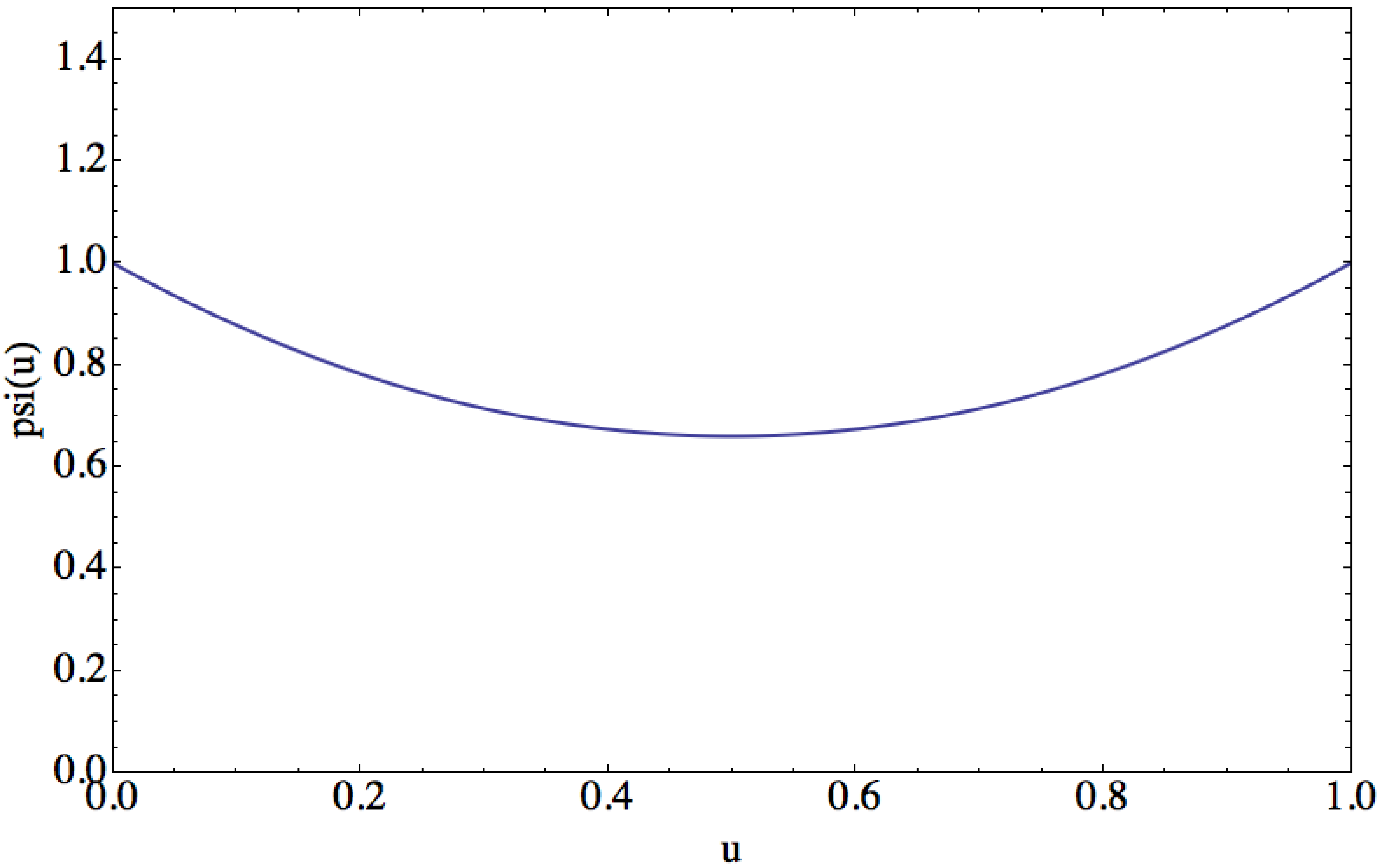
Probability for a photon of (any) energy E_γ
to generate one positron with fractional energy
$u = E_e/E_\gamma$ in the interval $[u, u + du]$
when traversing a layer of material of thickness dt

$$\psi(u) \quad dv\, dt$$

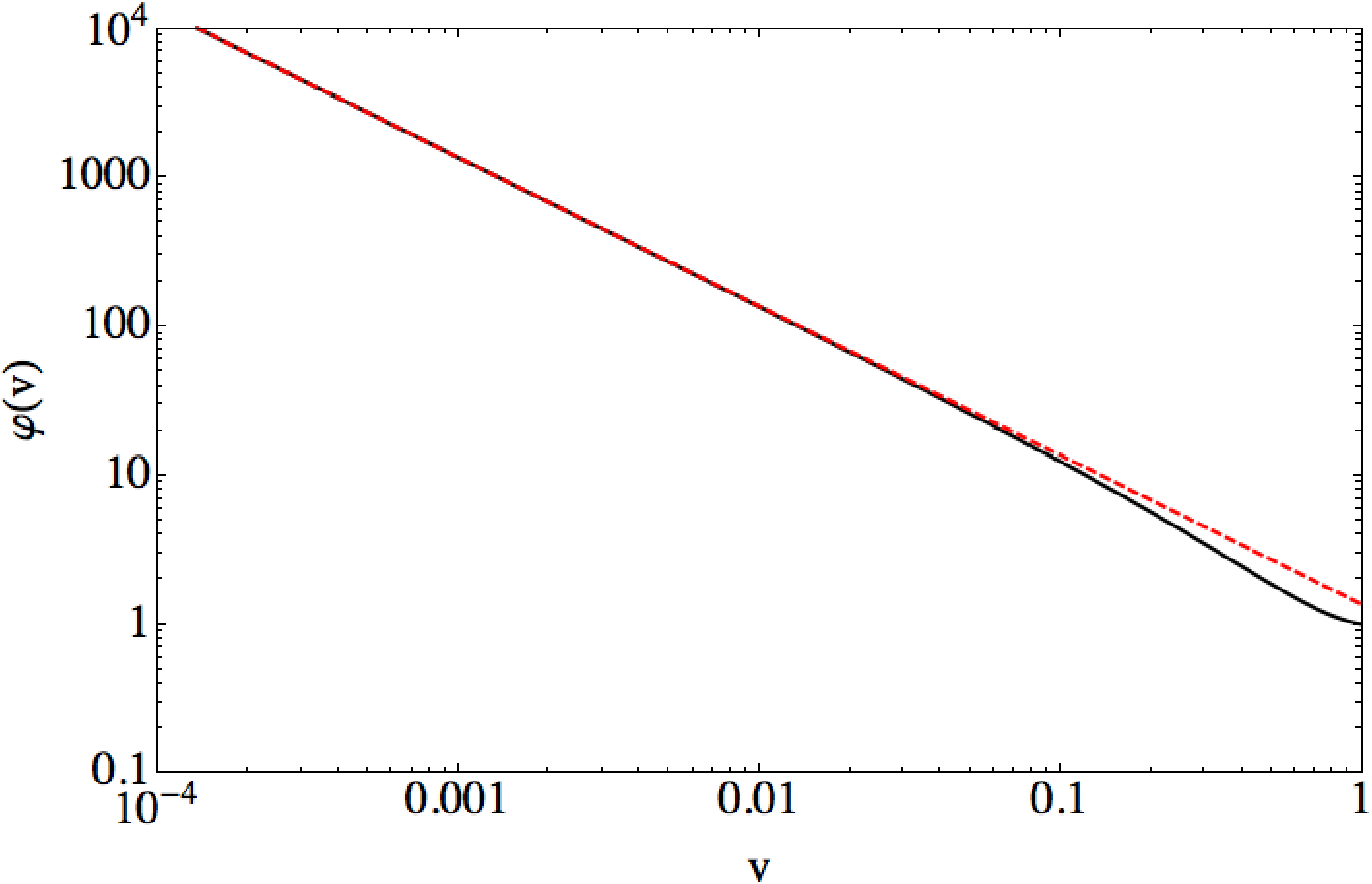
Probability for an electron of (any) energy E_e
to generate one photon with fractional energy
$v = E_\gamma/E_e$ in the interval $[v, v + dv]$,
when traversing a layer of material of thickness dt

$\psi(u)$

Pair Production

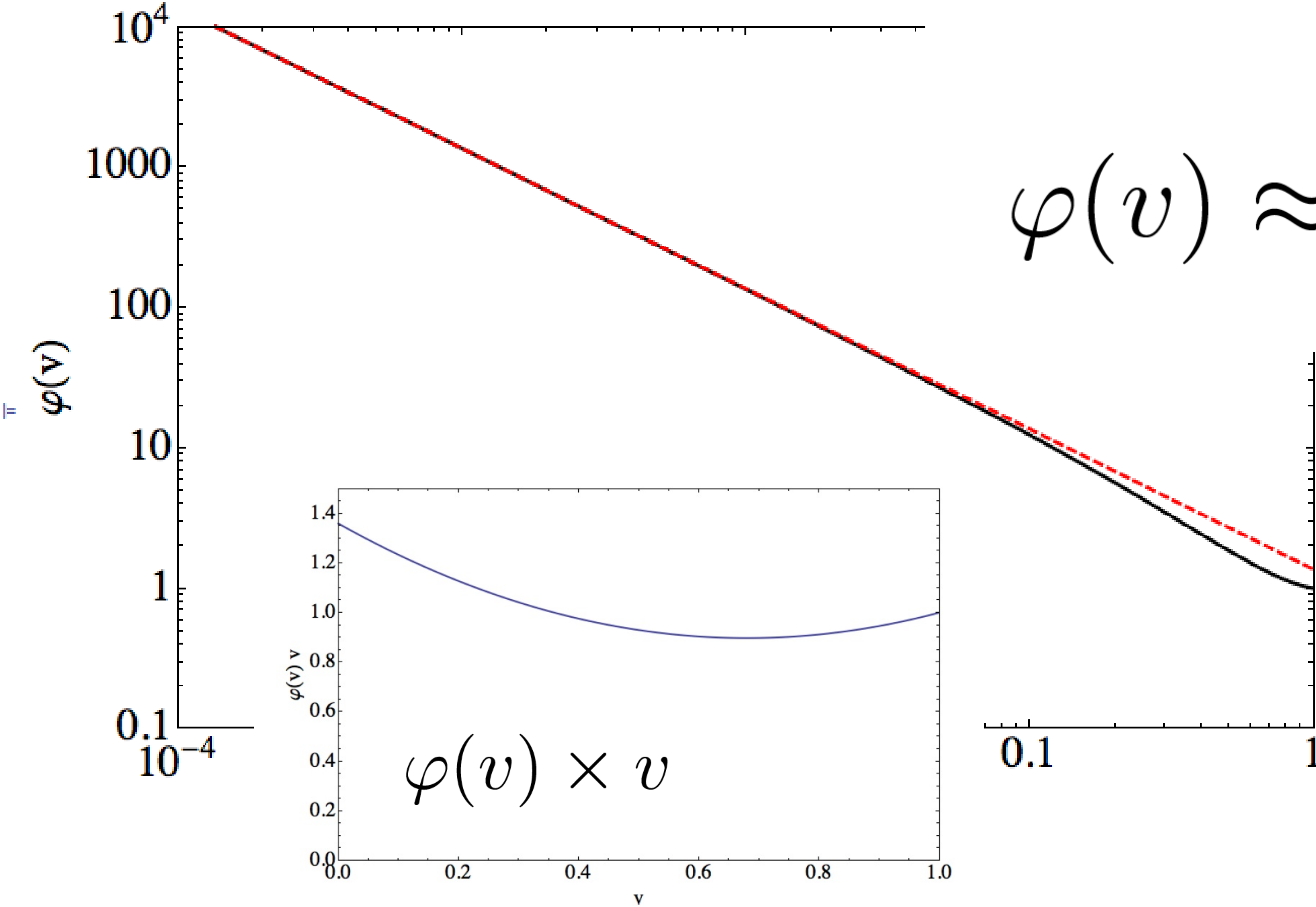


$\varphi(v)$ Bremsstrahlung



$$\varphi(v)$$

Bremsstrahlung



$$\varphi(v) \approx \frac{1}{v}$$

$$\varphi(v) \times v$$

Ionization losses

Coulomb scattering of shower relativistic electrons with the electrons [and the nuclei] of the medium.

The collisions generate (for the projectile particle):

[a] an *energy loss*
[b] and an *angular deviation*

First order approximation: describe the scattering with the “Rutherford formula”

$$\left.\frac{d\sigma}{d\Omega}\right|^{\mathrm{Rutherford}} = \left(\frac{q_1\, q_2}{Q^2}\right)^2 4m_e^2$$

$$Q^2 = |\vec{p}_i - \vec{p}_f|^2 \qquad\qquad 0 \leq Q^2 \lesssim (2m\,v)^2$$

$$Q^2 = -(p - p')^2$$

$$= -(E - E')^2 + (\vec{p} - \vec{p'})^2$$

$$Q^2 \simeq 2p^2(1 - \cos\theta) \simeq 2\, m_e\, T$$

$$d\Omega = 2\pi\, d\cos\theta$$

T = kinetic energy
given to the target
particle initially at rest

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|^{\text{Rutherford}} = \left(\frac{q_1\, q_2}{Q^2}\right)^2 4m_e^2 \qquad Q^2 = |\vec{p}_i - \vec{p}_f|^2 \qquad 0 \le Q^2 \lesssim (2m\, v)^2}$$

$$Q^2 = -(p - p')^2$$
$$= -(E - E')^2 + (\vec{p} - \vec{p'})^2$$

$$\frac{d\sigma}{d\theta} \propto \frac{1}{\theta^4}$$

$$Q^2 \simeq 2p^2(1 - \cos\theta) \simeq 2\, m_e\, T$$

$$\frac{d\sigma}{dT} \propto \frac{1}{T^2}$$

$$d\Omega = 2\pi\, d\cos\theta$$

$$\left.\frac{d\sigma}{dT}\right|_{\text{collision}} = 2\pi\,\frac{e^4}{m_e\,c^2\,\beta^2\,T^2}\,\left(1-\beta^2\frac{T}{T_{\max}}\right)$$

Rate of energy loss per unit length

$$\left.\frac{dE}{d\ell}\right|_{\text{collisions}} = n\;Z\;\int dT\;T\;\frac{d\sigma}{dT}$$

n = number density of target atoms

Z = number of electrons/atom

Energy loss per unit of column-density X

$$dX = d\ell\ \rho = d\ell\ n\ m_{\text{nucleon}}\, A = d\ell\ n\ \frac{A}{N_{\text{Avogadro}}}$$

$$\left.\frac{dE}{dX}\right|_{\text{collisions}} = N_{\text{Avogadro}}\,\frac{Z}{A}\,2\pi\frac{e^4}{m_e\,c^2}\,\frac{1}{\beta^2}\,\int_{T_{\min}}^{T_{\max}} dT\;T\;\frac{1}{T^2}\left[1-\beta^2\,\frac{T}{T_{\max}}\right]$$

$$\left.\frac{dE}{dX}\right|_{\text{collisions}} = N_{\text{Avogadro}}\,\frac{Z}{A}\,2\pi\frac{e^4}{m_e\,c^2}\frac{1}{\beta^2}\,\left[\ln\left(\frac{T_{\max}}{T_{\min}}\right)-\beta^2\right]$$

$$T_{\max} \simeq m_e\,c^2\,\beta\gamma$$

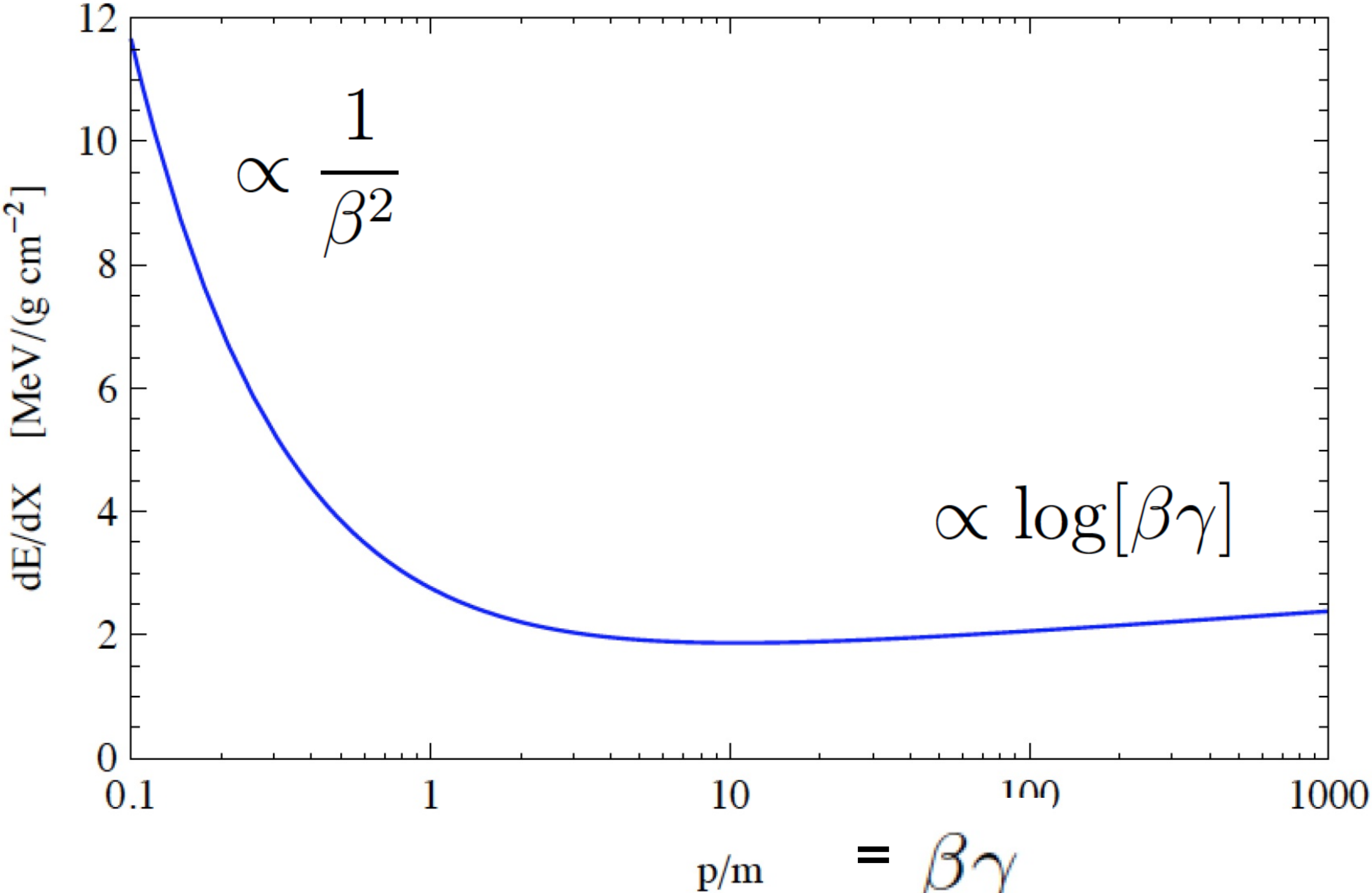
Kinematical limit
(for e-e scattering)

$$T_{\min} \simeq \langle I_{\text{ionization}}\rangle$$

Minimum energy transfer
to an atomic electron

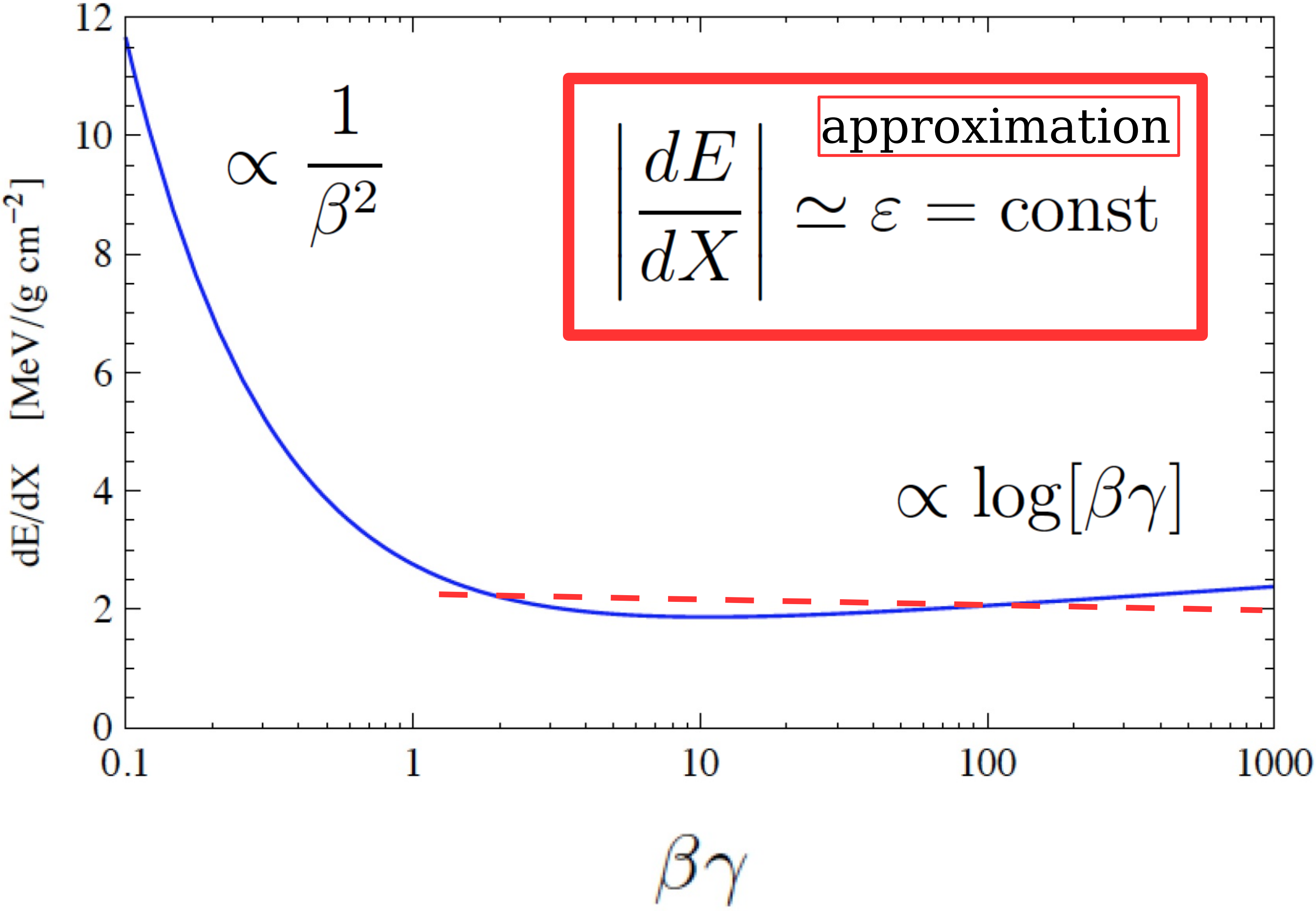
$$\left.\frac{dE}{dX}\right|_{\text{collisions}} \simeq N_{\text{Avogadro}}\,\frac{Z}{A}\,2\pi\frac{e^4}{m_e\,c^2}\frac{1}{\beta^2}\,\left[\ln\left(\frac{2m_e\,c^2\,\beta\gamma}{\langle I\rangle}\right)-\beta^2\right]$$

Rate of energy loss as a function of $\beta\gamma$



$= \beta\gamma$

Rate of energy loss
for collisions (ionization)



We have now constructed the
“fundamental elements”
for an electromagnetic shower

The two “vertices”
Bremsstrahlung
Pair production

Energy loss of electrons

Study Development of the Showers

Average Longitudinal Development of a purely electromagnetic shower

$$n_e(E, t)$$

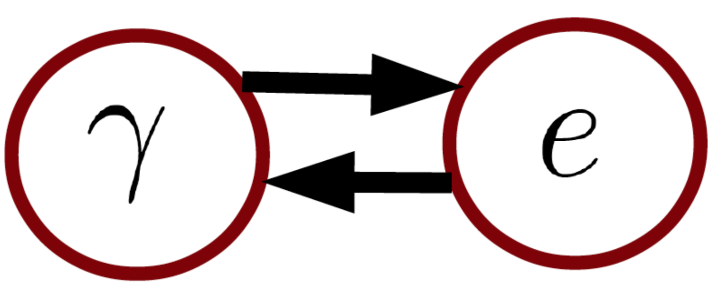
$$n_\gamma(E, t)$$

Spectra of

[*] electrons/positrons
[*] photons

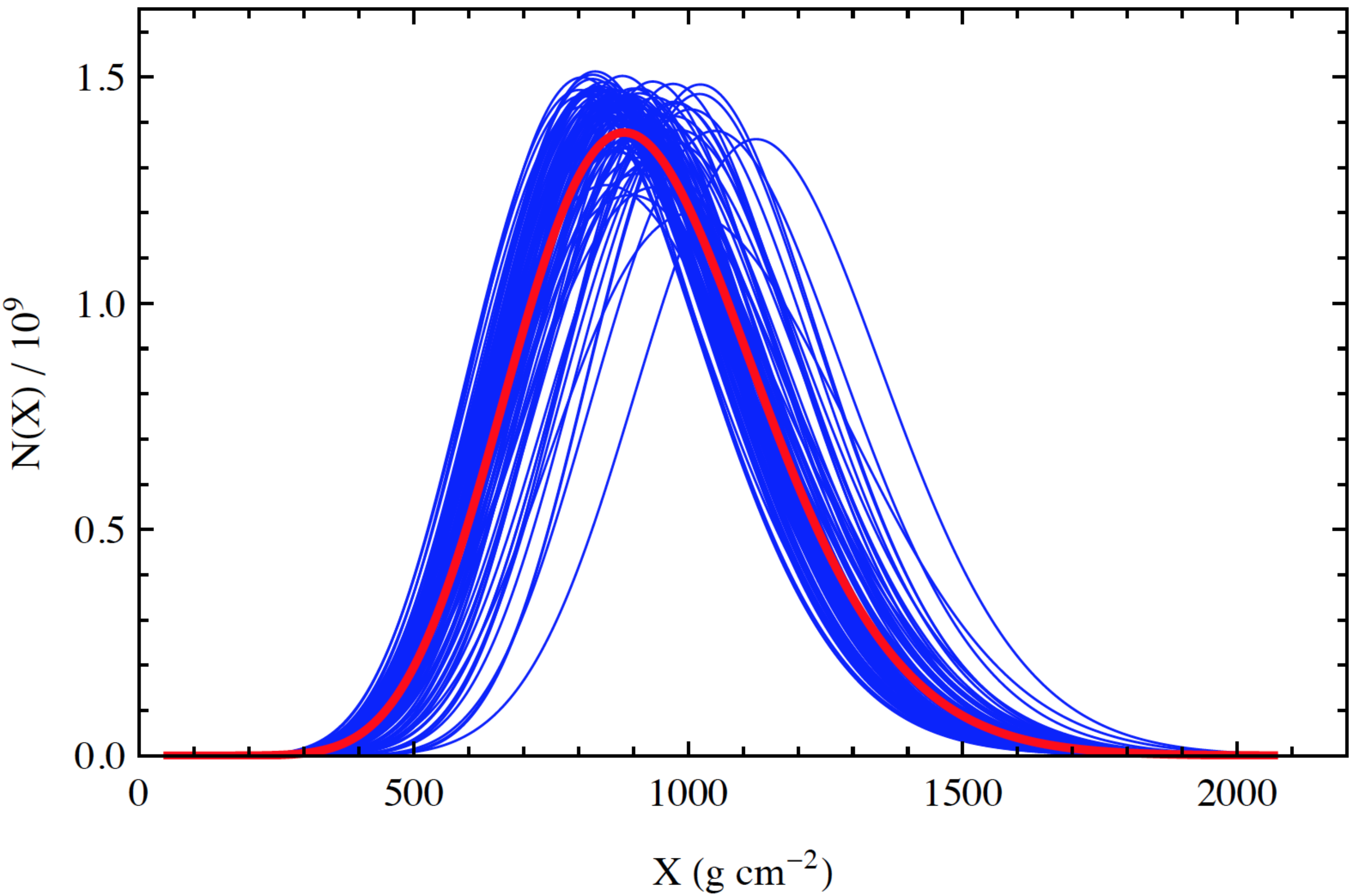
as a function of the depth along the shower

$$t = \frac{X}{\lambda_{\rm rad}}$$



The two particle types "generate each other"

Montecarlo calculation of the development of individual photon showers $E_0 = 10^{18.25}$ eV



SYSTEM of
INTEGRO-DIFFERENTIAL EQUATIONS

that describe the evolution with t
of the number of particles in the shower

$$n_e(E,t) \qquad n_\gamma(E,t)$$

[for a given initial condition]

OCTOBER, 1941 REVIEWS OF MODERN PHYSICS

Cosmic-Ray Theory

BRUNO ROSSI AND KENNETH GREISEN
Cornell University, Ithaca, New York

Bruno Rossi (1952)
"High Energy Particles"

“Approximation A”

Study the development of a shower with
these simplifying assumptions:

[1.] Describe *Bremsstrahlung* and *Pair production*
with the asymptotic expressions
[“splitting functions” that are scale invariant].

[2A.] Neglect the energy losses for collisions.

[3.] Neglect all other (less important) effects
(main correction Compton scattering)

“Approximation B”

Study the development of a shower with these simplifying assumptions:

[1.] Describe Bremsstrahlung and Pair production with the asymptotic expressions [“splitting functions” that are scale invariant].

[2B.] Energy loss for collisions is approximated as constant. New parameter: the “critical energy” (energy lost by e+- in one interaction length.)

$$\varepsilon = \left|\frac{dE}{dX}\right| \lambda_{\rm rad}$$

in air: $\varepsilon \approx 80 \text{ MeV}$

[3.] Neglect all other (less important) effects

Start studying Approximation A

Write down the equations that describe how the number of particles

change with depth:

$$\frac{\partial n_e(E,t)}{\partial t} = [\ldots]$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = [\ldots]$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = - \boxed{\begin{array}{l}\text{number of photons}\\ \text{(with energy E)}\\ \text{destroyed in dt}\end{array}}$$

$$+ \boxed{\begin{array}{l}\text{number of photons}\\ \text{(with energy E)}\\ \text{created by}\\ \text{all electrons}\\ \text{(with energy E' > E)}\end{array}}$$

Variation with t of the number of photons with energy E

$$\frac{\partial n_\gamma}{\partial t}(E,t) = \quad -n_\gamma(E,t) \quad \frac{\lambda_{\rm rad}}{\lambda_{\rm pair}}$$

$$+ \int_E^\infty dE' \int_0^1 dv \; n_e(E',t) \; \varphi(v) \; \delta[E - v\,E']$$

Integrate on
all electron energies
$E' > E$

All possible
gamma ray
radiations

Constraint:
Radiated
Photon
Energy E

Variation with t of the number of photons with energy E

$$\frac{\partial n_\gamma}{\partial t}(E,t) = -n_\gamma(E,t) \quad \frac{\lambda_{\rm rad}}{\lambda_{\rm pair}}$$

$$+ \int_E^\infty dE' \int_0^1 dv \; n_e(E',t) \; \varphi(v) \; \delta[E - v\, E']$$

$\boxed{\gamma \to \gamma}$

$\boxed{e \to \gamma}$

$$= \boxed{\begin{array}{l} -\sigma_0 \; n_\gamma(E,t) \qquad \boxed{\sigma_0 \simeq \frac{7}{9}} \\[2ex] \displaystyle\int_0^1 \frac{dv}{v} \; n_e\left(\frac{E}{v}, t\right) \; \varphi(v) \end{array}}$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = \quad -n_e(E,t) \int_0^1 dv \; \varphi(v)$$

$e \to e$ → $$+\int_E^\infty dE' \int_0^1 dv \; n_e(E',t) \; \varphi(v) \; \delta[E-(1-v)\,E']$$

$\gamma \to e$ → $$+\int_E^\infty dE' \int_0^1 du \; n_\gamma(E',t) \; \psi(u) \; \delta[E-u\,E']$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = \quad -n_e(E,t)\int_0^1 dv\;\varphi(v)$$

$$+\int_E^\infty dE'\int_0^1 dv\; n_e(E',t)\;\varphi(v)\;\delta[E-(1-v)\,E']$$

$$+\int_E^\infty dE'\int_0^1 du\; n_\gamma(E',t)\;\psi(u)\;\delta[E-u\,E']$$

Two divergent contributions (difference finite)

$$e \to e$$

$$\gamma \to e$$

$$\boxed{\begin{aligned}&-\int_0^1 dv\;\;\varphi(v)\;\left[n_e(E,t)-\frac{1}{1-v}\,n_e\left(\frac{E}{1-v},t\right)\right]\\ &+2\int_0^1\frac{du}{u}\,\psi(u)\,n_\gamma\left(\frac{E}{u},t\right)\end{aligned}}$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \;\; \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2 \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t) \; .$$

Approximation A

The most physically interesting solutions of the shower equations are those with initial conditions:

One photon of given energy

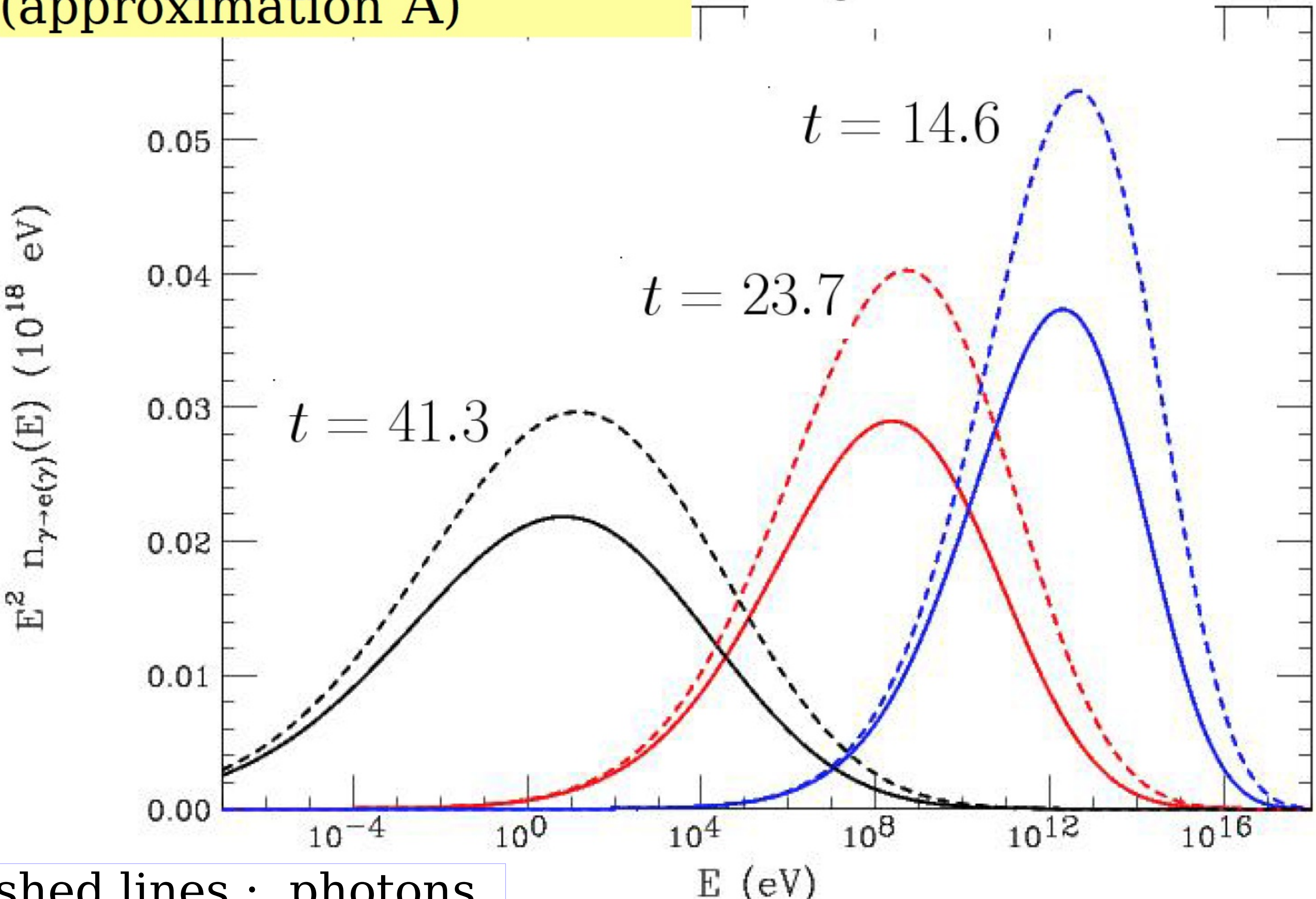
$$\begin{cases} n_\gamma(E,0) = \delta(E - E_0) \\ n_e(E,0) = 0 \end{cases}$$

One electron of given energy

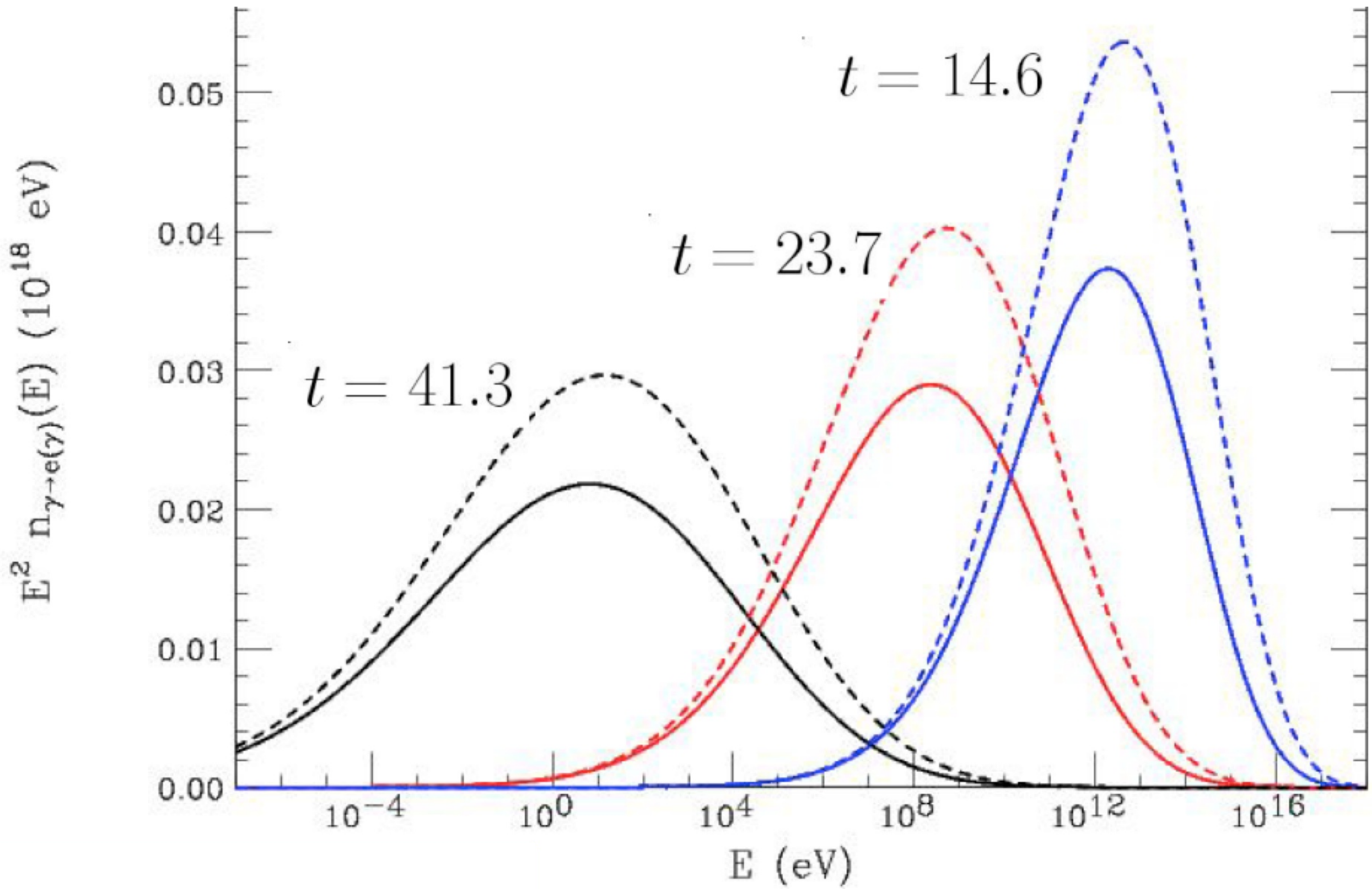
$$\begin{cases} n_\gamma(E,0) = 0 \\ n_e(E,0) = \delta(E - E_0) \end{cases}$$

Monochromatic Photon
(approximation A)

$E_0 = 10^{18}$ eV

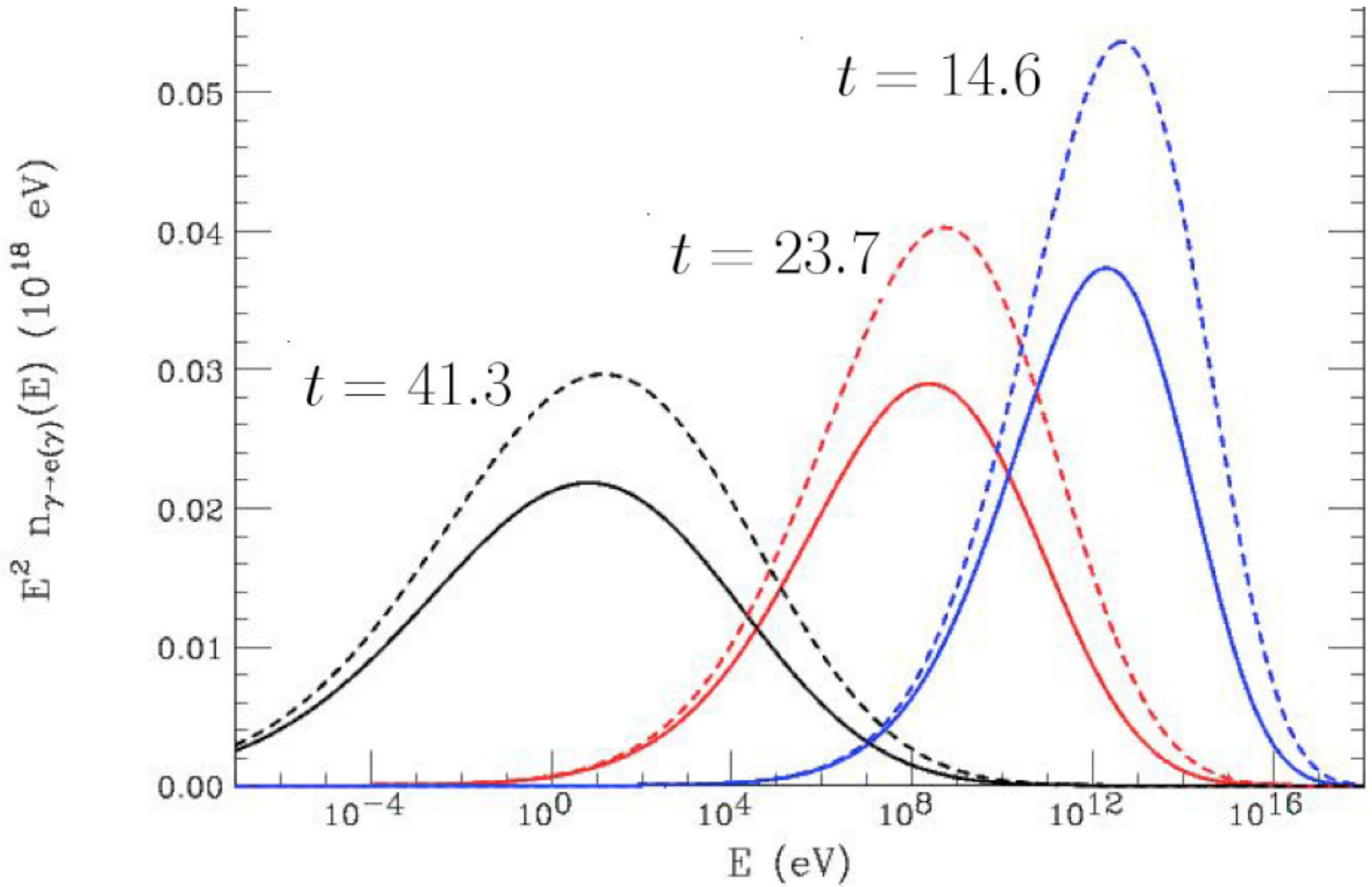


Dashed lines : photons
Solid lines : electrons



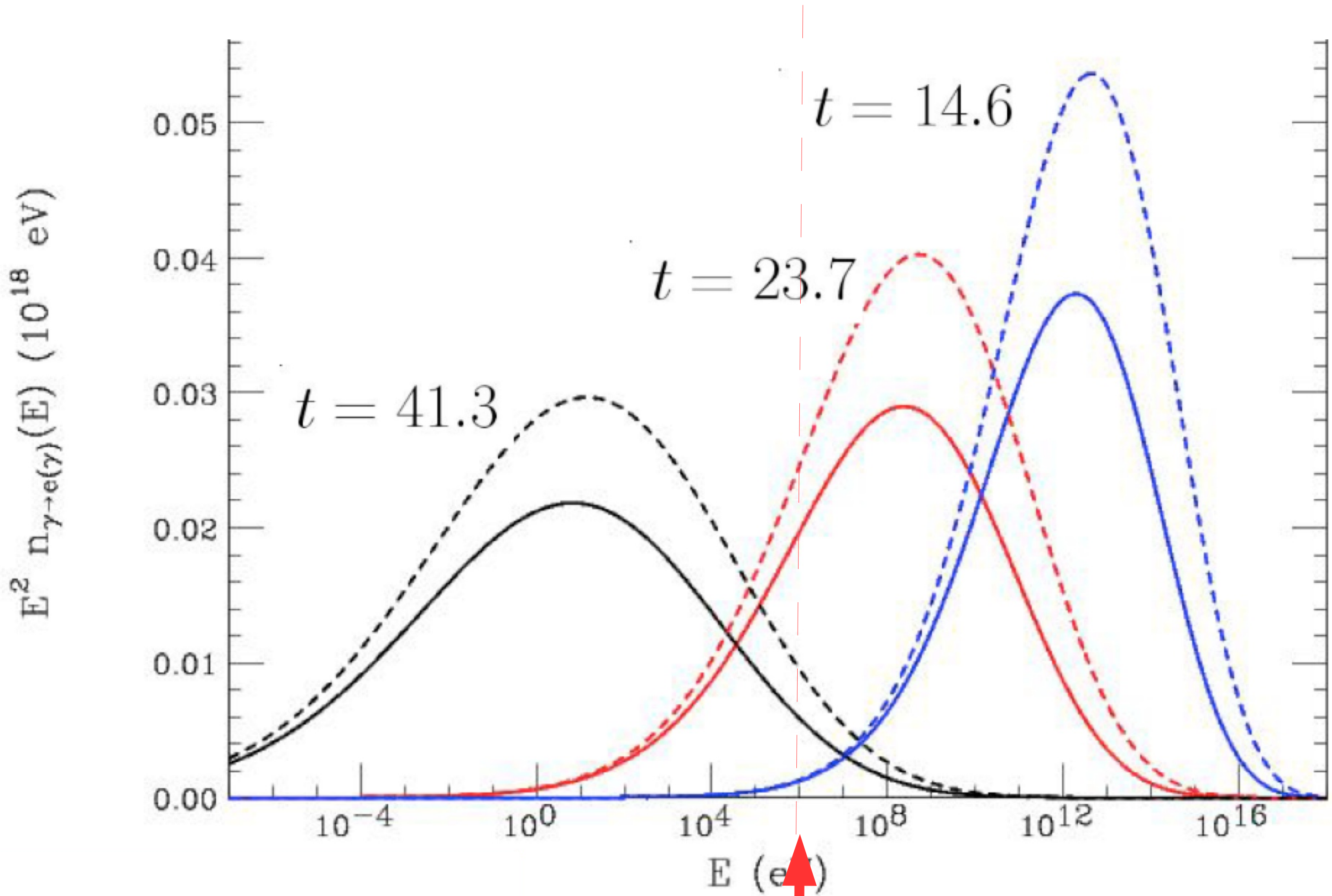
Note: spectra multiplied by E^2 : Area under curve = total energy carried by particles in the shower

$$\mathcal{E}_e = \int dE\ E\ n_e(E) = \int d\log E\ E^2\ n_e(E)$$



Note: spectra multiplied by E^2 : Area under curve = Total energy carried by particles in the shower

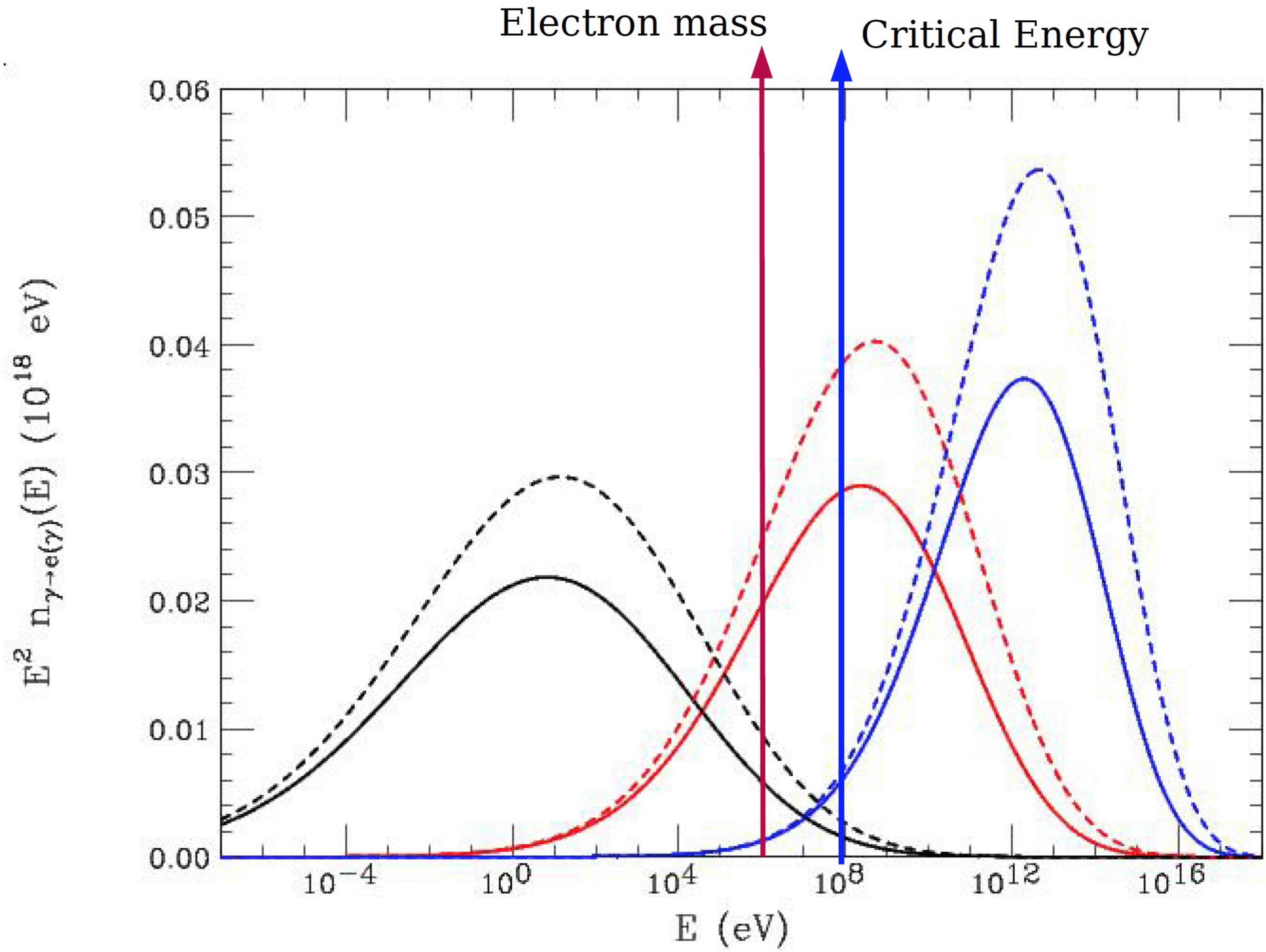
Initial energy of the primary particle (that initiate the shower) always entirely carried by the shower particles



Unphysical part
of the solution.

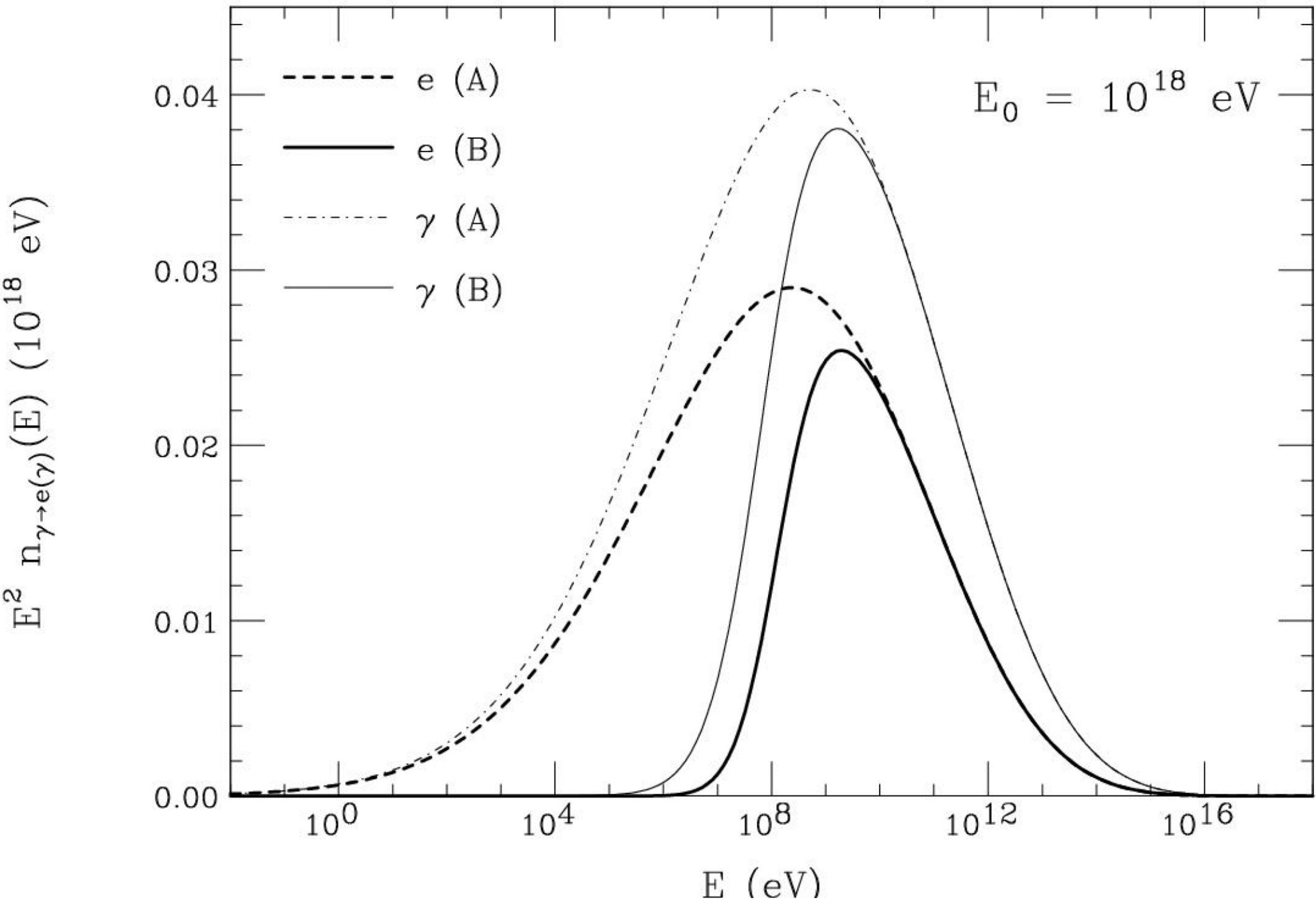
[The equations do not
“know” about
the electron mass]

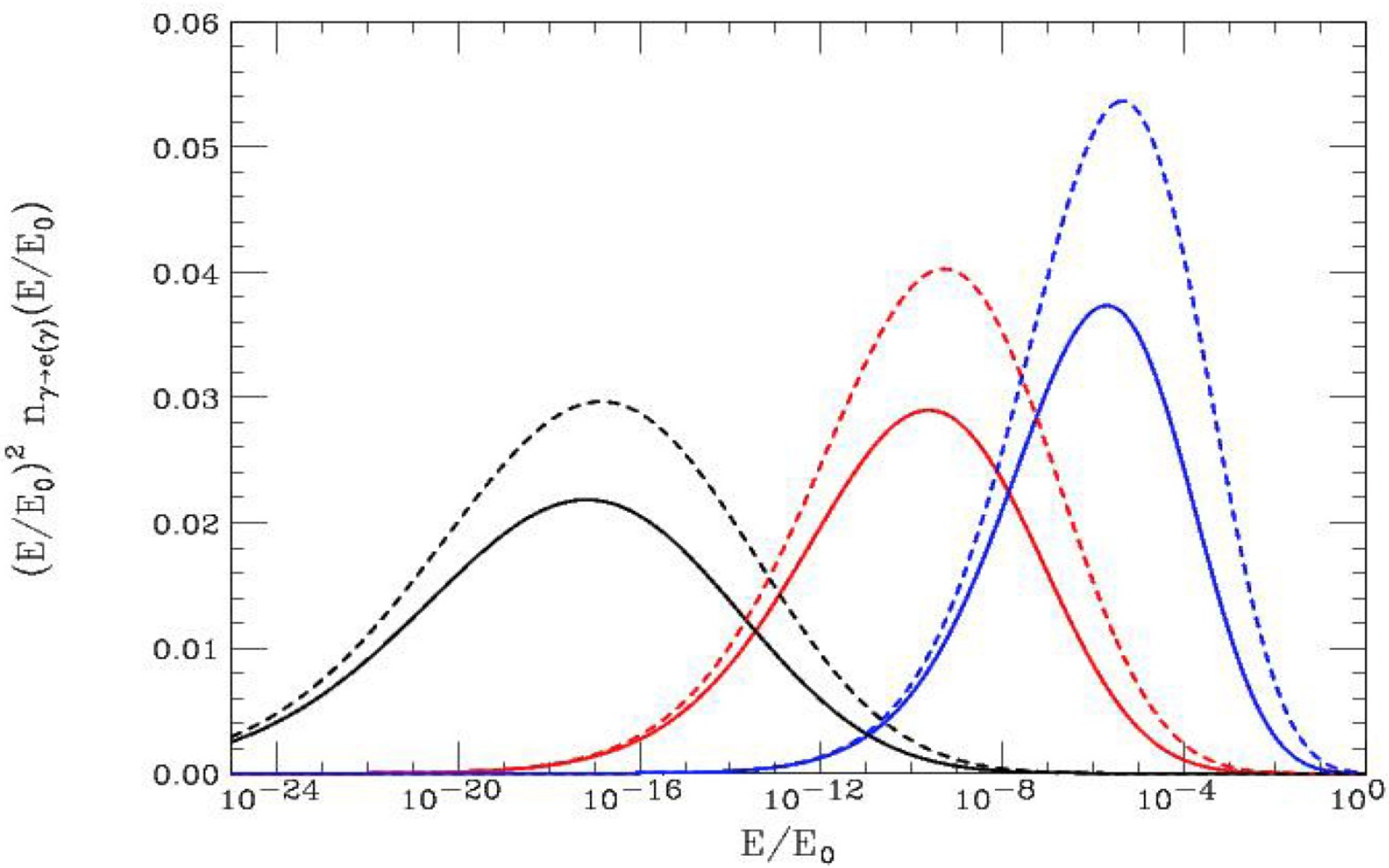
Electron mass
0.5 MeV



Electron mass
Critical Energy
0.06
0.05
0.04
0.03
0.02
0.01
0.00
$E^2\ n_{\gamma\to e(\gamma)}(E)\ (10^{18}\ \mathrm{eV})$
10^{-4}
10^{0}
10^{4}
10^{8}
10^{12}
10^{16}
E (eV)

Monochromatic Photon. Approximation A,B





Solution valid for any initial energy

Function of E/E_0

$$n_\alpha(E_0, E, t) = \frac{1}{E_0}\, f_\alpha\left(\frac{E}{E_0}, t\right)$$

$\gamma \to e$ $\quad e \to e$

$\gamma \to \gamma$ $\quad e \to \gamma$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \;\; \left[n_e(E,t) - \frac{1}{1-v} n_e \left(\frac{E}{1-v}, t \right) \right]$$

$$+2 \int_0^1 \frac{du}{u} \psi(u) \, n_\gamma \left(\frac{E}{u}, t \right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \varphi(v) \, n_e \left(\frac{E}{v}, t \right) - \sigma_0 \, n_\gamma(E,t) \; .$$

$\boxed{\text{Approximation A}}$

Homogeneous
"scaling" equations
[No explicit
Energy dependence]

“Elementary Solutions”

of the shower equations

Problem of shower development becomes very simple for

initial conditions
of power law form.

Solutions in the form of Power laws

$$n_e(E,t) = K_e(t)\ E^{-(s+1)}$$

$$n_\gamma(E,t) = K_\gamma(t)\ E^{-(s+1)}$$

put these forms
in the shower
equations (appr. A)

The shower equations take the form

$$\frac{\partial n_e(E,t)}{\partial t} = A(s)\, n_e(E,t) + B(s)\, n_\gamma(E,t)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = C(s)\, n_e(E,t) - \sigma_0\, n_\gamma(E,t)$$

Example for the source term $\gamma \to e$

$$2 \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

Substitute the form
$n_\gamma(E,t) = K_\gamma(t)\; E^{-(s+1)}$

$$2 \int_0^1 \frac{du}{u}\, \psi(u)\, K_\gamma(t)\, \left(\frac{E}{u}\right)^{-(s+1)}$$

$$K_\gamma(t)\; E^{-(s+1)} \boxed{\left[2 \int_0^1 \frac{du}{u}\, \psi(u)\; u^{(s+1)}\right]}$$

$$n_\gamma(E,t) \boxed{B(s)}$$

Momenta of the splitting functions

$$
\begin{aligned}
A(s) &= \int_0^1 dv\ \varphi(v)\ \left[1-(1-v)^s\right] \\
&= \left(\frac{4}{3}+2\,b\right)\ \left(\frac{\Gamma'(1+s)}{\Gamma(1+s)}+\gamma\right)+\frac{s\ (7+5\,s+12\,b\ (2+s))}{6\ (1+s)\ (2+s)}
\end{aligned}
$$

$$
B(s) = 2\int_0^1 du\ u^s\ \psi(u) = \frac{2\ \left(14+11\,s+3\,s^2-6\,b\ (1+s)\right)}{3\ (1+s)\ (2+s)\ (3+s)}
$$

$$
C(s) = \int_0^1 dv\ v^s\ \varphi(v) = \frac{8+7\,s+3\,s^2+6\,b\ (2+s)}{3\,s\ (2+3\,s+s^2)}
$$

$$
\frac{\partial n_e(E,t)}{\partial t} = A(s)\, n_e(E,t) + B(s)\, n_\gamma(E,t)
$$

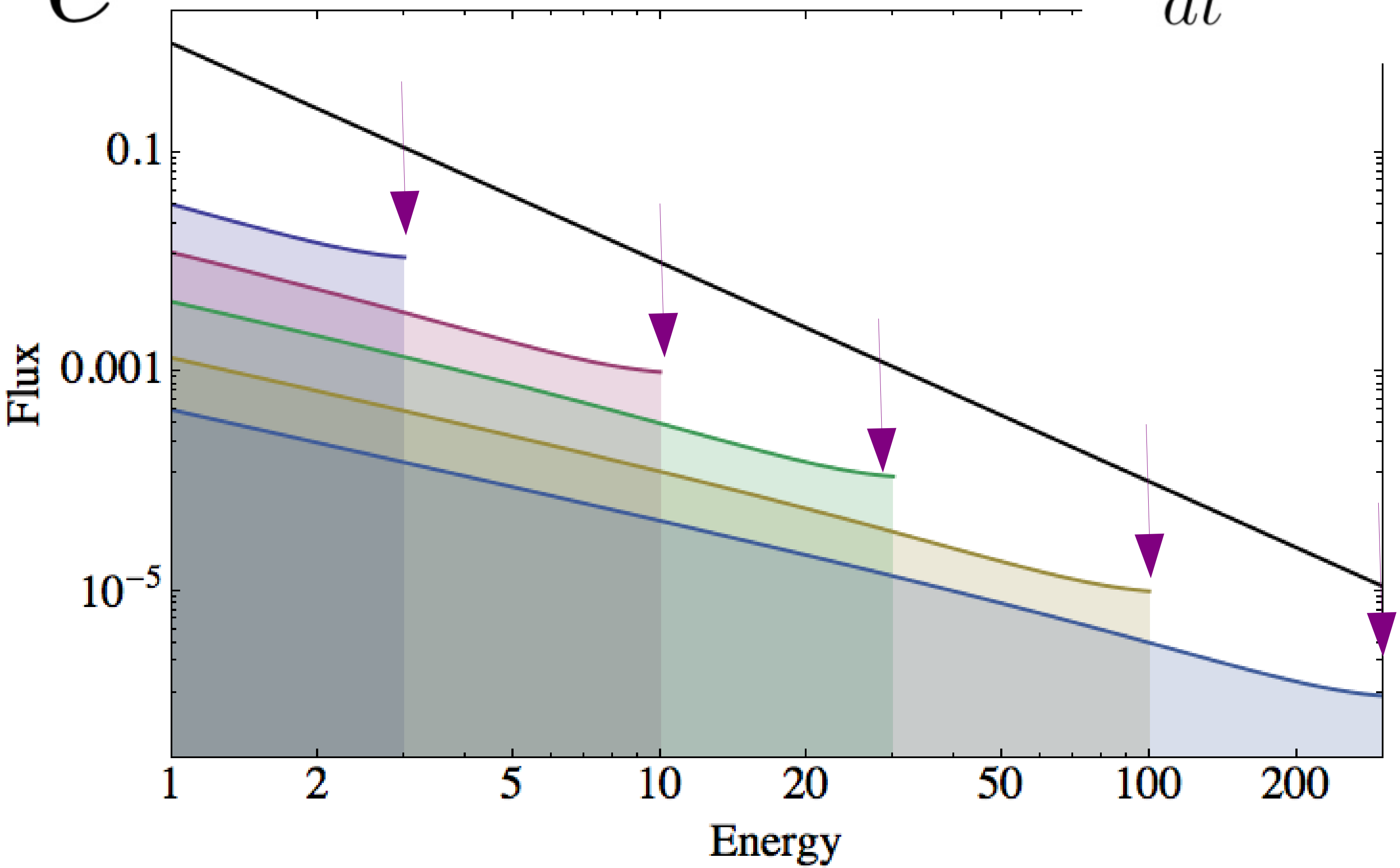
$$
\frac{\partial n_\gamma(E,t)}{\partial t} = C(s)\, n_e(E,t) - \sigma_0\, n_\gamma(E,t)
$$

Source of photons generated by
a power law electron spectrum is a
power law with the same exponent.

$$n_e(E) \propto E^{-(s+1)}$$

e

$$\frac{dn_{e\to\gamma}}{dt}(E) \propto E^{-(s+1)}$$

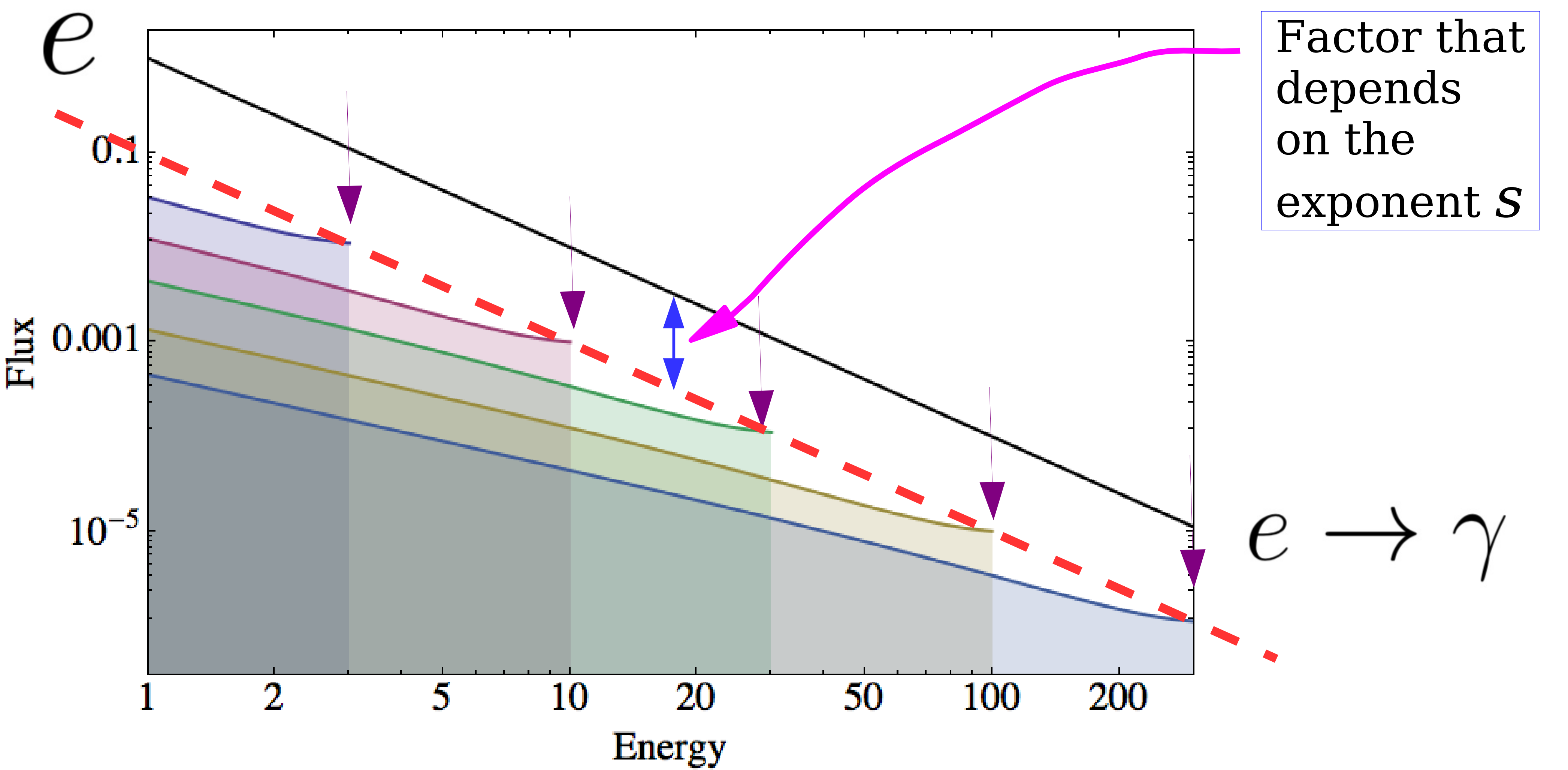


$e \to \gamma$

Source of photons generated by a power law electron spectrum is a *power law with the same exponent.*

$$n_e(E) \propto E^{-(s+1)}$$

$$\frac{dn_{e\to\gamma}}{dt}(E) \propto E^{-(s+1)}$$



(scaling) contributions combine to generate a power law [of equal exponent]

This implies that (in approximation A)
if the spectra are power laws at one depth t,
they are power laws at all t

Rewrite the two shower equations in matrix form

$$\frac{\partial}{\partial t} \left[\begin{array}{c} n_e(E,t) \\ n_\gamma(E,t) \end{array} \right] = \left[\begin{array}{cc} A(s) & B(s) \\ C(s) & -\sigma_0 \end{array} \right] \left[\begin{array}{c} n_e(E,t) \\ n_\gamma(E,t) \end{array} \right]$$

Substitute the forms

$K_e(t)\ E^{-(s+1)} \qquad K_\gamma(t)\ E^{-(s+1)}$

Can divide by

$E^{-(s+1)}$

$$\frac{\partial}{\partial t} \left[\begin{array}{c} K_e(t) \\ K_\gamma(t) \end{array} \right] = \left[\begin{array}{cc} A(s) & B(s) \\ C(s) & -\sigma_0 \end{array} \right] \left[\begin{array}{c} K_e(t) \\ K_\gamma(t) \end{array} \right]$$

For each value of the exponent (s+1)
there are two values of the ratio gamma/e
for which the t-evolution with t is a simple exponential

$$\begin{bmatrix} n_e(E,t) \\ n_\gamma(E,t) \end{bmatrix} = K\ e^{\lambda(s)\,t} \begin{bmatrix} 1 \\ r_\gamma(s) \end{bmatrix} E^{-(s+1)}$$

For each value of s there are two "elementary solutions"
where the shower develops exponentially in t

$$s \qquad \lambda_{1,2}(s) \qquad r_{1,2}(s)$$

Finding these "elementary solutions" is a simple eigenvalue-eigenvectors problem

$$\frac{\partial}{\partial t}\begin{bmatrix} K_e(t) \\ K_\gamma(t) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & -\sigma_0 \end{bmatrix} \begin{bmatrix} K_e(t) \\ K_\gamma(t) \end{bmatrix}$$

Substituting the exponential form:

$$\begin{bmatrix} K_e(t) \\ K_\gamma(t) \end{bmatrix} = K\, e^{\lambda(s)\,t} \begin{bmatrix} 1 \\ r_\gamma(s) \end{bmatrix}$$

One obtains an eigenvalue equation (2 solutions).

$$\begin{bmatrix} A(s) & B(s) \\ C(s) & -\sigma_0 \end{bmatrix} \begin{bmatrix} 1 \\ r_\gamma(s) \end{bmatrix} = \lambda(s) \begin{bmatrix} 1 \\ r_\gamma(s) \end{bmatrix}$$

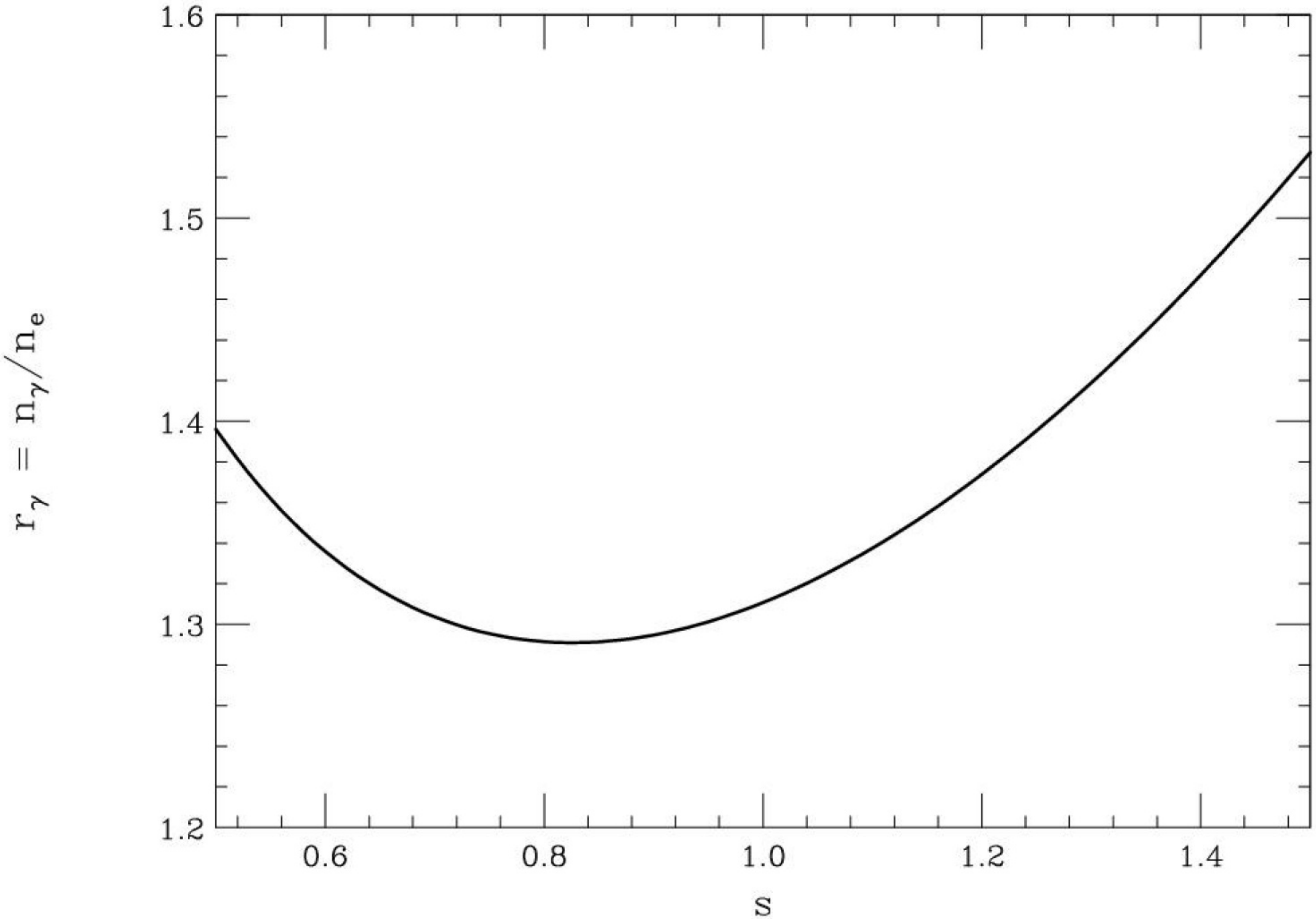
$$\boxed{\lambda_{1,2}(s) = -\frac{1}{2}\left(A(s)+\sigma_0\right) \pm \frac{1}{2}\sqrt{\left(A(s)-\sigma_0\right)^2 + 4\,B(s)\,C(s)}}$$

$$\boxed{r_\gamma^{(1,2)}(s) = \frac{C(s)}{\sigma_0 + \lambda_{1,2}(s)}}$$

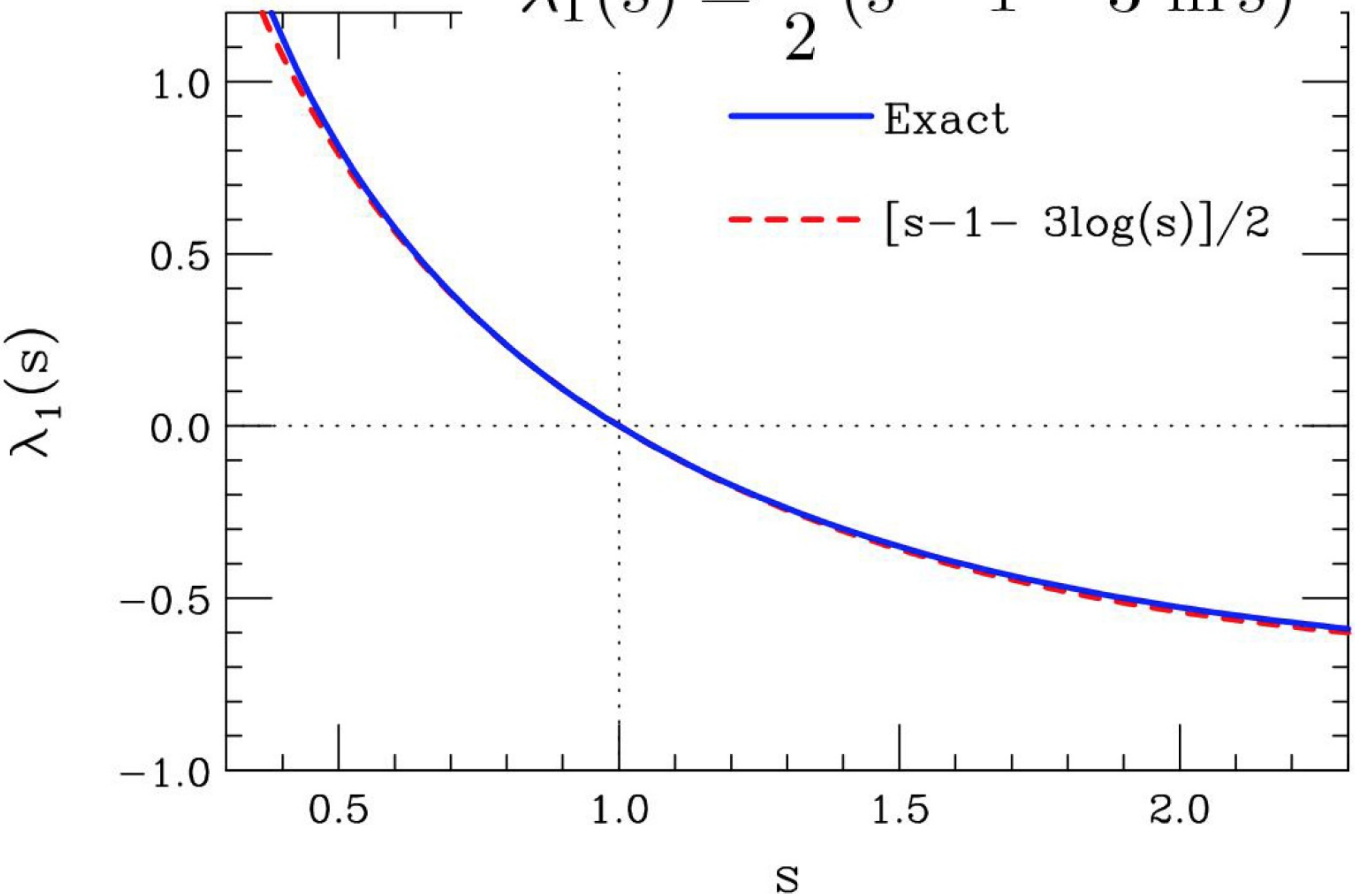
One eigenvalue large and negative
corresponds to a fast disappearing transient

The other eigenvalue correspond to the
more physically interesting solution

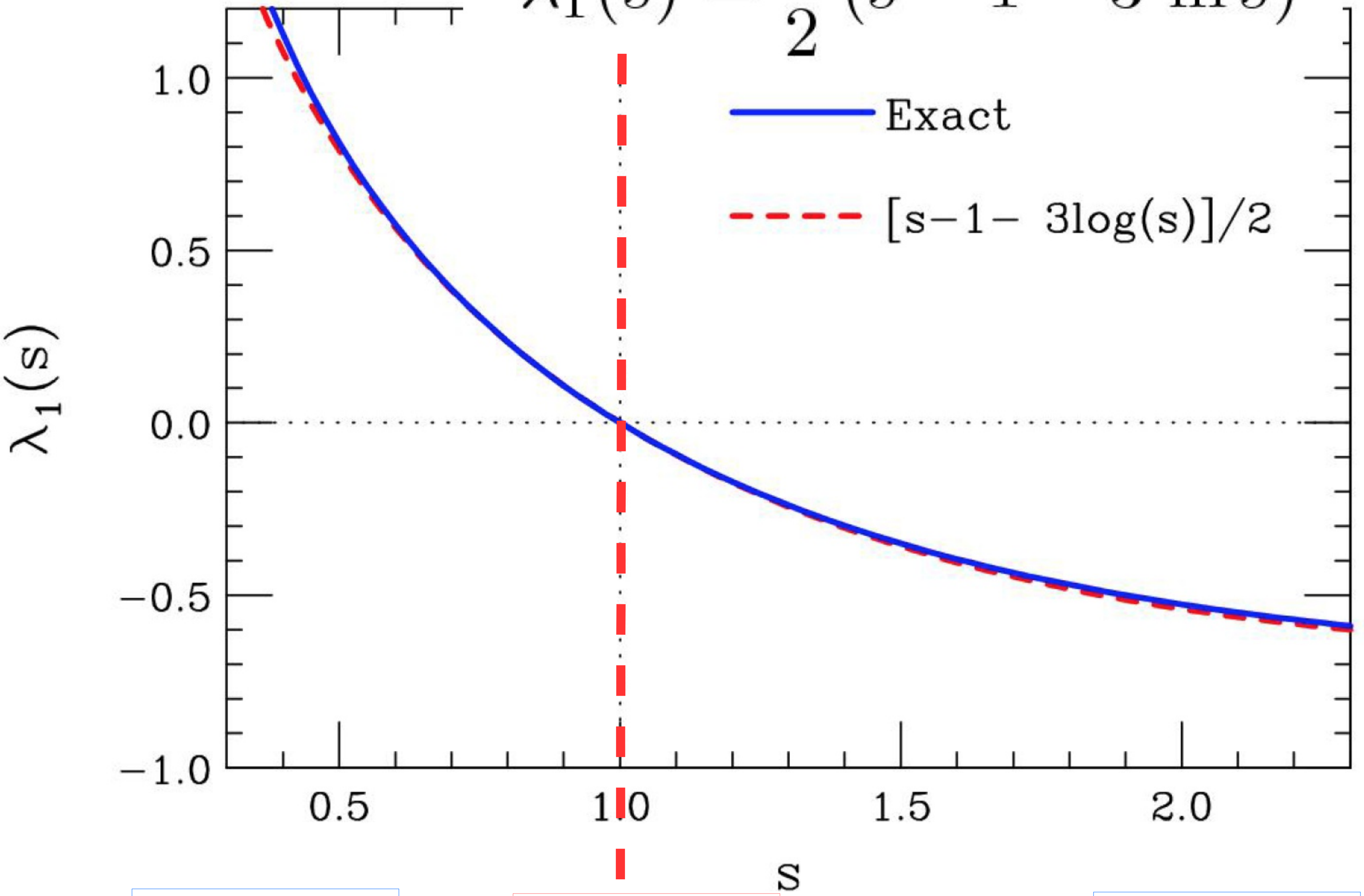
Ratio photon / electron   (of order unity)
[little dependence on s]



$$\bar{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$$



$$\overline{\lambda}_1(s) = \frac{1}{2}\left(s - 1 - 3\,\ln s\right)$$



$s < 1$

$\lambda > 0$

$s = 1$

$\lambda = 0$

$s > 1$

$\lambda < 0$

Physical meaning of the relation between

s and $\lambda_1(s)$

is not difficult to understand qualitatively

Important (and very instructive case)
[Exact solution]

$s = 1$ $\quad n_{e,\gamma}(E,t) \propto E^{-2}$

$\lambda_1(s) = 0$ $\quad n_{e,\gamma}(E,t) = \text{constant}$

$$n(E) = K\ E^{-2}$$

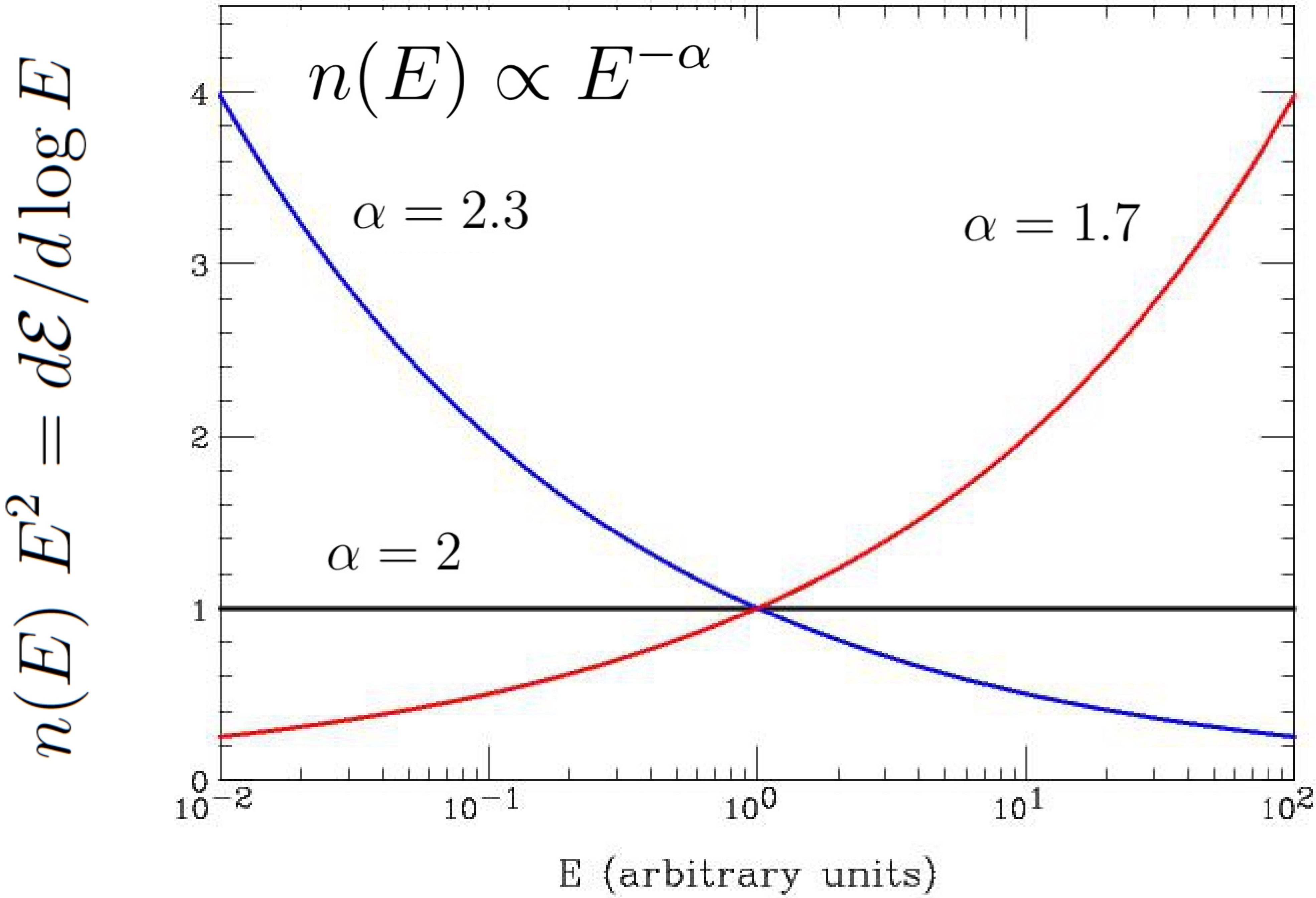
Special spectrum

same amount of energy
per decade of energy
(or unit of Log[E])

$$\frac{d\mathcal{E}}{d\ln E} = \frac{d\mathcal{E}}{dE}\ \frac{dE}{d\ln E}$$

$$= n(E)\ E\ \ E = n(E)\ E^2$$

Power Law Solutions : Spectral Energy Distribution



The particle "splitting"
during the shower development
Substitutes *[conserving energy]*

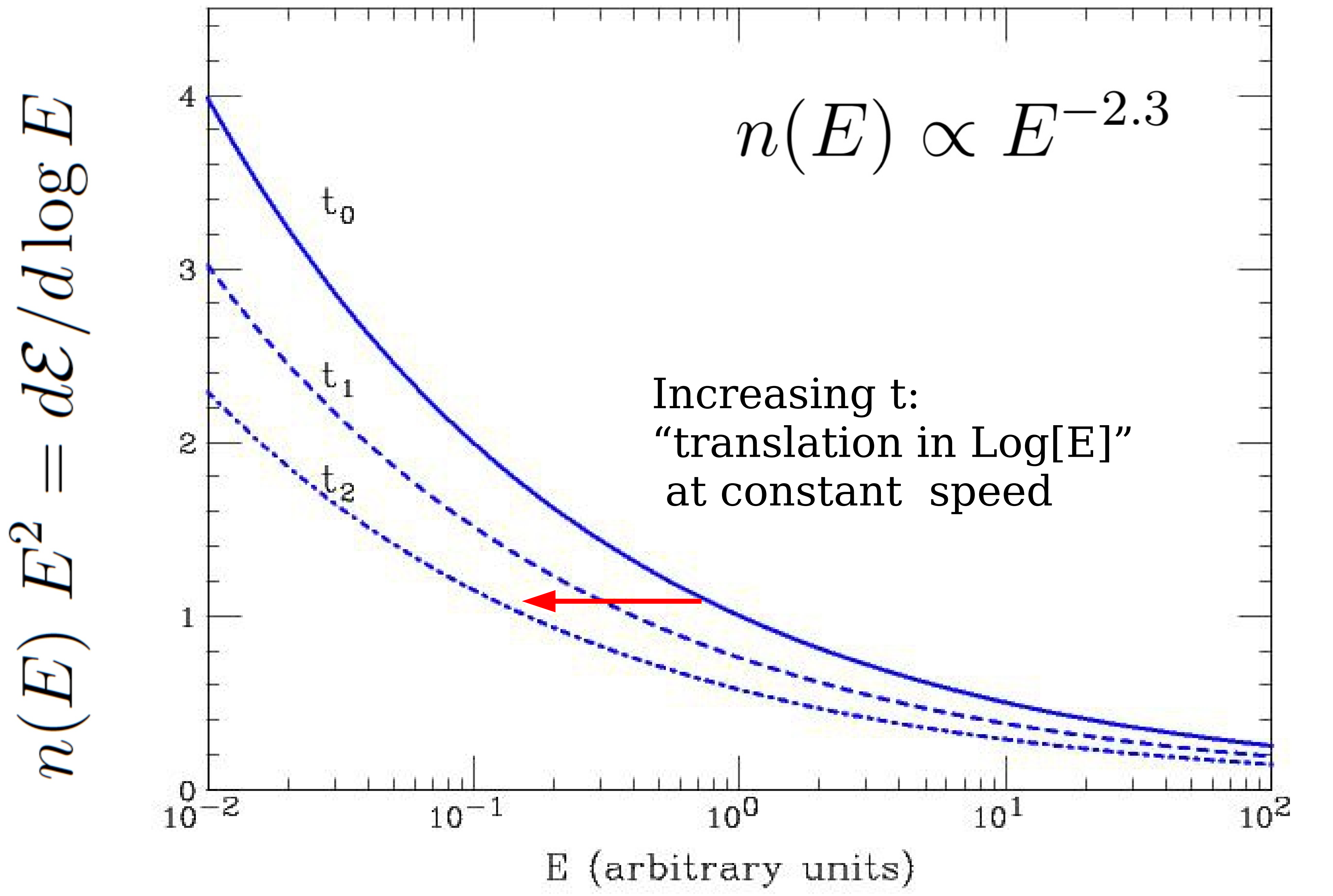
One particle of energy E

with

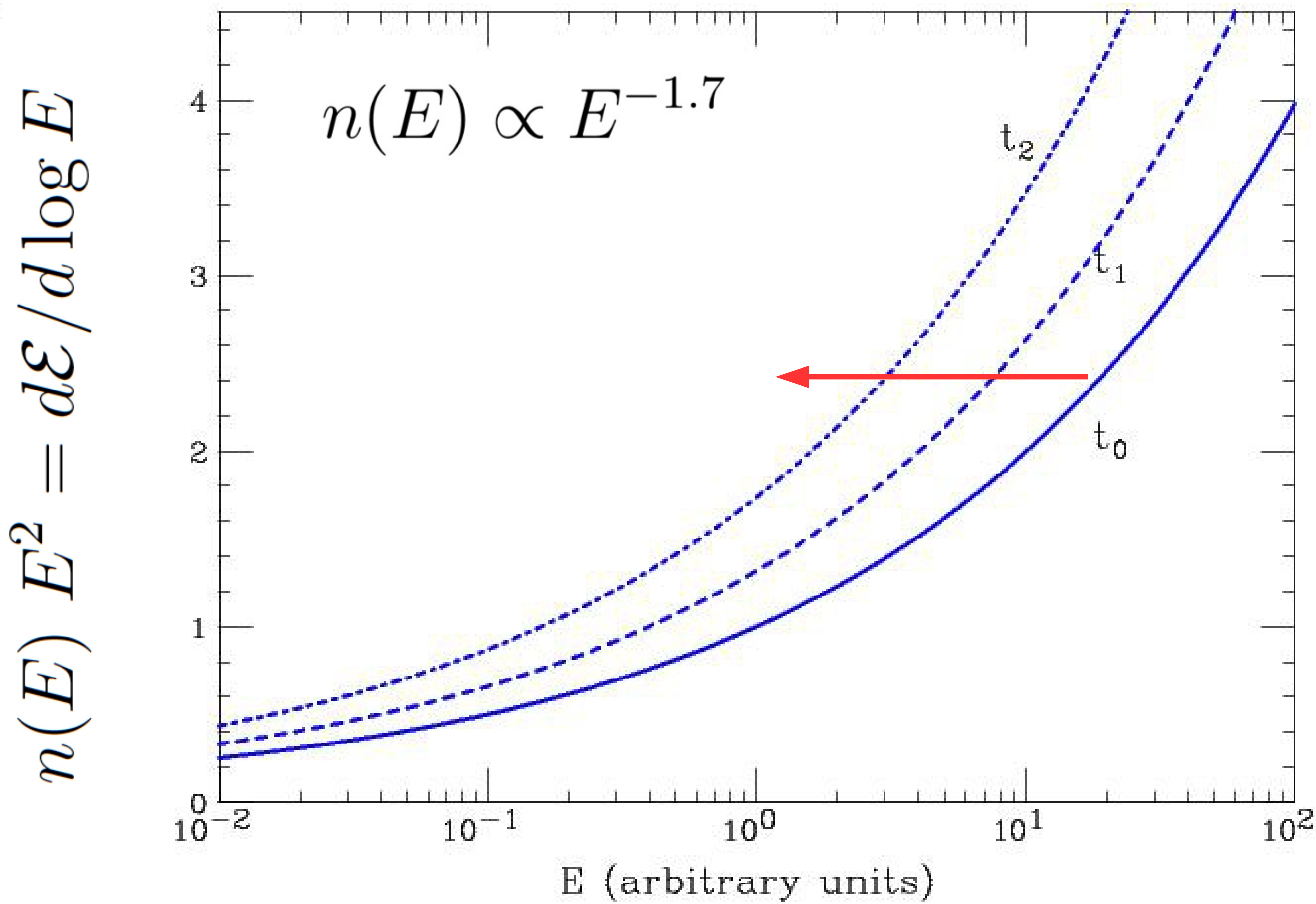
Two particles
of lower energy $E_1 + E_2 = E$

For power-law spectra this implies
that the spectrum:

Grows exponentially for exponent $s < 1$
Remains constant for $s = 1$
Decreases exponentially for $s > 1$



$n(E) \propto E^{-2.3}$
t_0
t_1
t_2
Increasing t:
"translation in Log[E]"
at constant speed
$n(E)\ E^2 = d\mathcal{E}/d\log E$
E (arbitrary units)
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}



$n(E)\ E^2 = d\mathcal{E}/d\log E$
$n(E) \propto E^{-1.7}$
t_2
t_1
t_0
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
E (arbitrary units)

$$s = 1 \iff \lambda_1(s) = 0$$

$$s < 1 \iff \lambda_1(s) > 0$$

$$s > 1 \iff \lambda_1(s) < 0$$

Calculation of Shower Development in approximation B

New ingredient:

Electrons and Positrons lose energy continuously because of collisions (ionization).

Need to incorporate this effect
in the shower development equation

Introduction of the energy loss for collisions in the shower equations.

$$\frac{dE}{dt} = \beta(E)$$

Energy variation Law

$$n(E,t) \qquad\qquad n(E,t+dt)$$

$$n(E,t+dt)\; dE = n(E', \;\; t)\; dE'$$

$$E' = E - \beta(E)\; dt \qquad\qquad dE' = \left(1 - \frac{d\beta(E)}{dE}\right)\; dE$$

$$\boxed{n(E, t + dt)\; dE = n(E', dt)\; dE'}$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial t}\, dt\right]\; dE \quad = \quad \left[n(E,t) - \frac{\partial n(E,t)}{\partial E}\, \beta(E)\, dt\right] \left(1 - \frac{d\beta(E)}{dE}\right)\; dE$$

$$\boxed{\frac{\partial n(E,t)}{\partial t} = -\frac{\partial}{\partial E}\, [n(E,t)\; \beta(E)]}$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv\ \varphi(v)\ \left[n_e(E,t) - \frac{1}{1-v}\,n_e\left(\frac{E}{1-v},t\right)\right]$$

$$+2\int_0^1 \frac{du}{u}\,\psi(u)\,n_\gamma\left(\frac{E}{u},t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\,\varphi(v)\,n_e\left(\frac{E}{v},t\right) - \sigma_0\,n_\gamma(E,t)$$

Approximation A

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \; \left[n_e(E,t) - \frac{1}{1-v} n_e\left(\frac{E}{1-v}, t\right)\right]$$

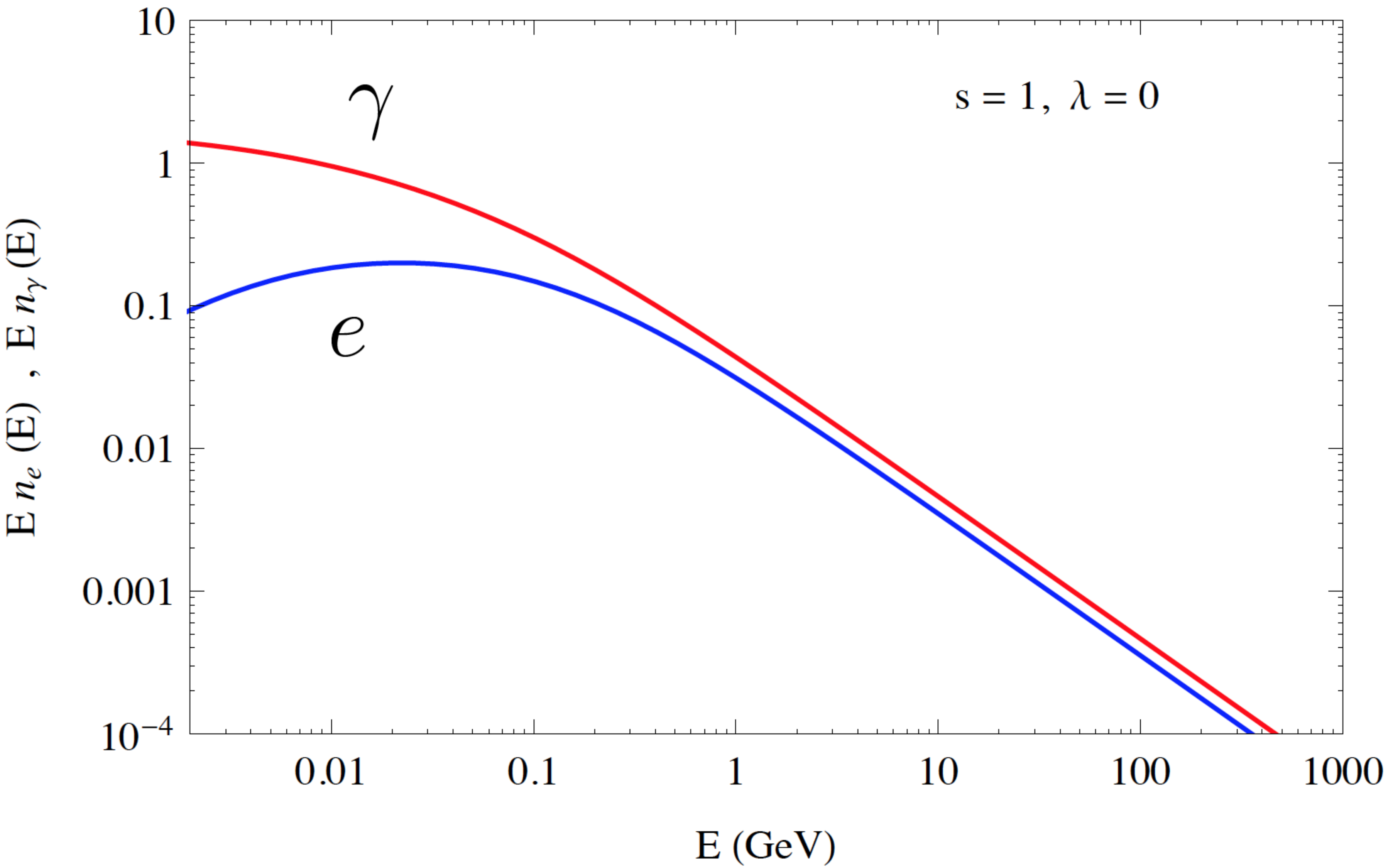
$$+2 \int_0^1 \frac{du}{u} \psi(u) \, n_\gamma \left(\frac{E}{u}, t\right)$$

$$+ \varepsilon \, \frac{\partial n_e(E,t)}{\partial E}$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \varphi(v) \, n_e \left(\frac{E}{v}, t\right) - \sigma_0 \, n_\gamma(E,t)$$

Approximation B

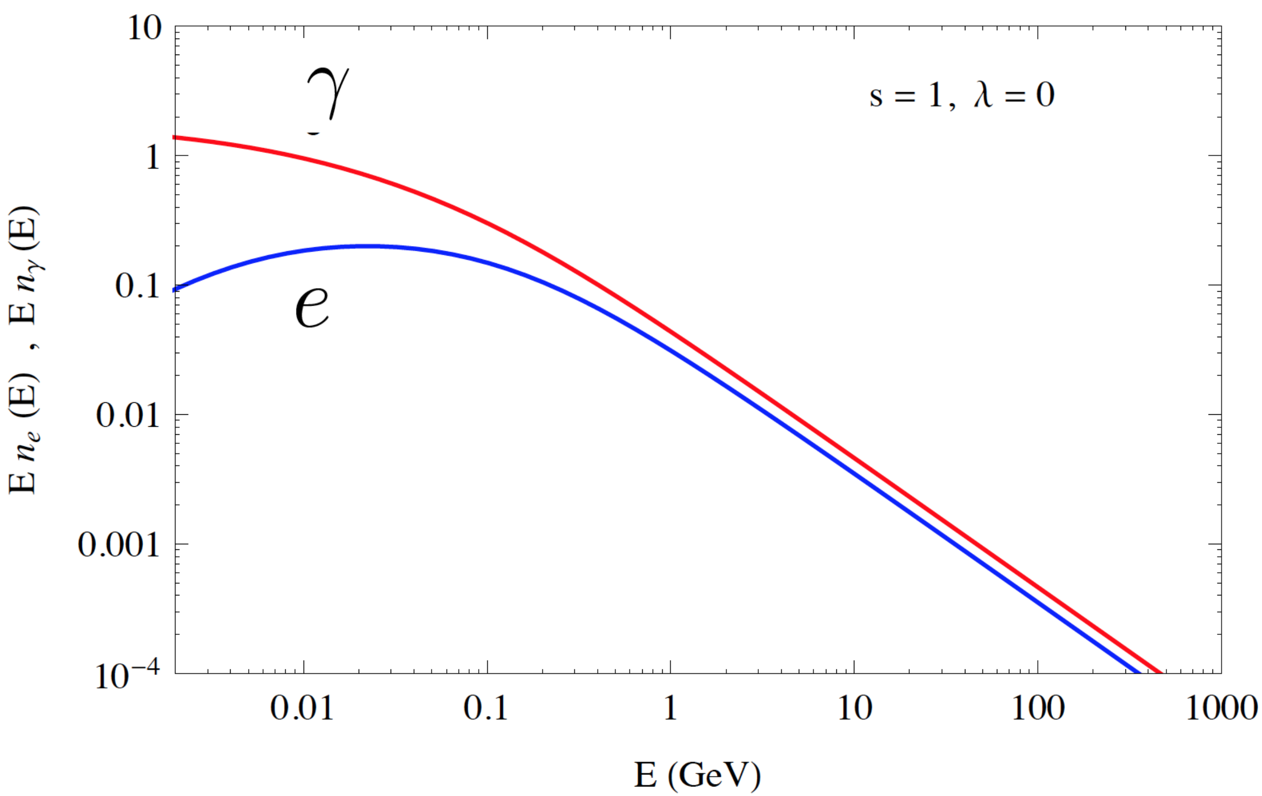
"Elementary solutions" of the electromagnetic shower equations in approximation B



Asymptotic form (large energy)

$$n_e(E,t) = K\ E^{-2}$$

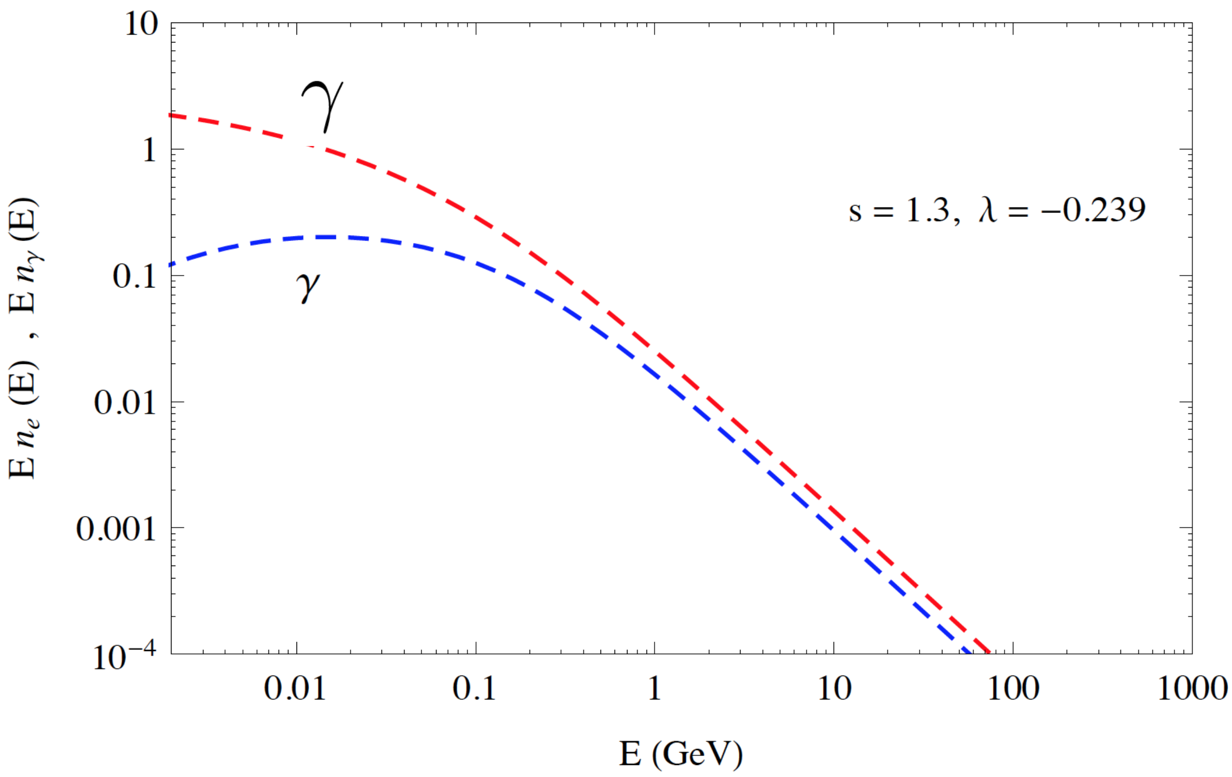
$$n_\gamma(E,t) = 1.310\ K\ E^{-2}$$



Spectra of electrons and photons remain constant at all t

$$n_e(E,t) = K\ E^{-2.3}\ e^{-0.239\,t}$$

$$n_\gamma(E,t) = 1.419\ K\ E^{-2.3}\ e^{-0.239\,t}$$

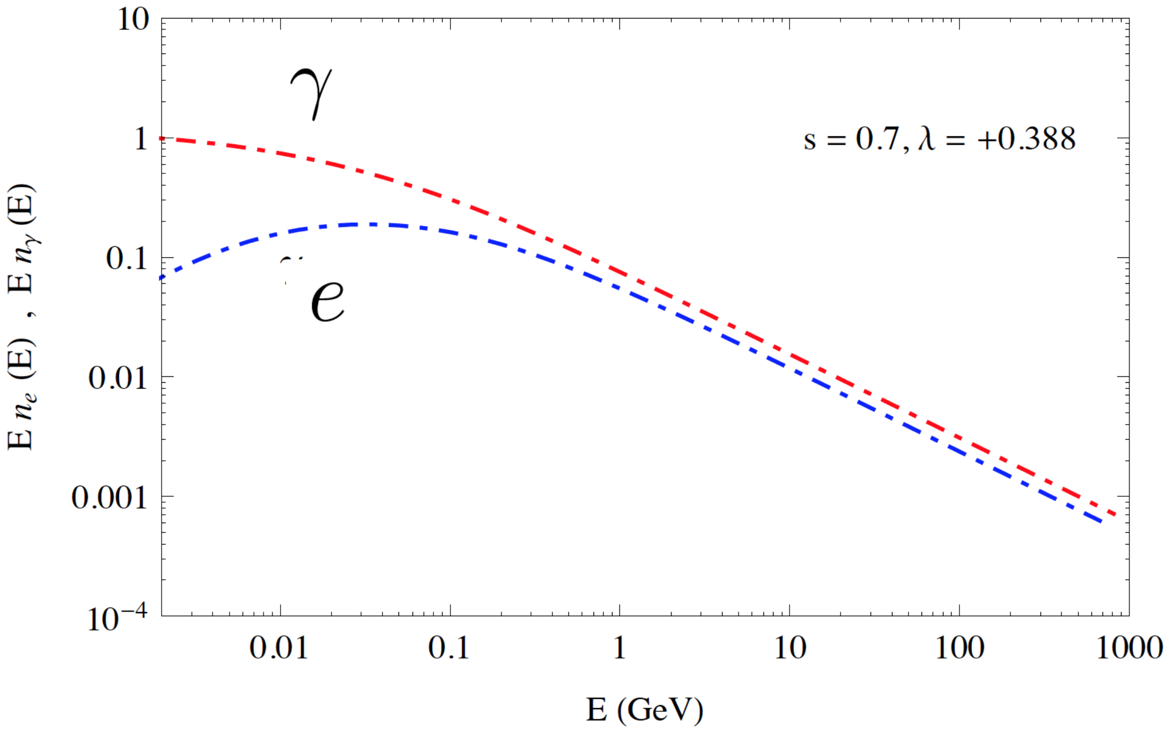


Spectra of electrons and photons have constant shape at all t

Exponentially decreasing normalization

$$n_e(E,t) = K\ E^{-1.7}\ e^{+0.388\,t}$$

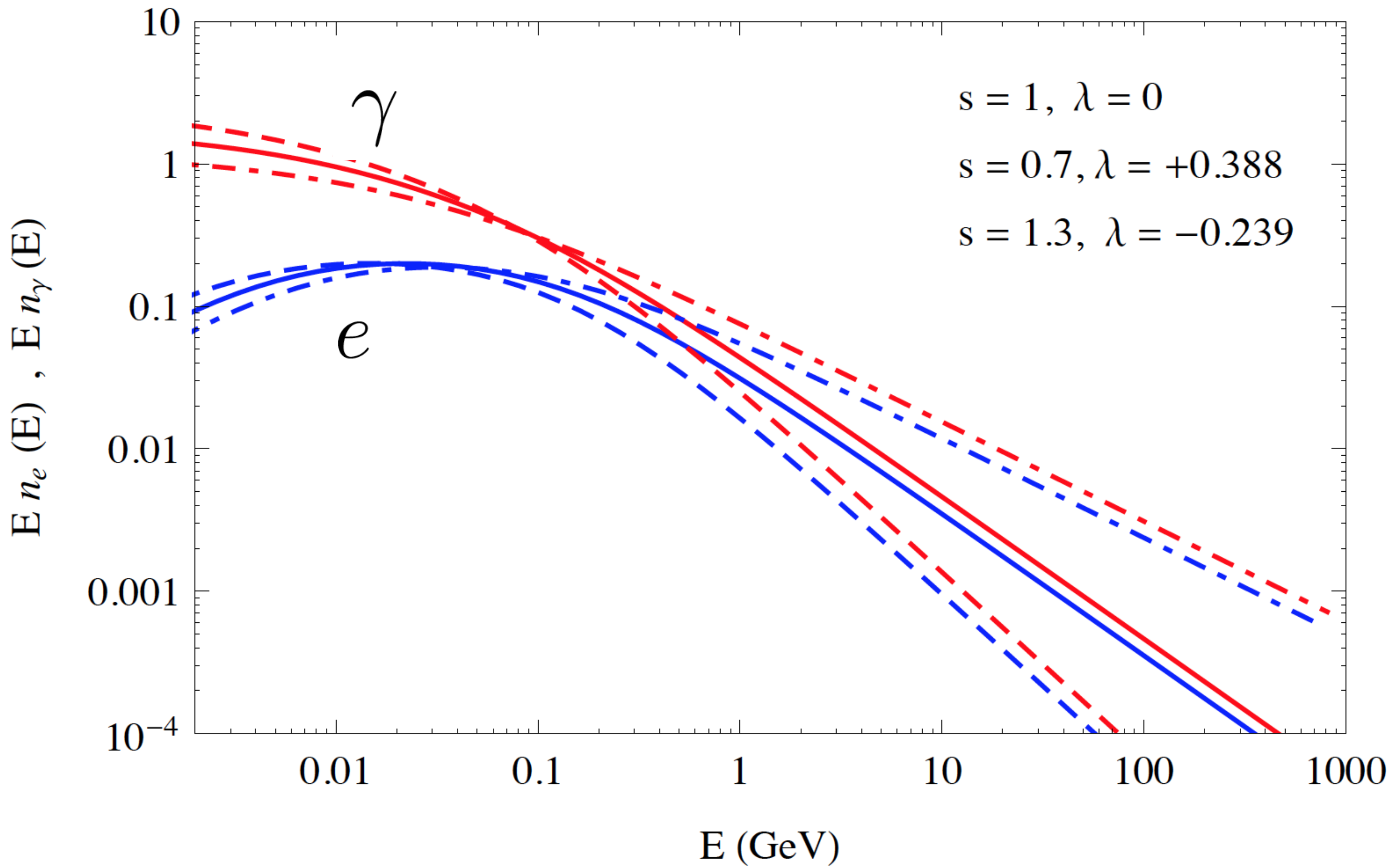
$$n_\gamma(E,t) = 1.303\ K\ E^{-1.7}\ e^{+0.388\,t}$$



Spectra of electrons and photons have constant shape at all t

Exponentially increasing normalization

$\varepsilon \simeq 81$ MeV



"Elementary solutions" in approximation-B coincide with solutions in approximation -A for $E \gg \varepsilon$

The elementary solutions of the shower equations in approximation B, have a simple and very useful

phenomenological application
for real showers

(electromagnetic and also hadronic).

$$N_e(t)$$ Development of one real shower

Shower *"stage of development"* at the depth $t_{\rm obs}$

$$\frac{1}{N_e(t)} \frac{dN_e}{dt}\bigg|_{t=t_{\rm obs}} = \lambda$$

$$N_e(t) \approx e^{\lambda\,(t-t_{\rm obs})}\; N_e(t_{\rm obs})$$

Mapping $\lambda \leftrightarrow s$

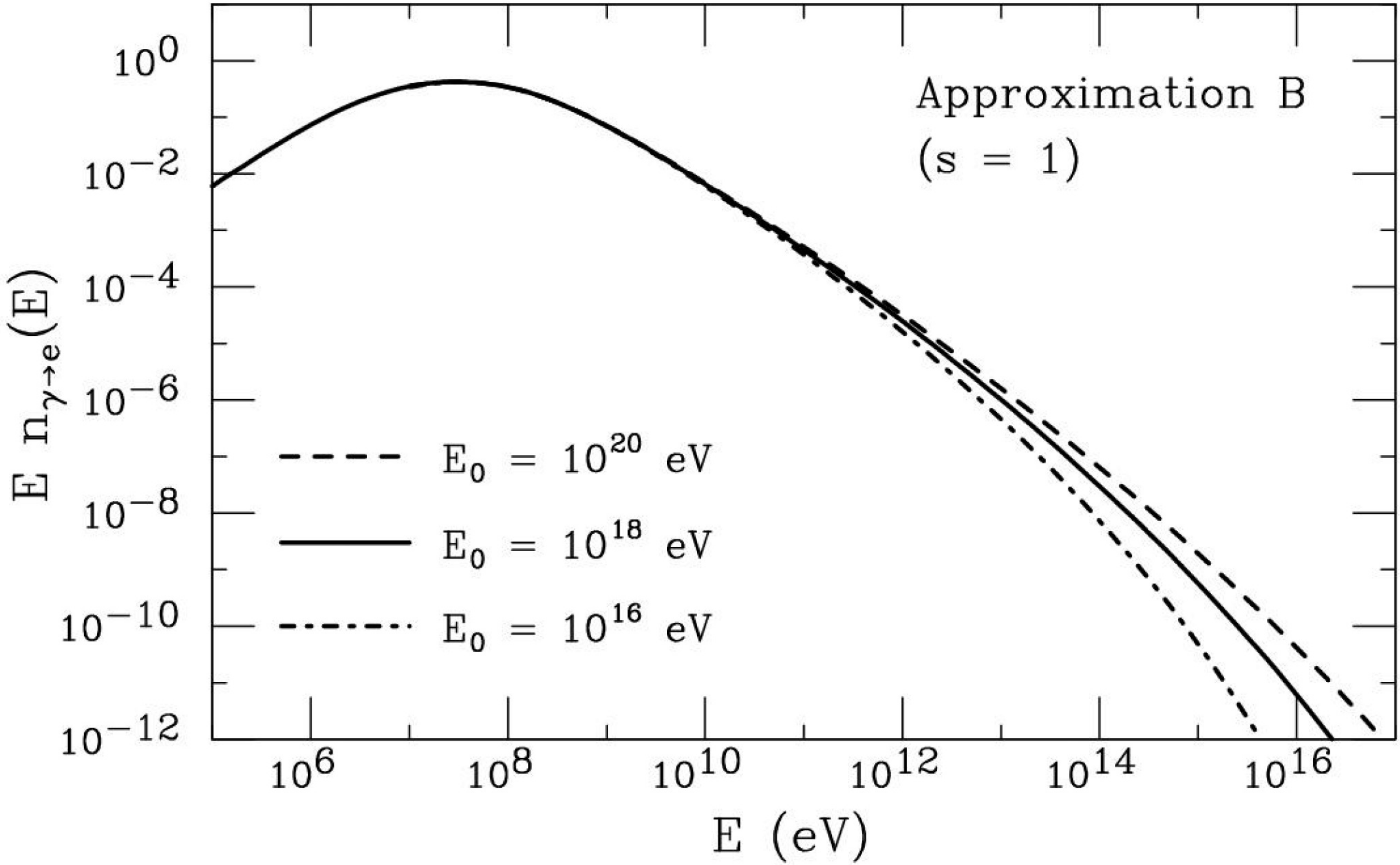
$$\lambda = \lambda(s)$$

s = shower *age*

Shower at maximum: s = 1
Shower growing s < 1
Shower decreasing s > 1

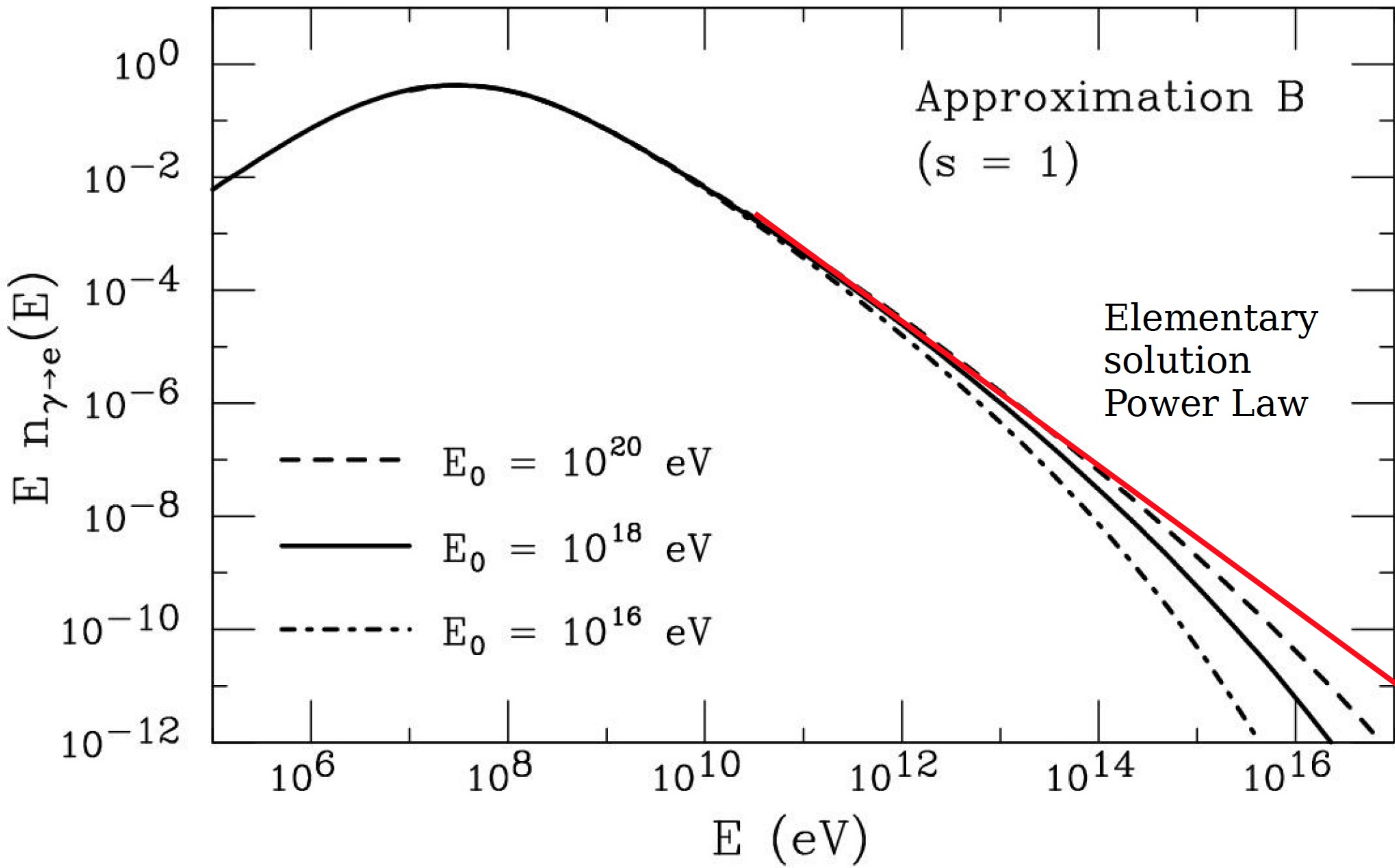
Different Energy : Same Age (Shower Maximum)

[normalized - average development of gamma ray showers]



Different Energy : Same Age (Shower Maximum)

[average development of gamma ray showers]



Most (by number) of the electron/positrons
and of photons in a "mature" shower
(that is those not at high energy)
have spectra of a "universal shape"
that is determined by the *"age"*
(that is by the stage of shower development)

Growing" (pre-maximum) showers
have harder spectra , with respect to
Decreasing (post-maximum) showers

Result extensively tested
by numerical simulations

"Euristic method" (developed by Greisen) to compute the average evolution of an electromagnetic shower.

Size of a shower
number of electrons above a minimum energy $E_{\rm min}$

$$N_e(E_{\rm min}, t) = \int_{E_{\rm min}}^{E_0} dE \; n_e(E, t)$$

In first approximation for a short interval of t one can write:

$$N_e(E_{\rm min}, t) \approx K \left(\frac{E_{\rm min}}{E_0} \right)^{-s} e^{\lambda(s)\, t}$$

$$N_e(E_{\min}, t) \approx K \left(\frac{E_{\min}}{E_0} \right)^{-s} e^{\lambda(s)\, t}$$

This form is valid in the energy range

$\varepsilon \lesssim E \lesssim E_0$ (and not too close to the limits)

A reasonable first order approximation for the total size

$$\lim_{E_{\min}\to 0} N_e(E_{\min}, t) \approx K \left(\frac{\varepsilon}{E_0} \right)^{-s} e^{\lambda(s)\, t}$$

$$N(t) = N_e(E_{\min}, t)$$

Use a more compact notation

$$\boxed{\frac{dN(t)}{dt} \simeq \lambda(s)\; N(t)}$$

$s(t)$

$$N_e(t) \approx K \left(\frac{E_{\min}}{E_0}\right)^{-s} e^{\lambda(s)\,t}$$

$$\boxed{\frac{dN(t)}{dt} = \left[\lambda(s) + [\lambda'(s)\,t + \ln(E_0/E_{\min})]\,\frac{ds}{dt}\right]\; N(t)}$$

$$\boxed{\lambda'(s)\,t + \ln(E_0/E_{\min}) = 0}$$

We obtain this “constraint”

$$\boxed{\lambda'(s)\,t + \ln(E_0/E_{\min}) = 0}$$

Constraint
on the evolution
of the shower

$$\lambda'(s) = \frac{1}{2}\left(1 - \frac{3}{s}\right)$$

Approximate
form of $\lambda(s)$

$$\lambda(s) = \frac{1}{2}(s - 1 - 3\ \ln s)$$

$$\boxed{\frac{1}{2}\left(1 - \frac{3}{s}\right)\ t + \ln\left(\frac{E_0}{E_{\min}}\right) = 0}$$

Can rewrite the
constraint in
simple, explicit form

$$\frac{1}{2}\left(1-\frac{3}{s}\right)t+\ln\left(\frac{E_0}{E_{\min}}\right)=0$$

$$t = t_{\max}$$

$$s = 1$$

$$\boxed{t_{\max} = \ln\left(\frac{E_0}{E_{\min}}\right)}$$

The position of the shower maximum grows with the logarithm of the energy of the primary particle

$$\boxed{\frac{1}{2}\left(1-\frac{3}{s}\right)t+\ln\left(\frac{E_0}{E_{\rm min}}\right)=0}$$

More in general we can solve for s

$$s(t)=\frac{3\,t}{t+2\,\ln(E_0/E_{\rm min})}$$

$$s(t)=\frac{3\,t}{t+2\,t_{\rm max}}$$

This means that we
can estimate the shape
of the particle spectrum
knowing the “age” of the shower

$$\frac{dN}{dt} = \lambda[s(t)]\, N(t)$$

We know know $s(t)$

Can use the approximate form $\lambda(s)$

Obtain a simple closed form equation

$$\frac{dN}{dt} = \frac{1}{2}\left[\frac{3t}{t + 2\,t_{\max}} - 1 - 3\,\ln\left(\frac{3t}{t + 2\,t_{\max}}\right)\right]$$

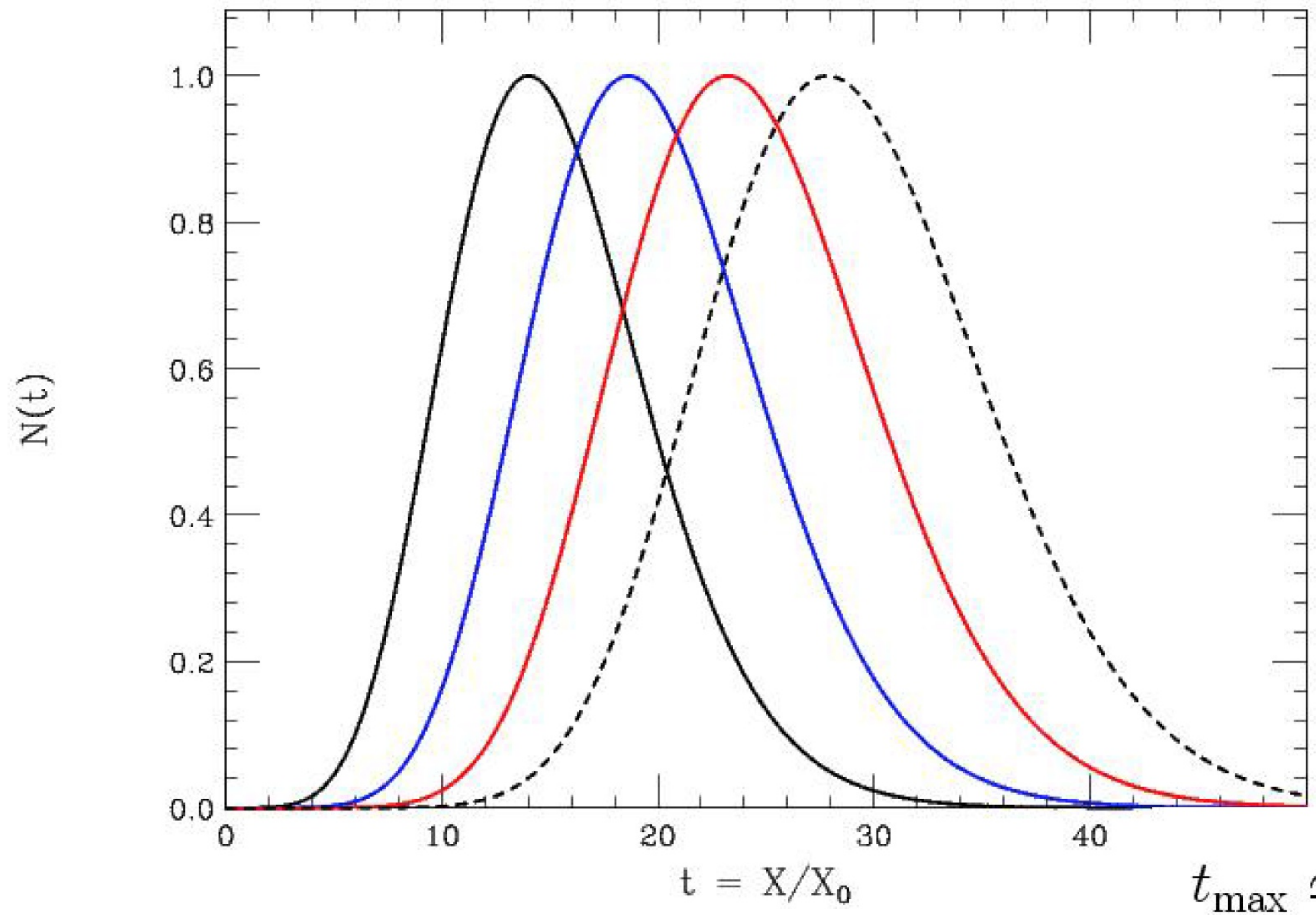
$$\frac{dN}{dt} = \frac{1}{2}\left[\frac{3t}{t+2\,t_{\mathrm{max}}} - 1 - 3\ \ln\left(\frac{3t}{t+2\,t_{\mathrm{max}}}\right)\right]$$

The equation has an analytic integral
and the result is

$$\boxed{N(t) = N_{\mathrm{max}}\ \exp\left[t\ \left(1 - \frac{3}{2}\ \ln\left(\frac{3t}{t+2\,t_{\mathrm{max}}}\right)\right)\right]}$$

“Greisen profile”

$$N(t) = N_{\max}\ \exp\left[t\ \left(1 - \frac{3}{2}\ \ln\left(\frac{3t}{t + 2\,t_{\max}}\right)\right)\right]$$



$$t_{\max} \simeq \ln\left(\frac{E_0}{E}\right)$$

“Greisen profile”

The "Greisen profile"

[average development of electromagnetic shower]

$$N_{\text{Greisen}}(t, E_0) = N_0 \ \exp\left[t\left(1 - \frac{3}{2}\log\left(\frac{3t}{t + 2\ \ln(E_0/\varepsilon)}\right)\right)\right]$$

$$t_{\max} = \ln\frac{E_0}{\varepsilon}$$

Position of Maximum

$$\sigma = \sqrt{\frac{3}{2}\ \ln\frac{E_0}{\varepsilon}}$$

Width of distribution

$$-\frac{1}{\sigma^2} = \left.\frac{d^2 N(t)}{dt^2}\right|_{t=t_{\max}}$$

$$\int_0^\infty dt\ N(t) \simeq \sqrt{2\pi}\, N_{\max}\, \sigma\ \left[1 + \frac{0.176}{\sigma^2} + \ldots\right]$$

$$\varepsilon\ \int_0^\infty dt\ N(t) = E_0$$

$$N_{\max} \approx \frac{E_0}{\varepsilon}\ \frac{1}{\sqrt{\ln(E_0/\varepsilon)}}$$

$$\tau = t - t_{\max}$$

$$\ln \frac{N(t)}{N_{\max}} = -\frac{\tau^2}{2\,\sigma^2} + \frac{5\,\tau^3}{12\,\sigma^4} - \frac{3\,\tau^4}{8\,\sigma^8} + \ldots$$

The Greisen profile can be understood qualitatively as a Gaussian

centered at $t_{\max}$

With width σ

with a small asymmetric distortion (tail at large t) that become negligible with at very large energy

Main results for e.m. shower development

$$X_{\mathrm{max}}(E) \simeq \lambda_{\mathrm{rad}} \ \ln\left(\frac{E}{\varepsilon}\right)$$

Logarithmic growth of the penetration.

$$N_{\mathrm{max}}(E) \simeq \frac{E}{\varepsilon} \frac{1}{\sqrt{\ln(E/\varepsilon)}}$$

Energy Conservation

Elongation rate = 85 (g/cm^2)/decade

$$X_{\max}(E) \approx D \;\log_{10} E + X_0$$

D Elongation rate

$$X_{\max}(E) \simeq \lambda_{\mathrm{rad}} \;\ln\left(\frac{E}{\varepsilon}\right)$$

e.m. showers

$$D \simeq \lambda_{\mathrm{rad}} \;\ln 10 \simeq 85 \text{ g cm}^2$$

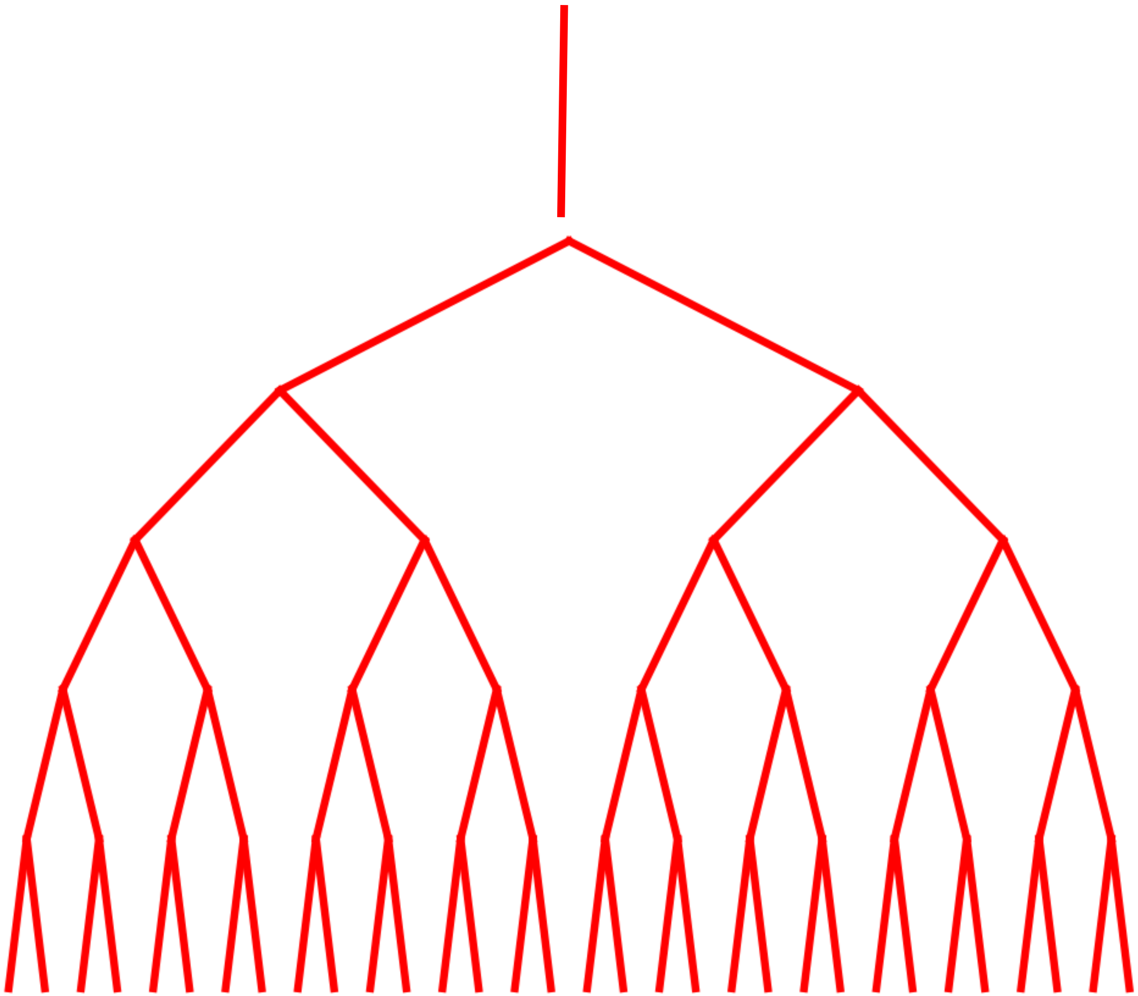
Elongation rate for electromagnetic shower

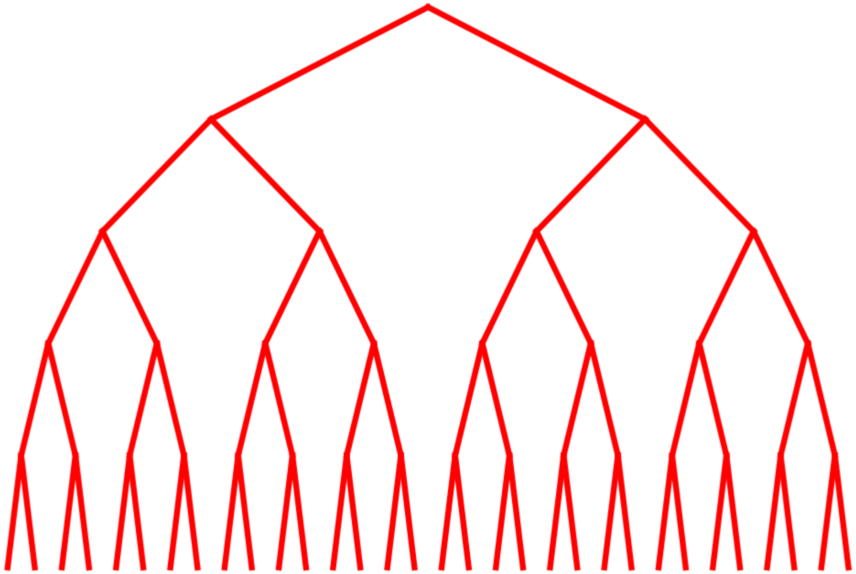
Heitler toy model for electromagnetic showers

“Electron-photon” particle

Splitting length λ

Critical energy ε





Development stops at n:

$$\frac{E}{2^n} = \varepsilon$$

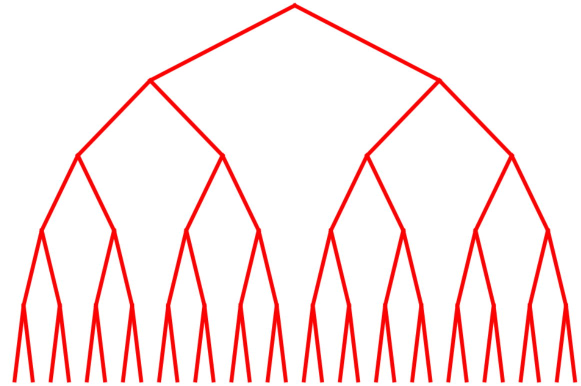
$$n = \log_2 \left(\frac{E}{\varepsilon} \right)$$

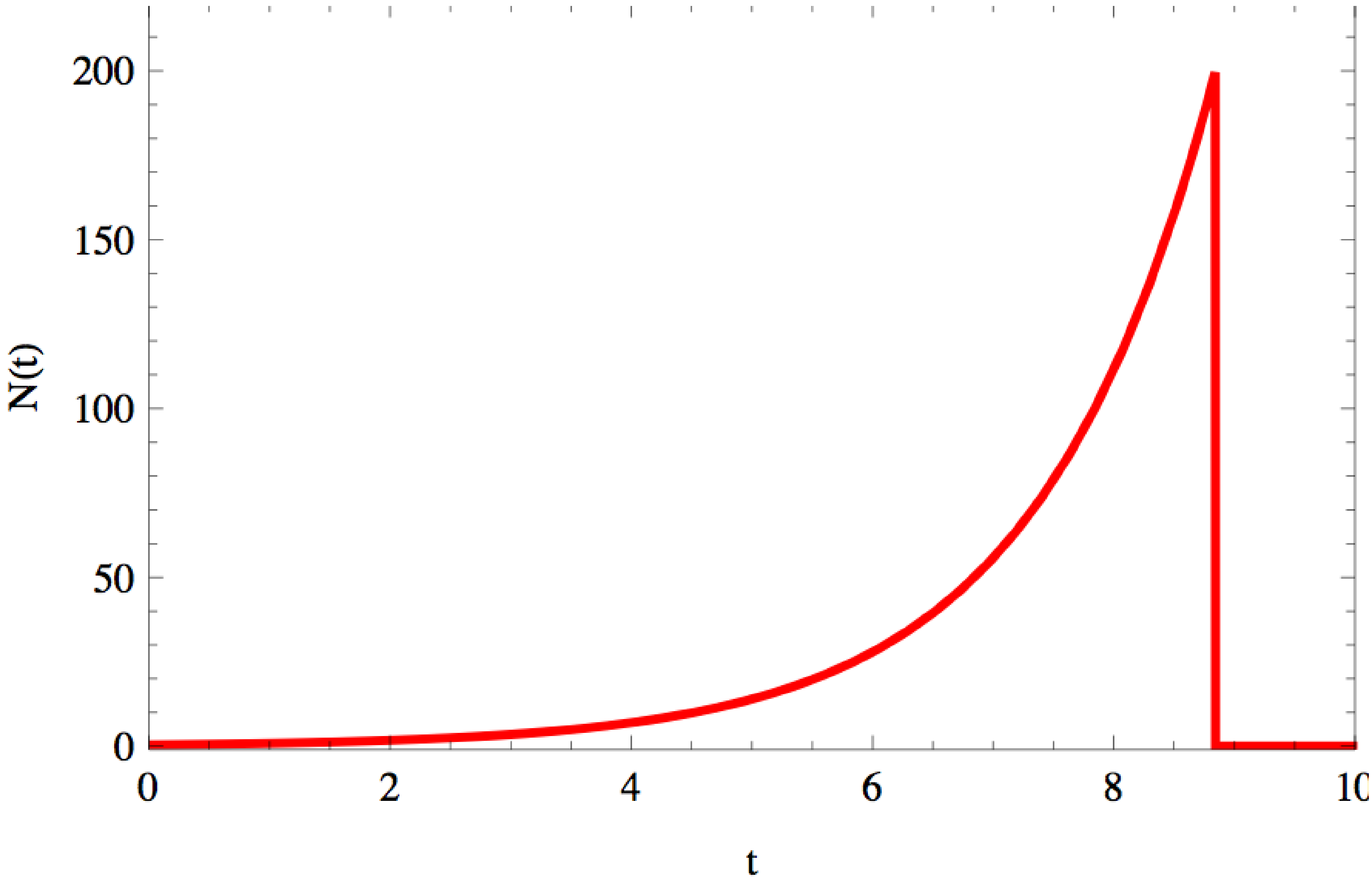
$$N_{\max} = 2^n = 2^{\log_2 E/\varepsilon} = \frac{E}{\varepsilon}$$

$$X_{\max} = \lambda\, n = \lambda\ \log_2 \left(\frac{E}{\varepsilon} \right) \quad = \frac{\lambda}{\ln 2}\ \ln \left(\frac{E}{\varepsilon} \right)$$

$$= \lambda_{\mathrm{rad}}\ \ln \left(\frac{E}{\varepsilon} \right)$$

Shower development
in Heitler toy model:

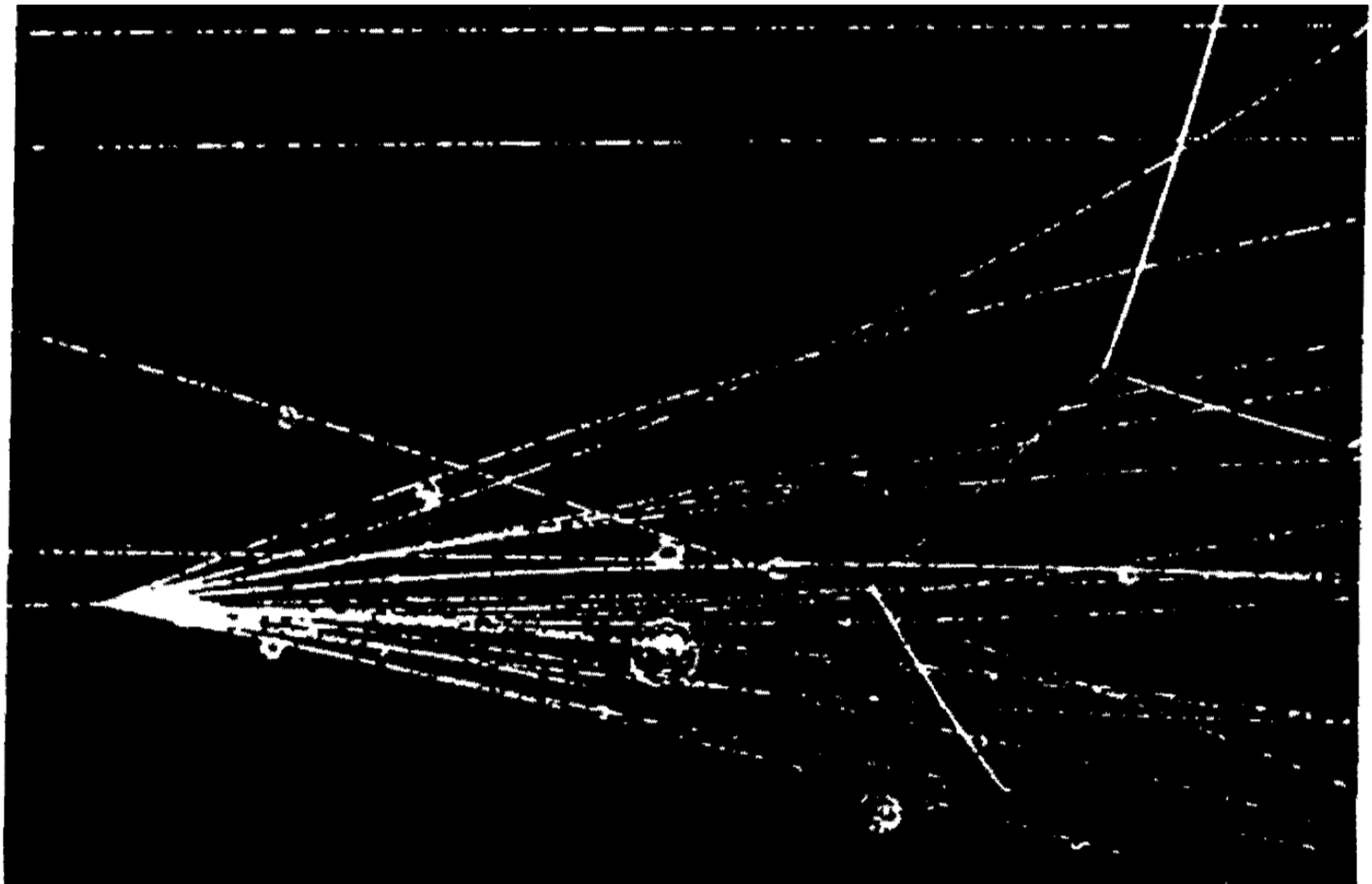




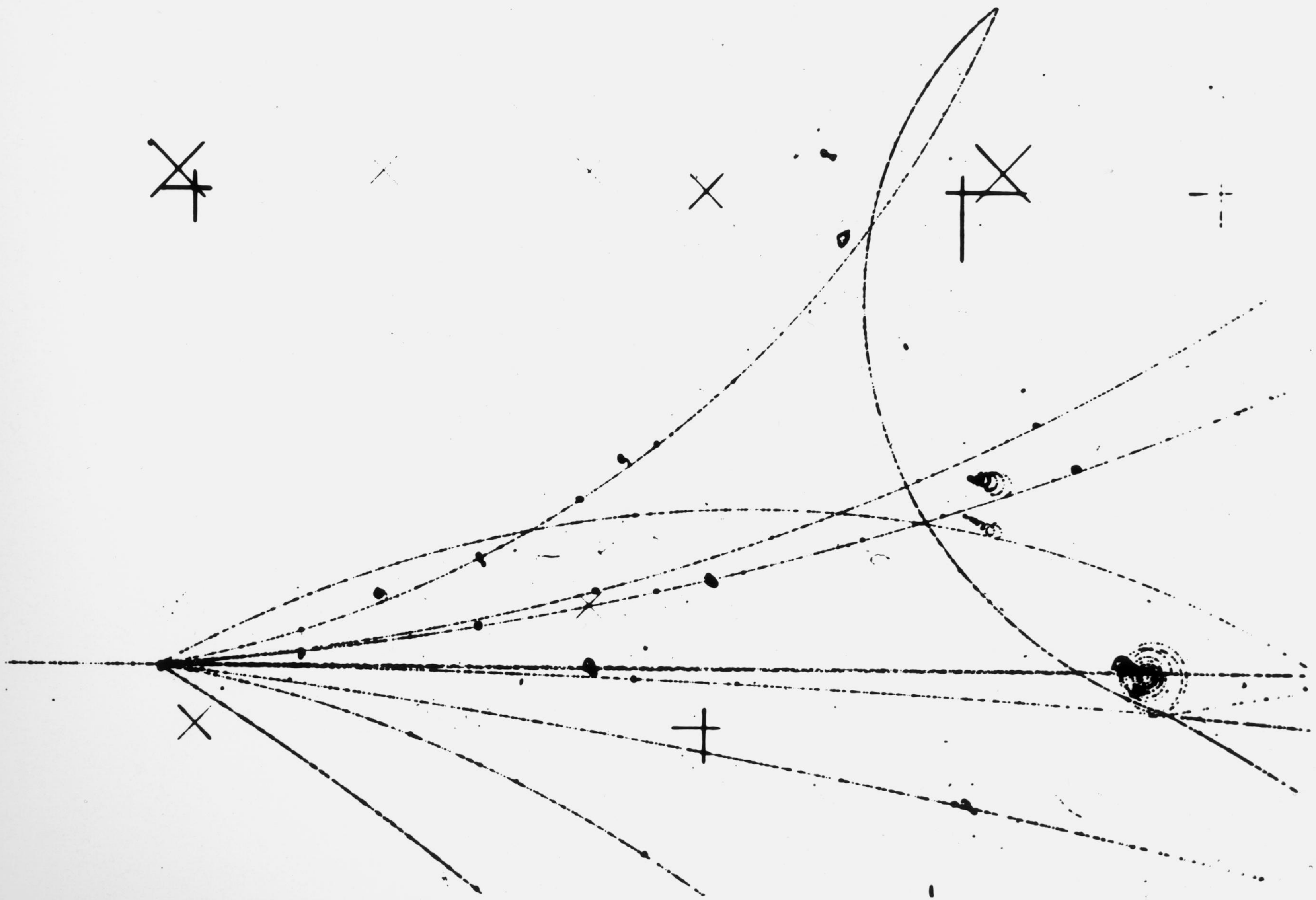
Shower generated by hadrons [protons and Nuclei]

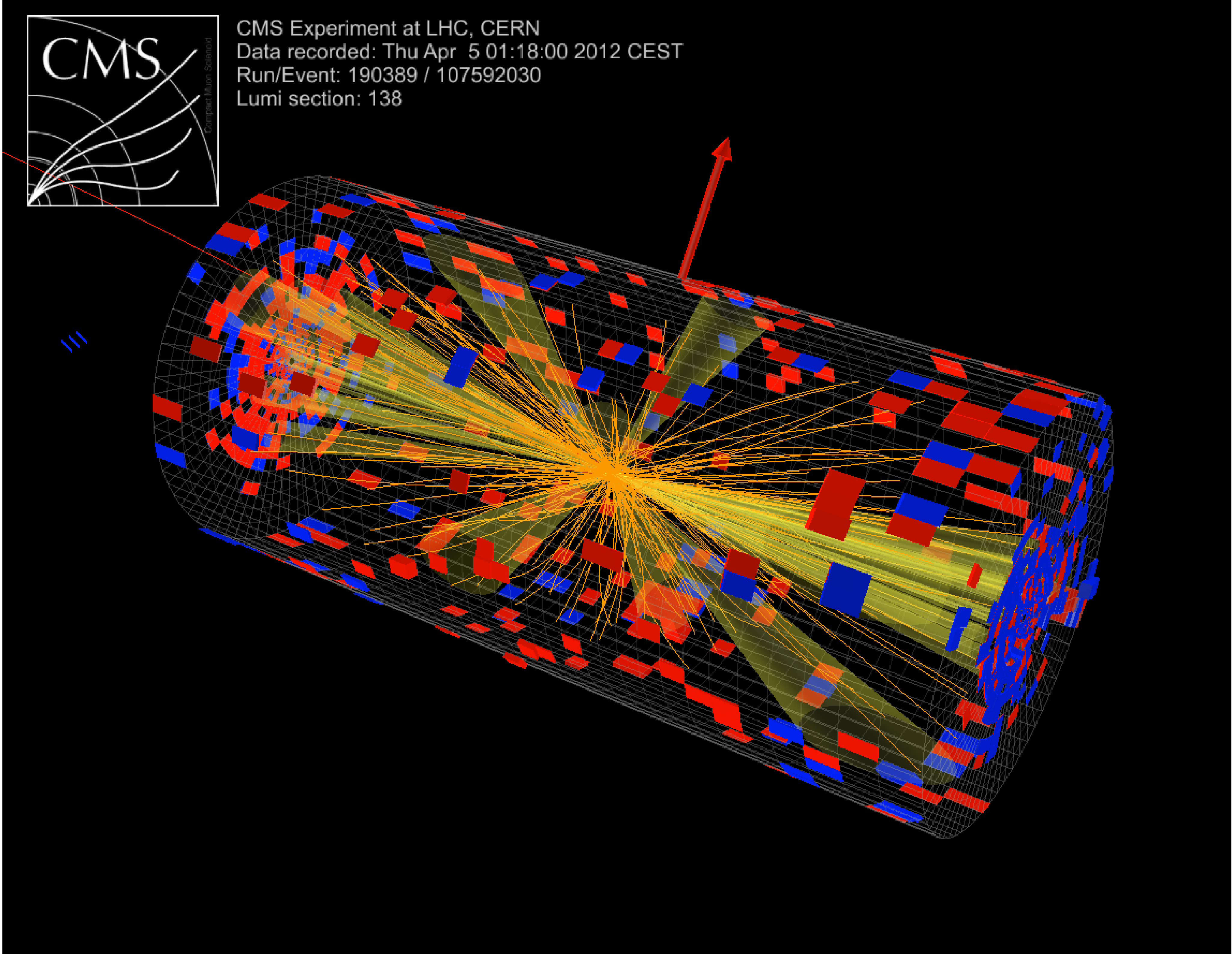
A much more complicated problem
where we are not able to reliably
compute from first principles the
properties of the shower development

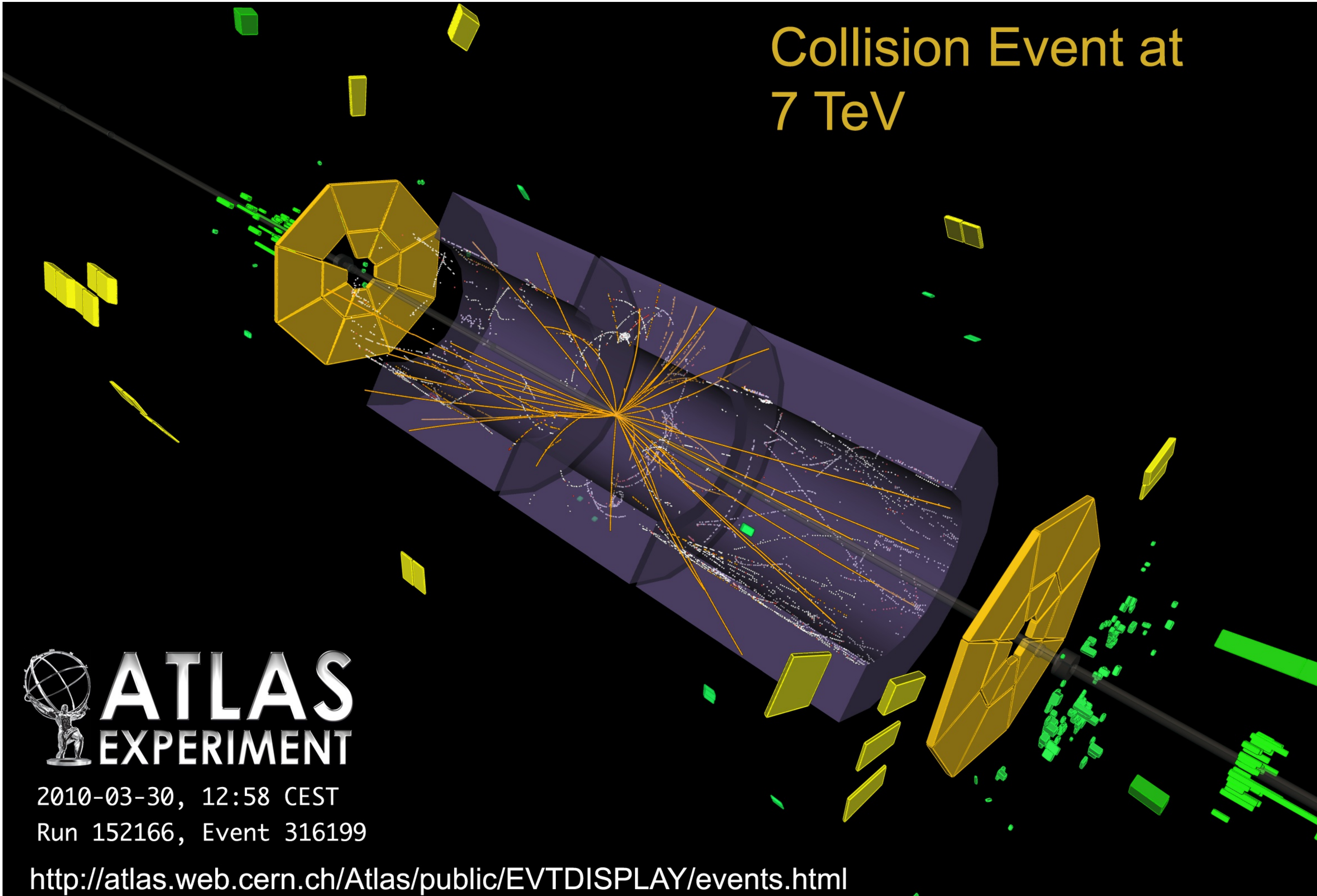
Interaction between a proton of 28 GeV/c and a proton of the hydrogen bubble chamber. In the interaction are produced 16 charged particles beside several neutral particles (by courtesy of CERN, Geneva).



One of the first photographs taken with the thirty-inch bubble chamber at NAL. June 15, 1972. A two-hundred GeV proton enters the chamber and interacts with the liquid hydrogen. The resulting collision produces a spectacular event with ten visible nuclear fragments emerging. The tracks are nearly thirty inches long.




CMS
CMS Experiment at LHC, CERN
Data recorded: Thu Apr 5 01:18:00 2012 CEST
Run/Event: 190389 / 107592030
Lumi section: 138



Collision Event at
7 TeV
ATLAS
EXPERIMENT
2010-03-30, 12:58 CEST
Run 152166, Event 316199
http://atlas.web.cern.ch/Atlas/public/EVTDISPLAY/events.html

$$p + p \rightarrow p(n) + p(n)$$

$$+ \pi^{+} + \pi^{-} + \pi^{\circ}$$

$$+ K^{+} + K^{-} + K^{\circ} + \overline{K^{\circ}}$$

$$+ p(n) + \overline{p}(\overline{n})$$

$$+ \ldots [e^{\pm}, \mu^{\pm}, \text{charm, heavy quarks}]$$

$$\rho^{+} \rightarrow \pi^{+} + \pi^{0}$$

$$\omega_{0} \rightarrow \pi^{+} + \pi^{-} + \pi^{\circ}$$

$$\Delta^{++} \rightarrow p + \pi^{+}$$

$$p + p \rightarrow p(n) + p(n)$$

"Leading" baryons
40-50% of energy

$$+ \pi^{+} + \pi^{-} + \pi^{\circ}$$

most abundant particles

$$+ K^{+} + K^{-} + K^{\circ} + \overline{K^{\circ}}$$

strange mesons

$$+ p(n) + \overline{p}(\overline{n})$$

baryon/
anti-baryon pairs

$$+ \ldots [e^{\pm}, \mu^{\pm}, \mathrm{charm}, \mathrm{heavy\ quarks}]$$

$$p + p \to p(n) + p(n)$$

$$+ \pi^{+} + \pi^{-} + \pi^{\circ}$$

$$[u\bar{d}] \quad [d\bar{u}] \quad \left([u\bar{u}] + [d\bar{d}]\right) / \sqrt{2}$$

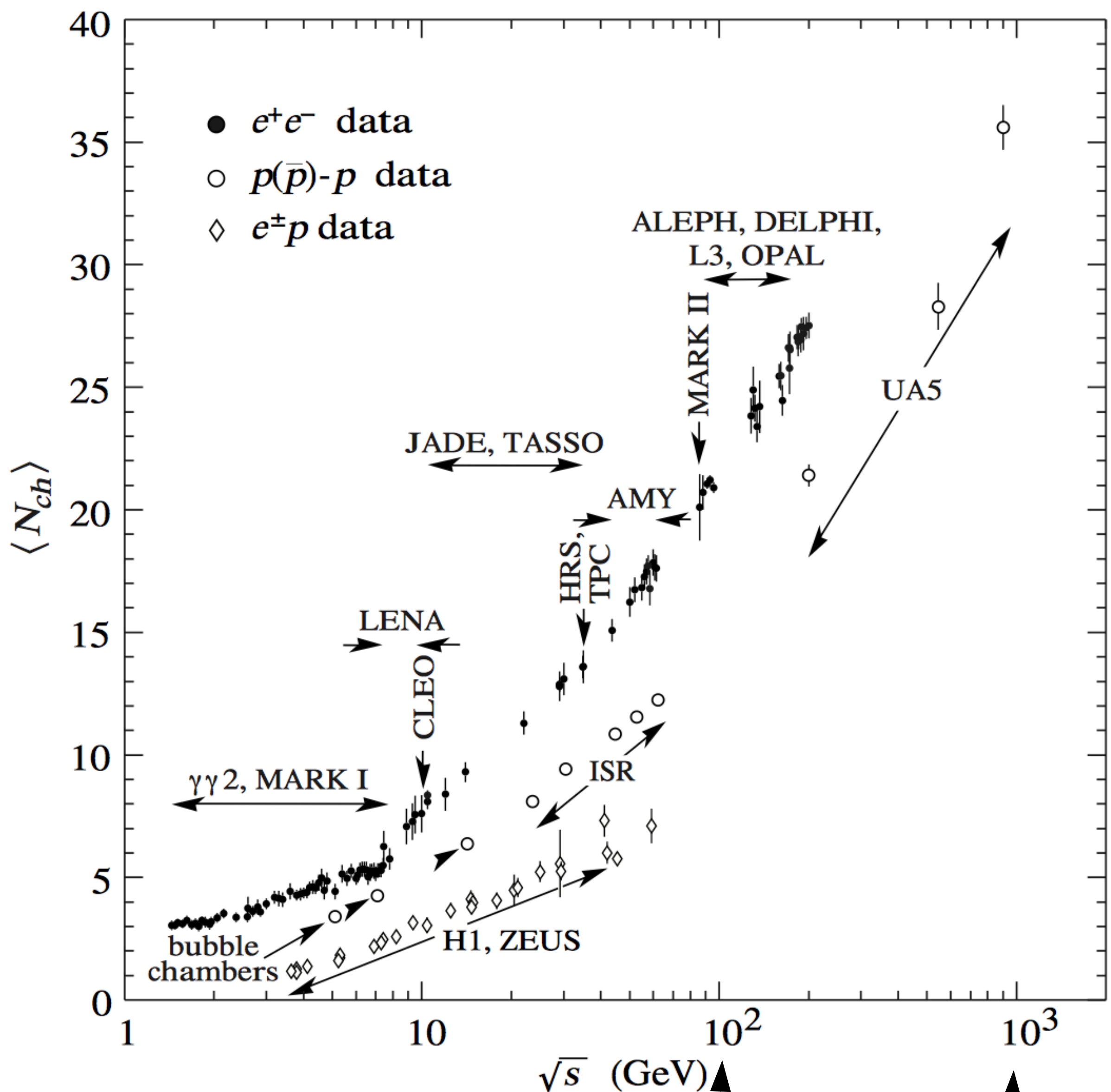
$$+ K^{+} + K^{-} + K^{\circ} + \overline{K^{\circ}}$$

$$[u\bar{s}] \quad [s\bar{u}] \quad [d\bar{s}] \quad [s\bar{d}]$$

$$+ p(n) + \overline{p}(\overline{n})$$

$$+ \ldots [e^{\pm}, \mu^{\pm}, \text{charm, heavy quarks}]$$

Average Charged Multiplicity



5.3 TeV

530 TeV

$$\pi^{\circ} \to \gamma\ \gamma$$

Neutral pions are the source
of an electromagnetic component
(very rapid decay)

$$\pi^{+} \to \mu^{+}\ \nu_{\mu}$$

$$\pi^{-} \to \mu^{-}\ \overline{\nu}_{\mu}$$

Competition between
interaction and decay

"Critical energy"
[separate decay from interactions]

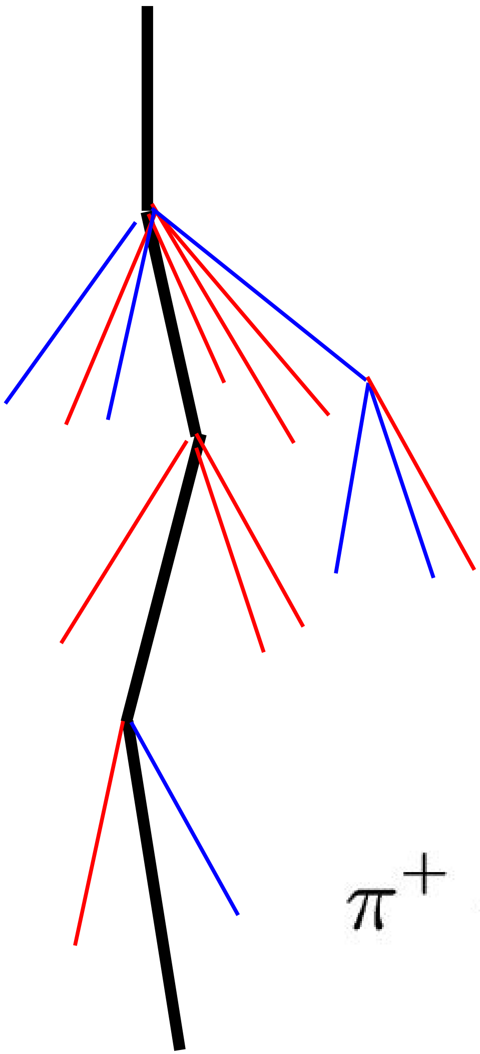
$$E^{*}_{\pi} \approx 120\ \mathrm{GeV}$$

$$\lambda^{\mathrm{int}}_{\pi^{\pm}} \approx 120\ \mathrm{gcm}^{-2}$$

$$\lambda_{\mathrm{decay}}(E_{\pi}) = \langle\rho\rangle\ \frac{1}{\beta\,\gamma} = \langle\rho\rangle\ \frac{m_{\pi}}{\beta\,E_{\pi}}$$

Proton Shower

Vertices : theoretically not understood (and not exactly scaling)



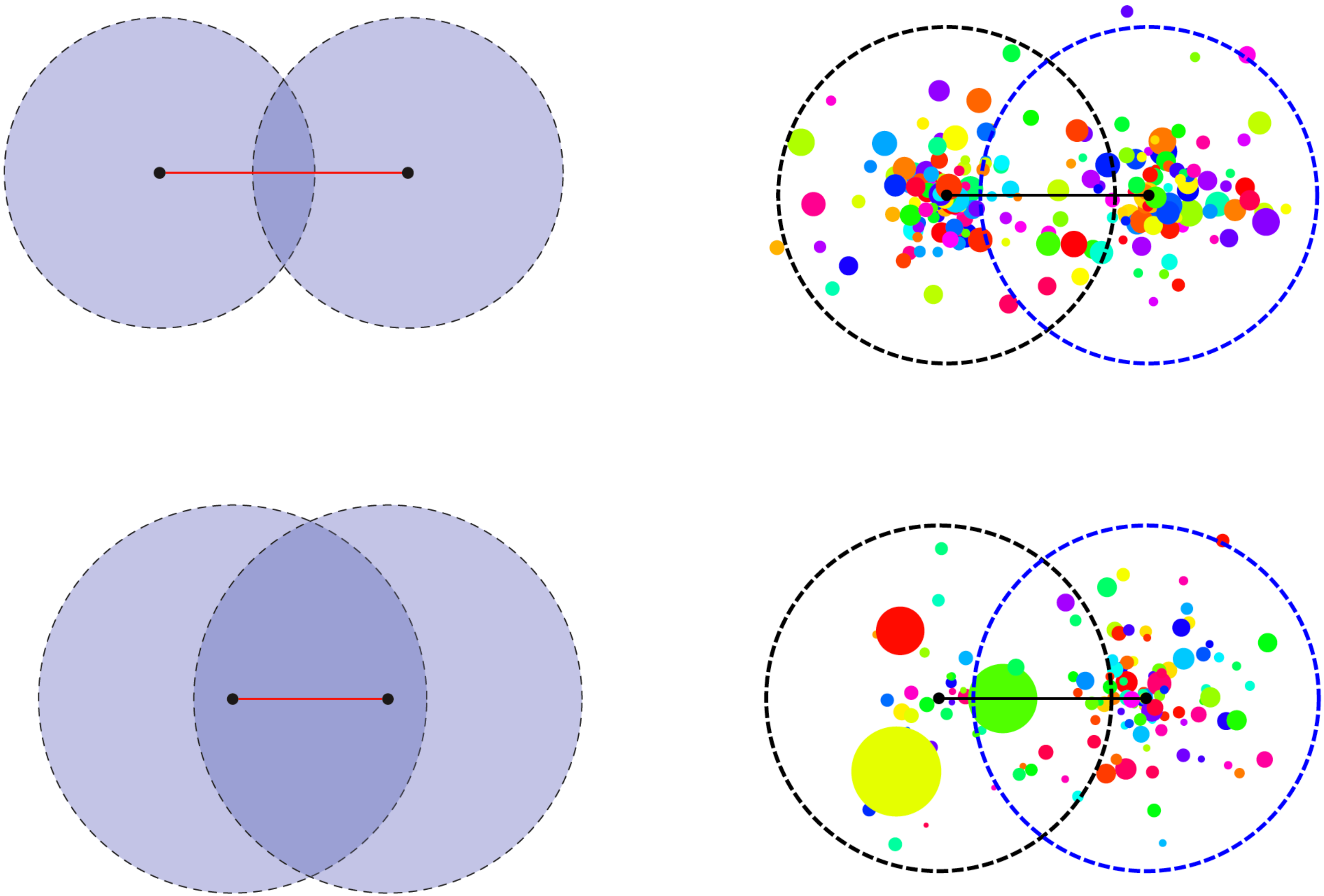
$$\pi^0 \to \gamma\gamma$$

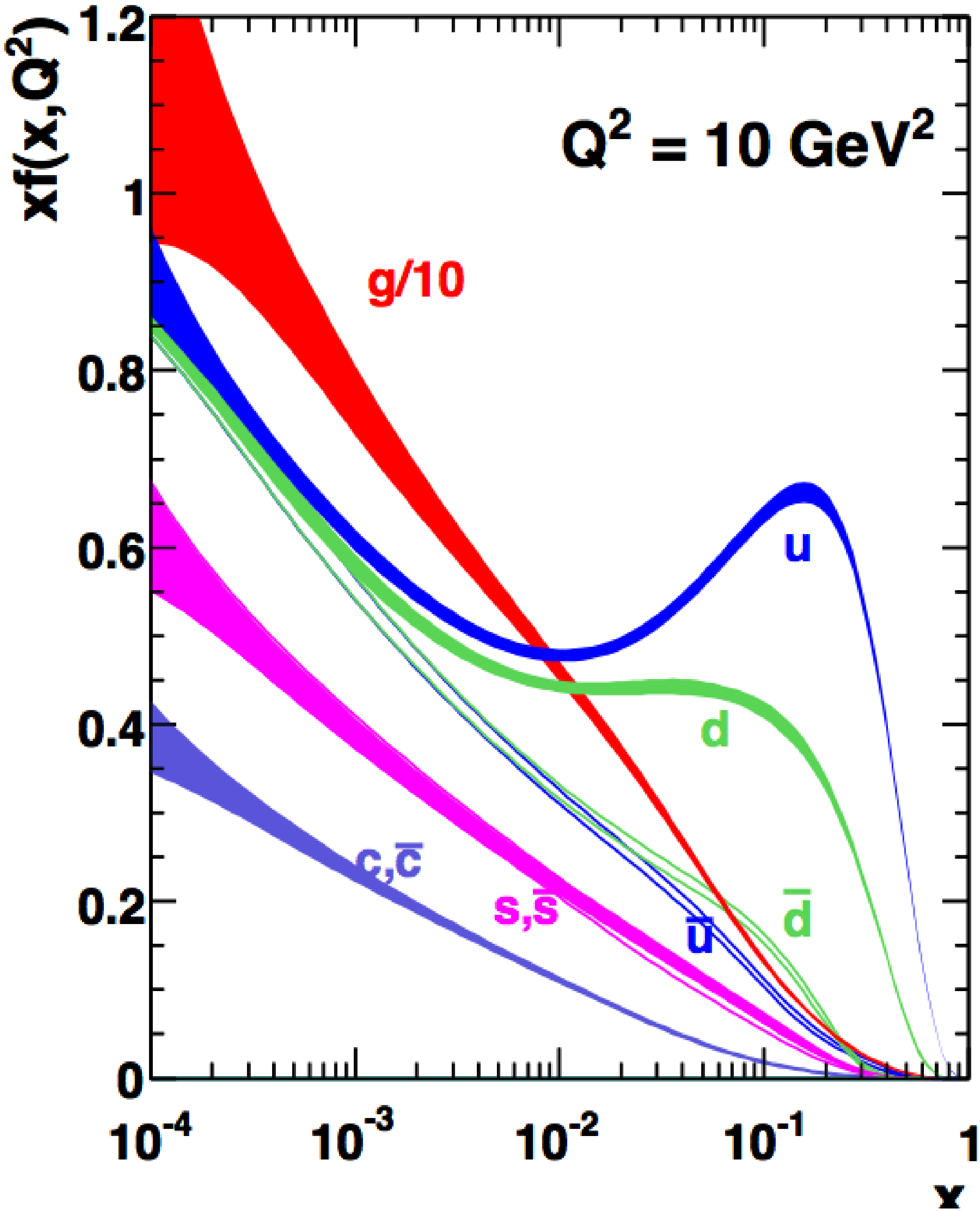
$$\pi^+ \to \mu^+ \nu_\mu$$

Quantum
Chromo
Dynamics

“Confinement”

“Asymptotic freedom”





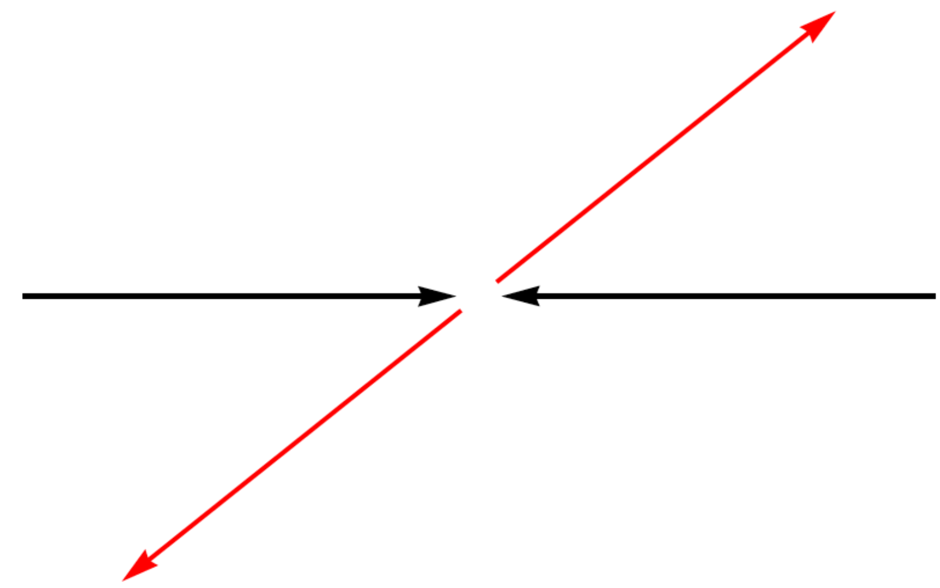
Parton Distribution Functions

$$f_j(x) \propto \frac{1}{x^{1+\delta}}$$

Rapid growth for x → 0

$$e^{+} + e^{-} \to \text{hadrons}$$

$$e^{+} + e^{-} \to q + \overline{q}$$



$$\frac{d\sigma}{d\cos\theta} = \frac{2\ \alpha^2\ q^2}{3\ E^2}\ (1 + \cos^2\theta)$$

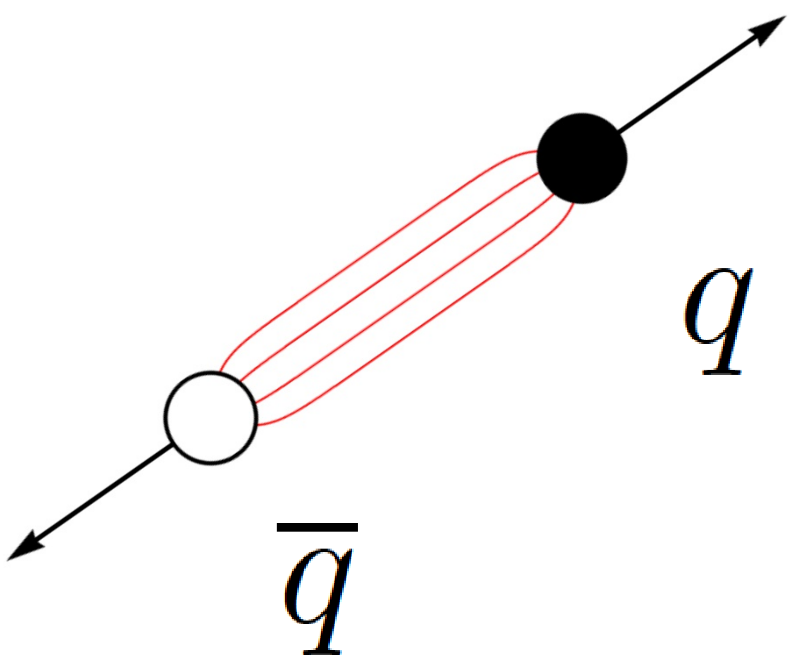
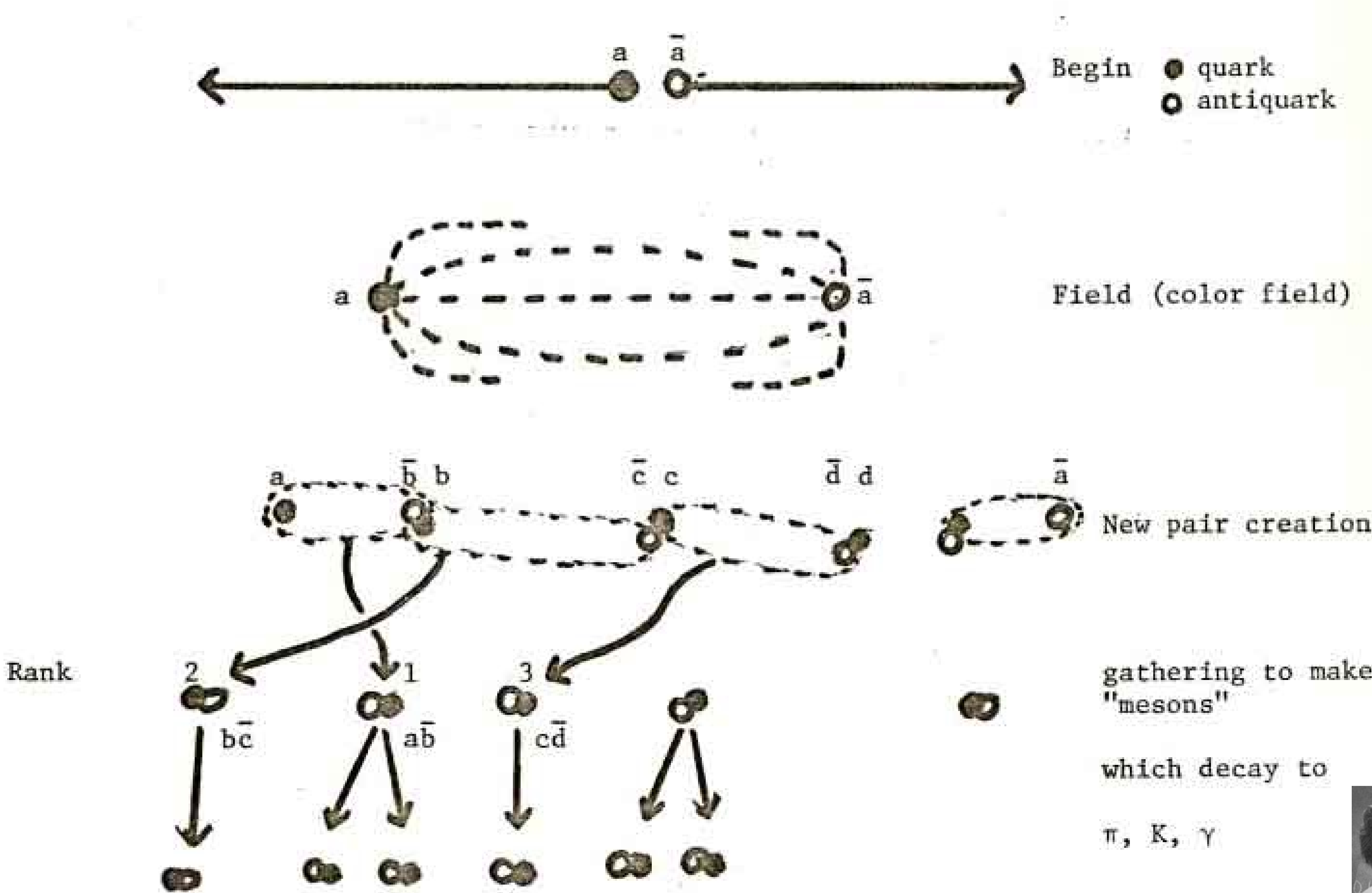
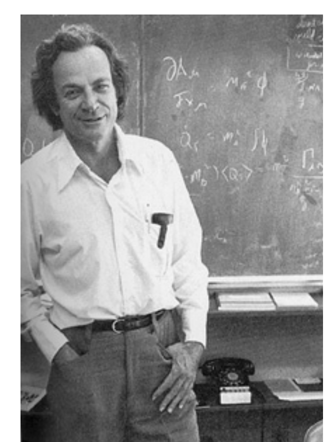
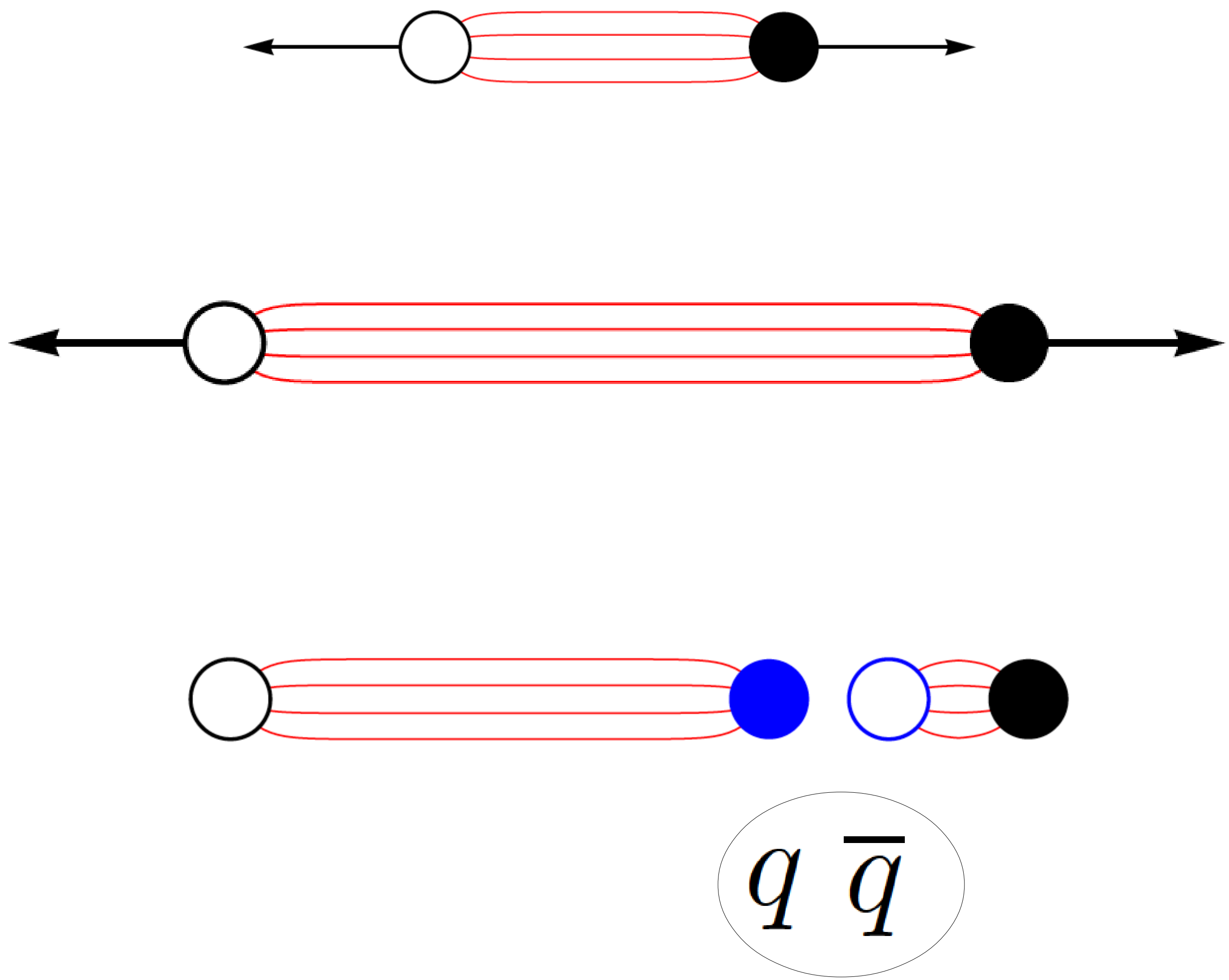


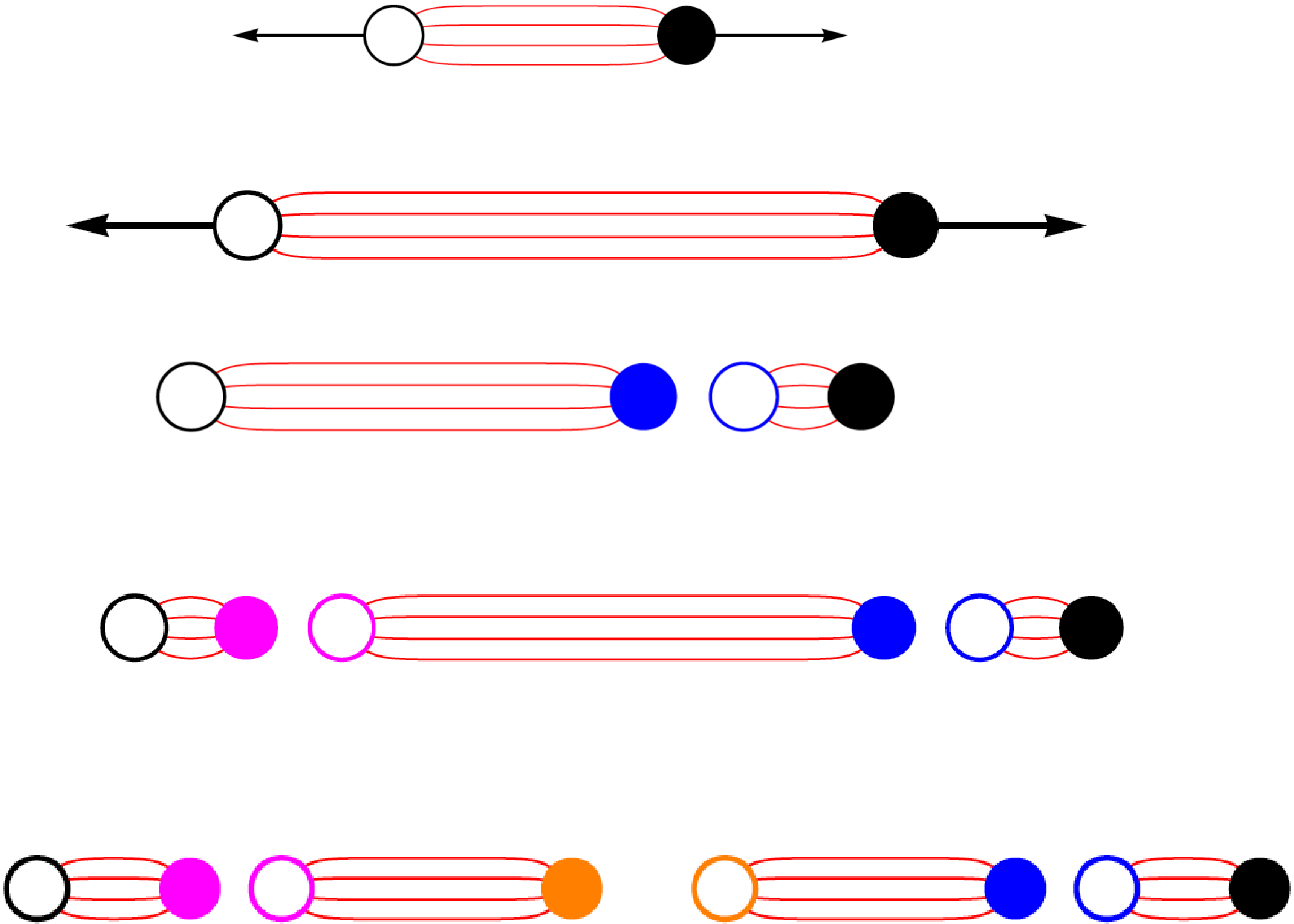
Fig. 1. An e^+e^- Annihilation

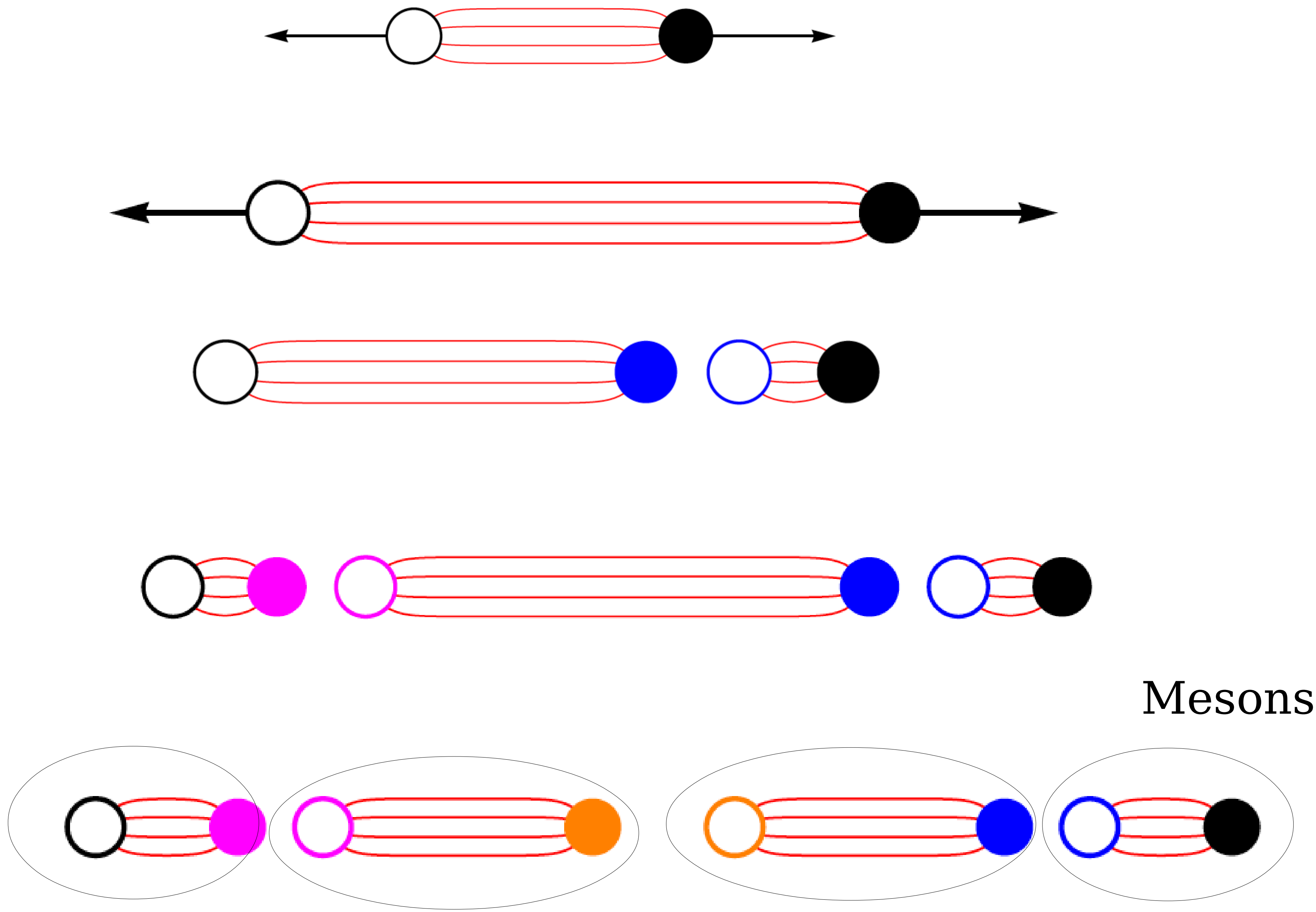




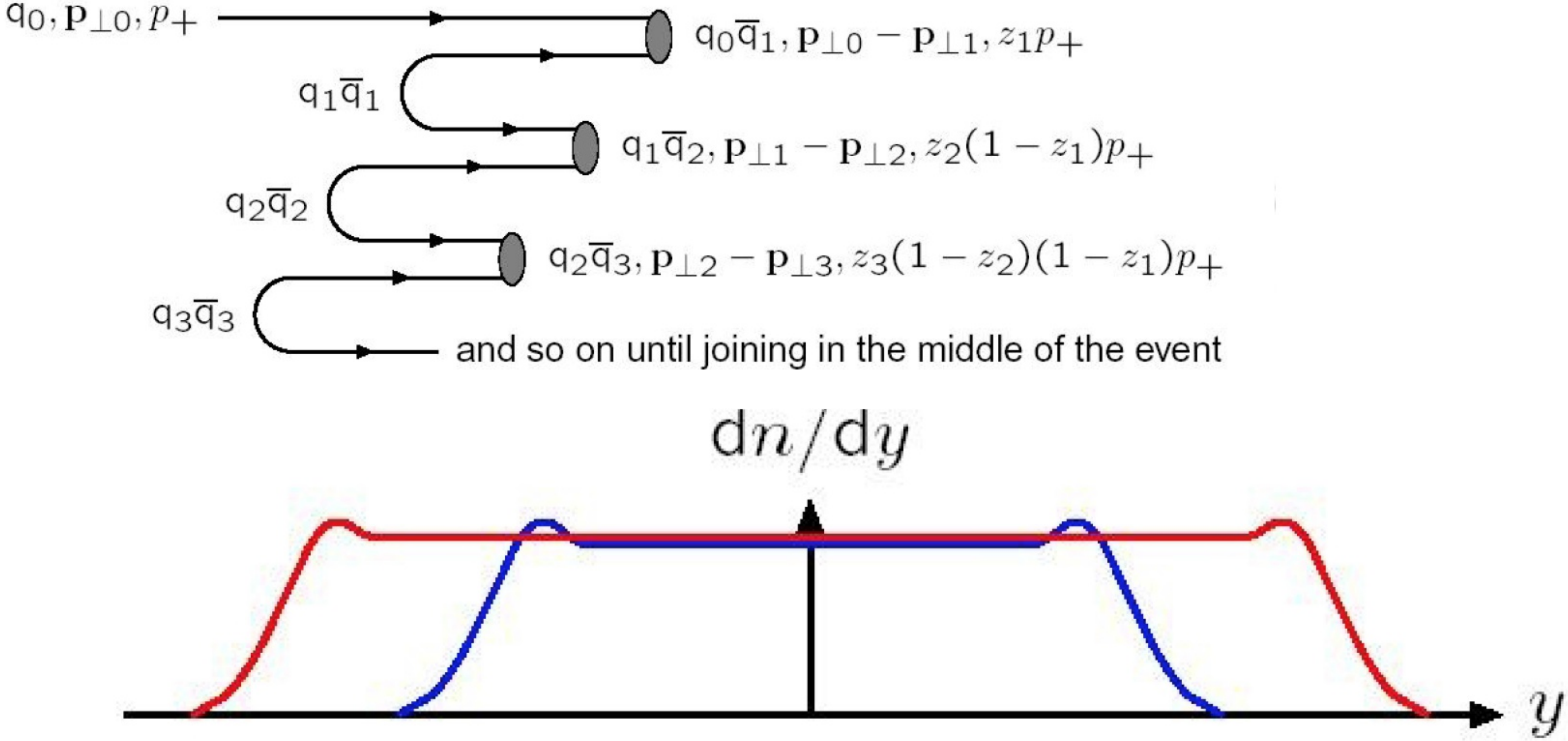
Field -Feynman : Quark - Fragmentation


$q\ \bar{q}$

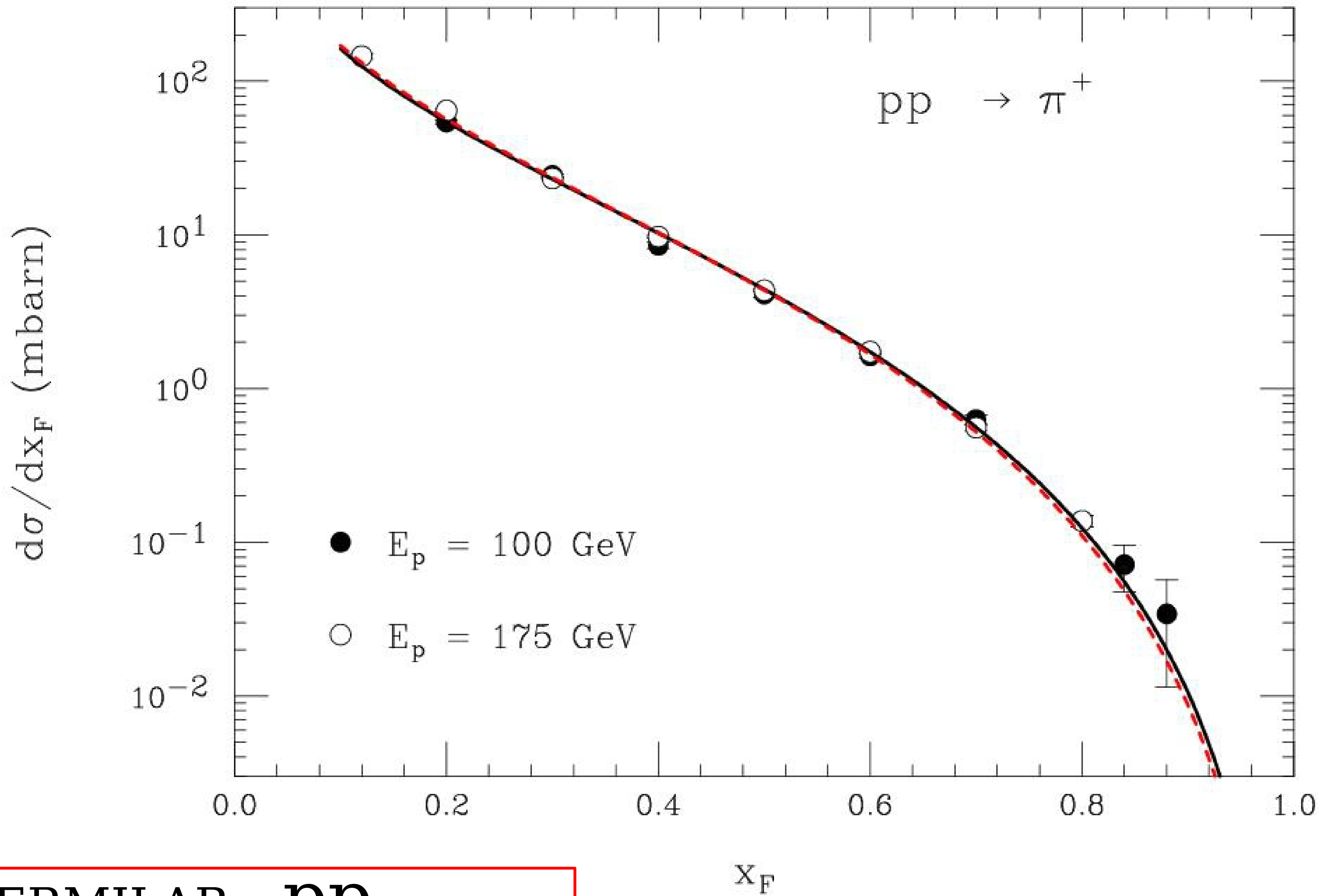



Mesons

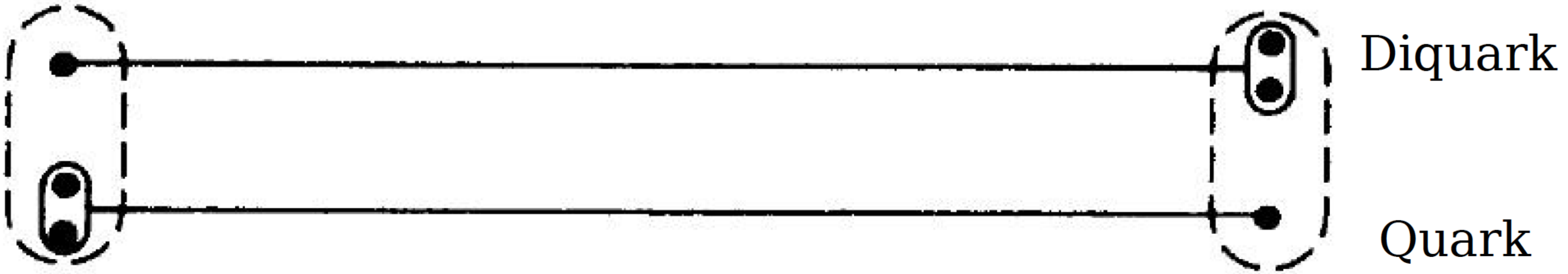
The (iterative) Fragmentation of one COLOR STRING produces a SCALING SPECTRUM of HADRONS



Phenomenological Evidence for SCALING



FERMILAB: pp
Brenner et al (1982)



Basic Structure of
a NON diffractive PP interactions
is made of TWO STRINGS

Color Structure

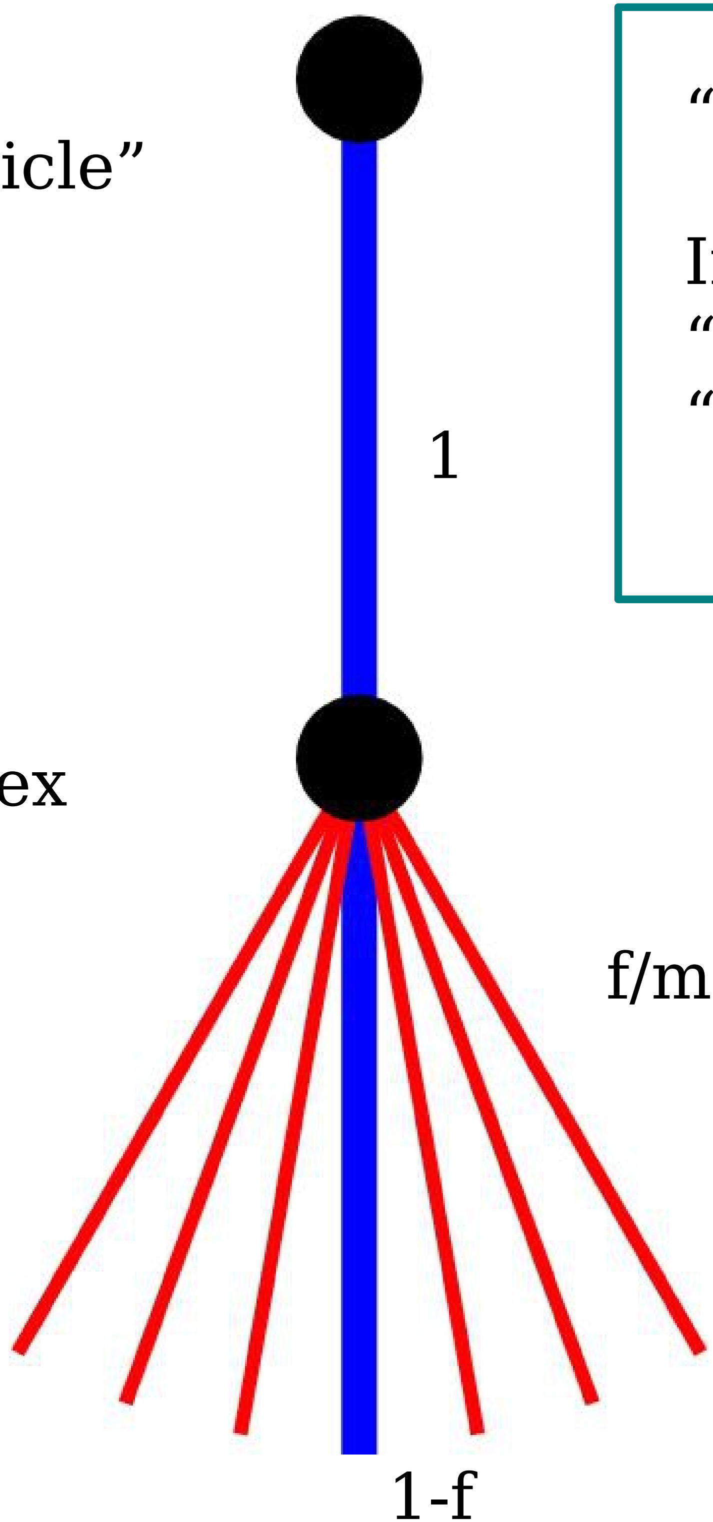
$$3 \otimes 3 = \overline{3} \oplus 6$$

$$3 \otimes \overline{3} = 1 \oplus 8$$

“Hadronic particle”

Hadronic
Interaction
Length

Hadronic vertex

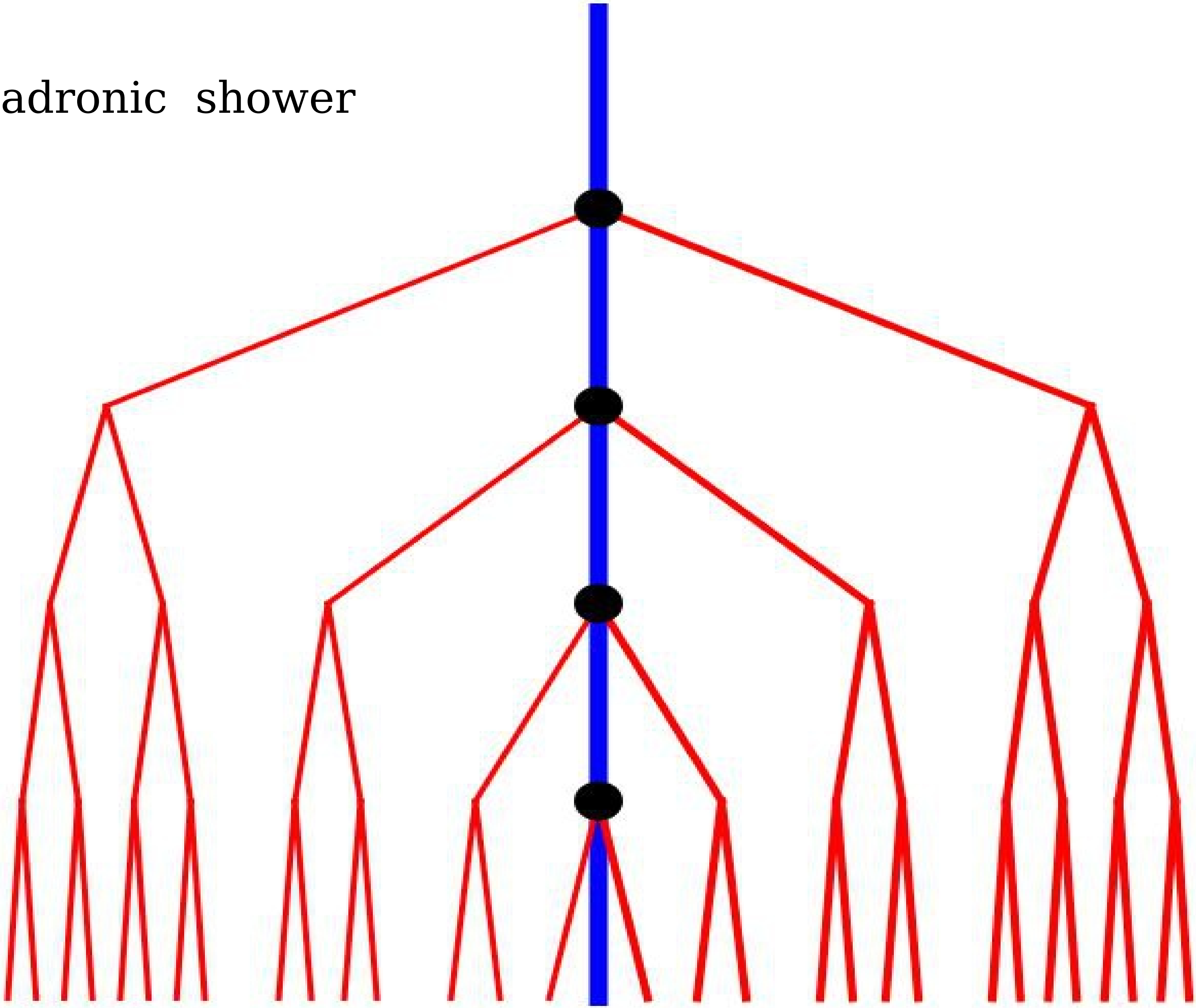


“Hadron”

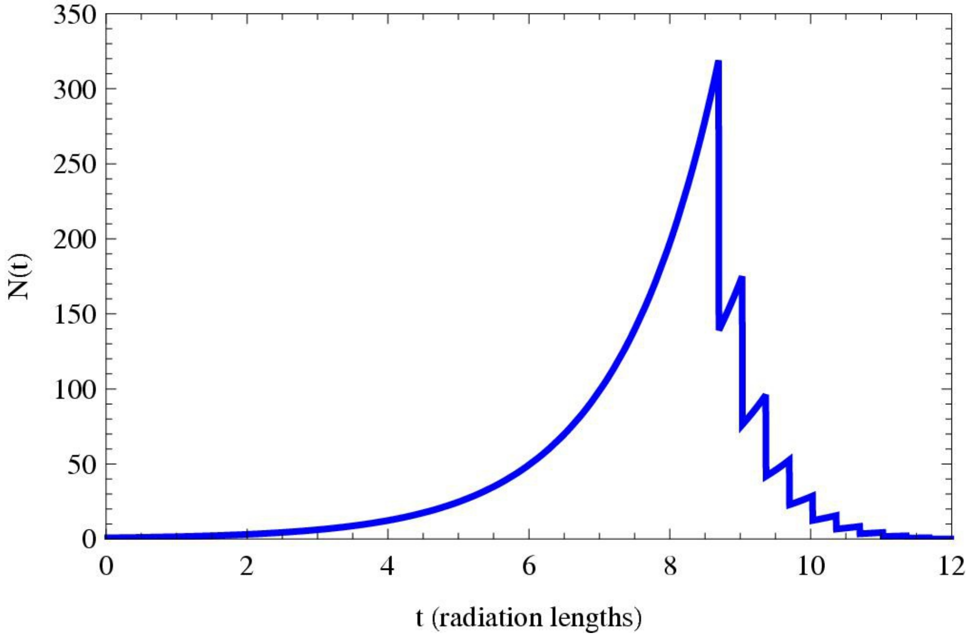
Interaction Length π
“Inelasticity” f
“multiplicity m

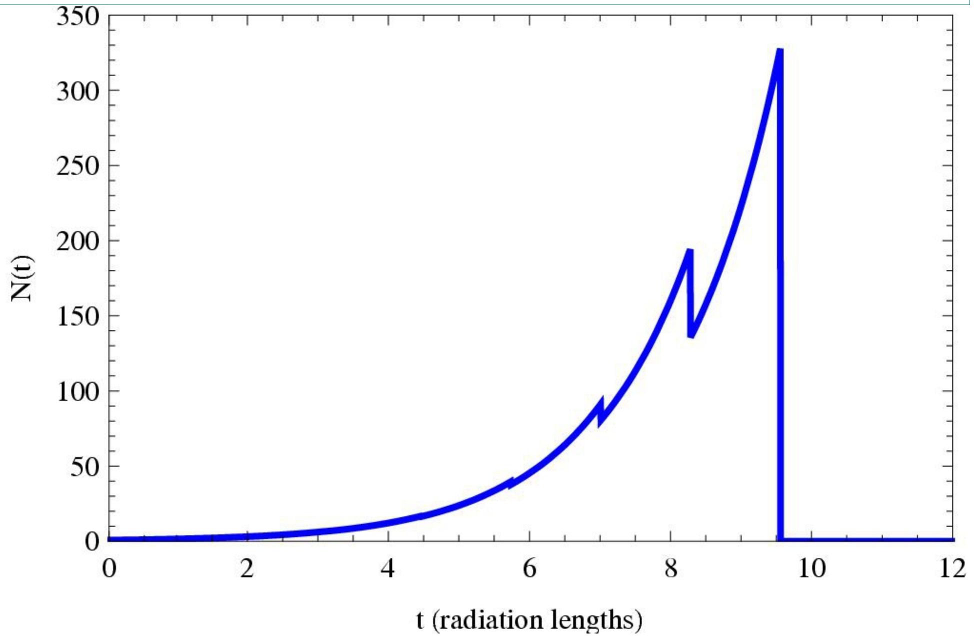
Energy sharing
$1 \rightarrow (1\text{-}f) +$
$f/m +$
$f/m +$
$f/m +$
$f/m +$
…….

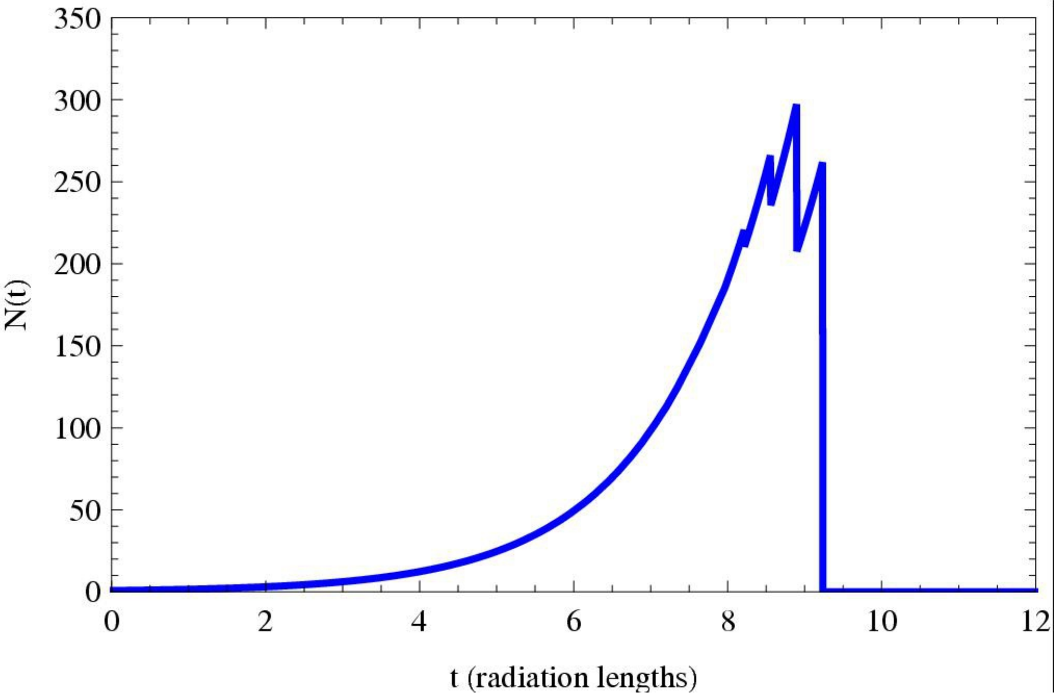
Hadronic shower

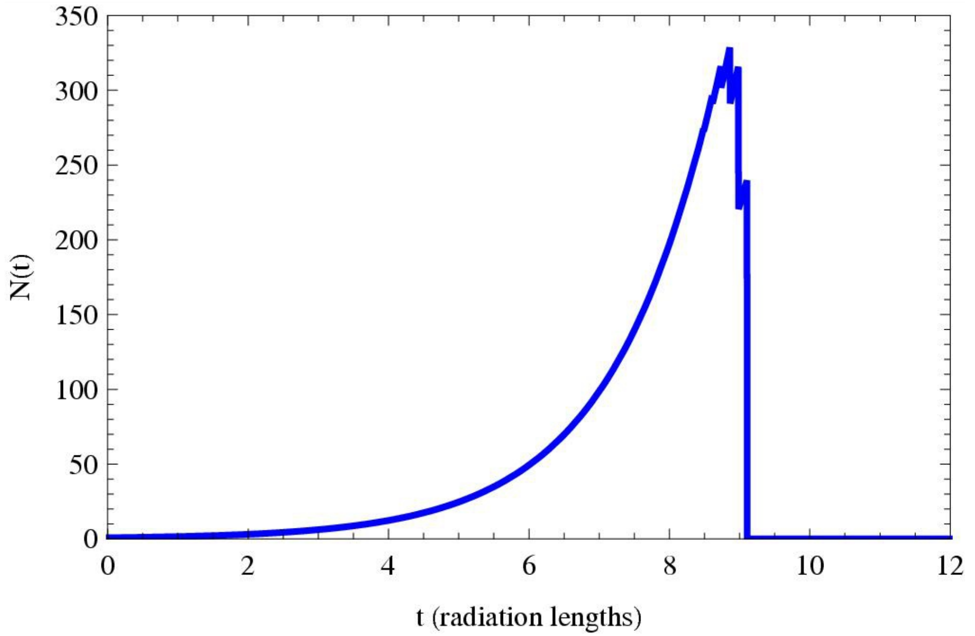


Hadronic parameters π, inelasticity, hardness

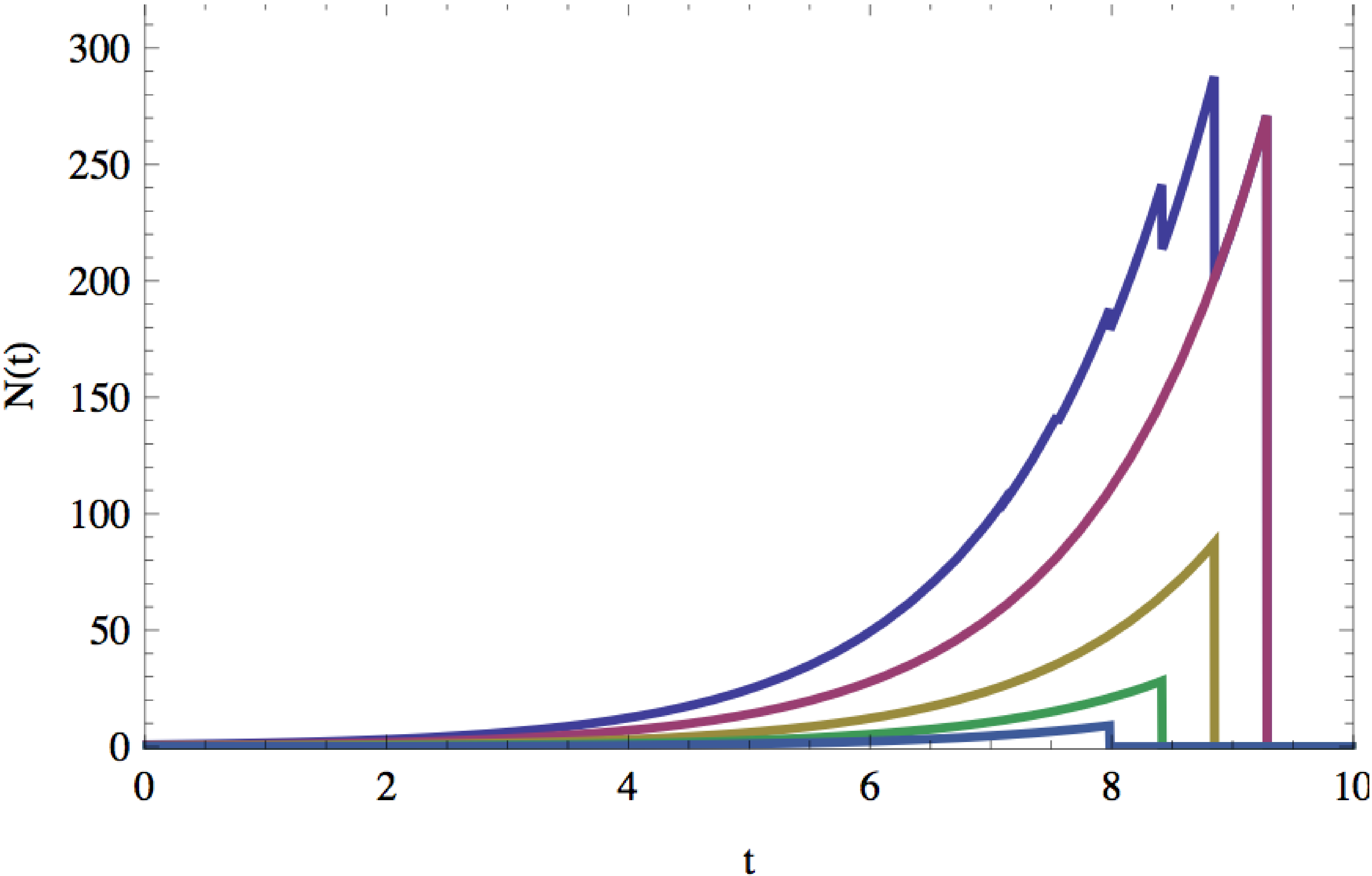


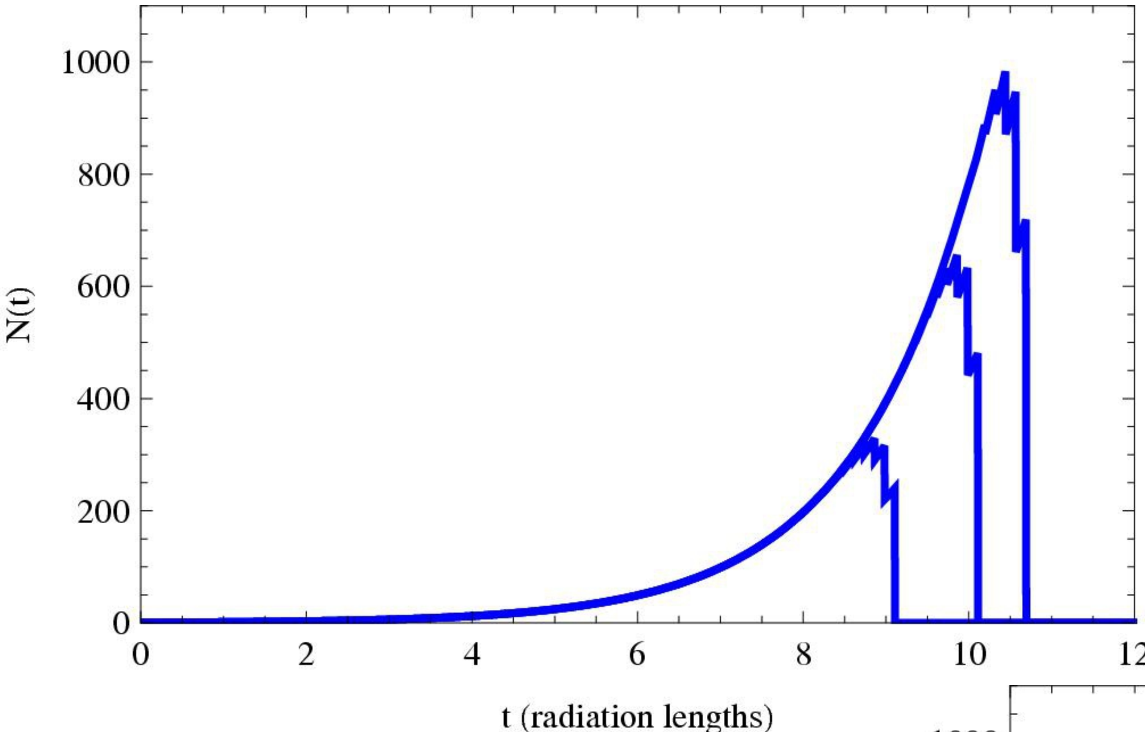




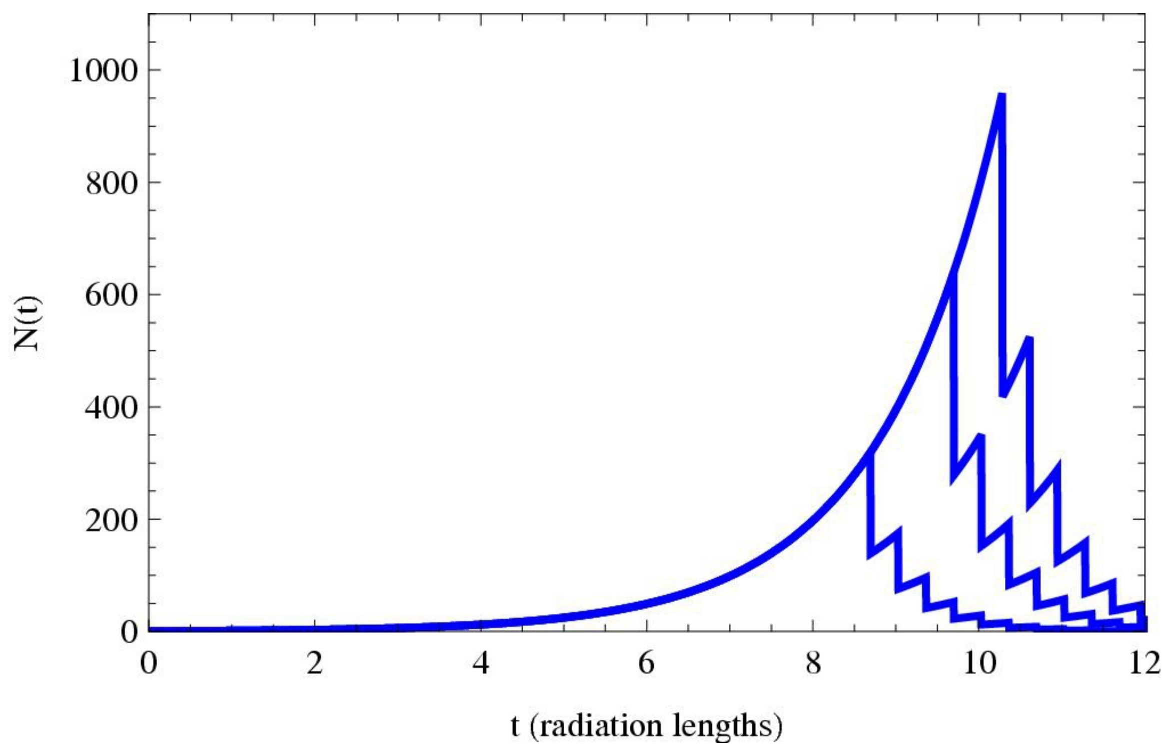


Hadronic shower in toy model.





1000
800
600
400
200
0
N(t)
0
2
4
6
8
10
12
t (radiation lengths)



1000
800
600
400
200
0
N(t)
0
2
4
6
8
10
12
t (radiation lengths)

$$X_{\max}(E) = k\,\Lambda + \lambda\,\log_2\left[\frac{E\,(1-f)^k\,f}{m\,\varepsilon}\right]$$

[integer]

$$\boxed{X_{\max}(E) = \lambda_{\text{rad}}\,\ln E + \text{constant}}$$

$$k = k(\Lambda/\lambda, f)$$

$$k = \begin{cases} 1 & \text{for } 0 \le f < 1 - 2^{-x} \\ \left[-\frac{1}{x}\,\log_2\left(\frac{1-2^x\,(1-f)}{f}\right)\right] & \text{for } 1 - 2^{-x} < f \le 1 \end{cases}$$

Hadronic interaction parameters

IF π , and the other hadronic interactions parameters are energy independent

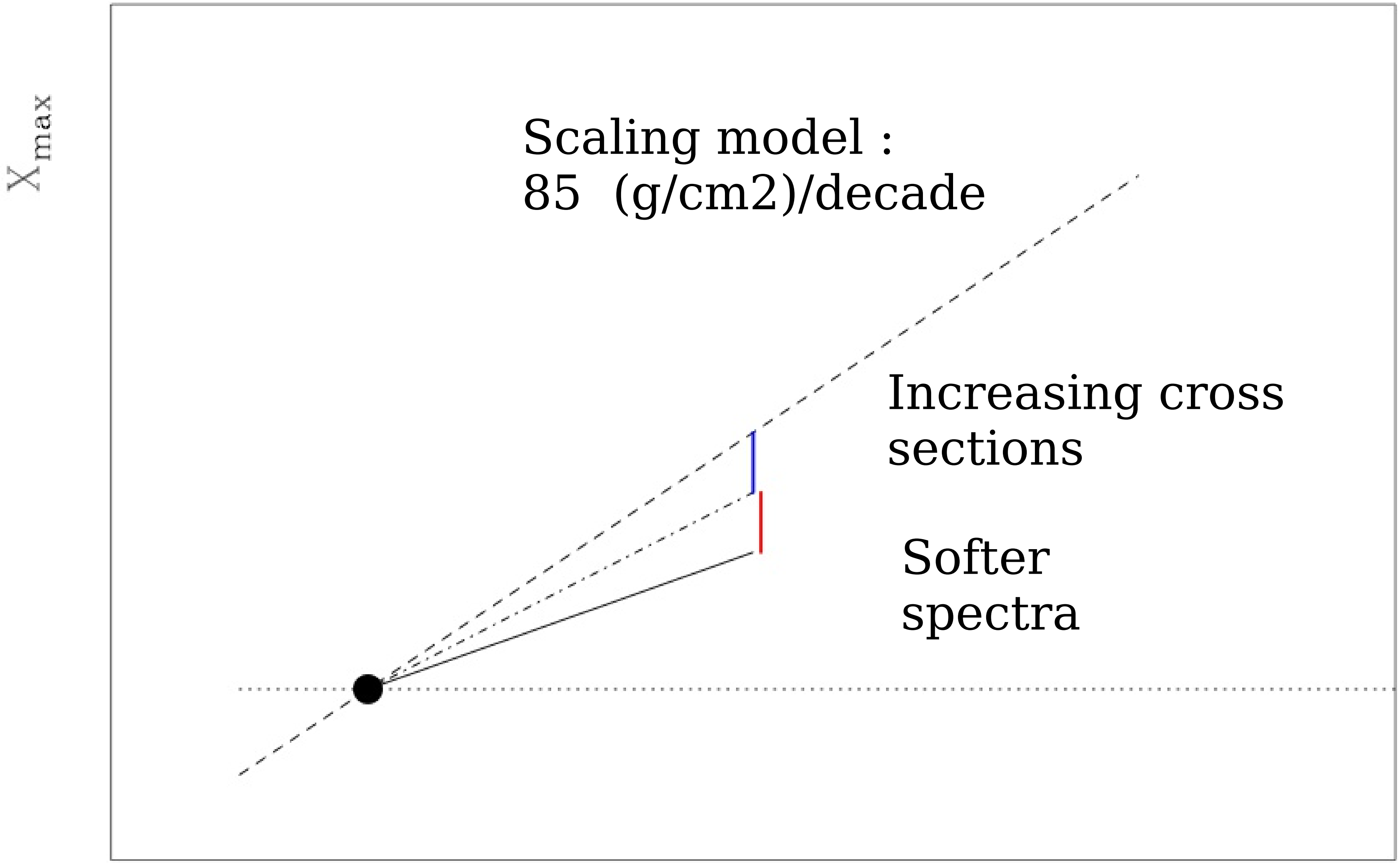
$$\frac{dX_{\max}}{d\ln E} = \frac{\lambda}{\ln 2} \equiv \lambda_{\text{rad}}$$

"Elongation rate" is equal to the radiation length

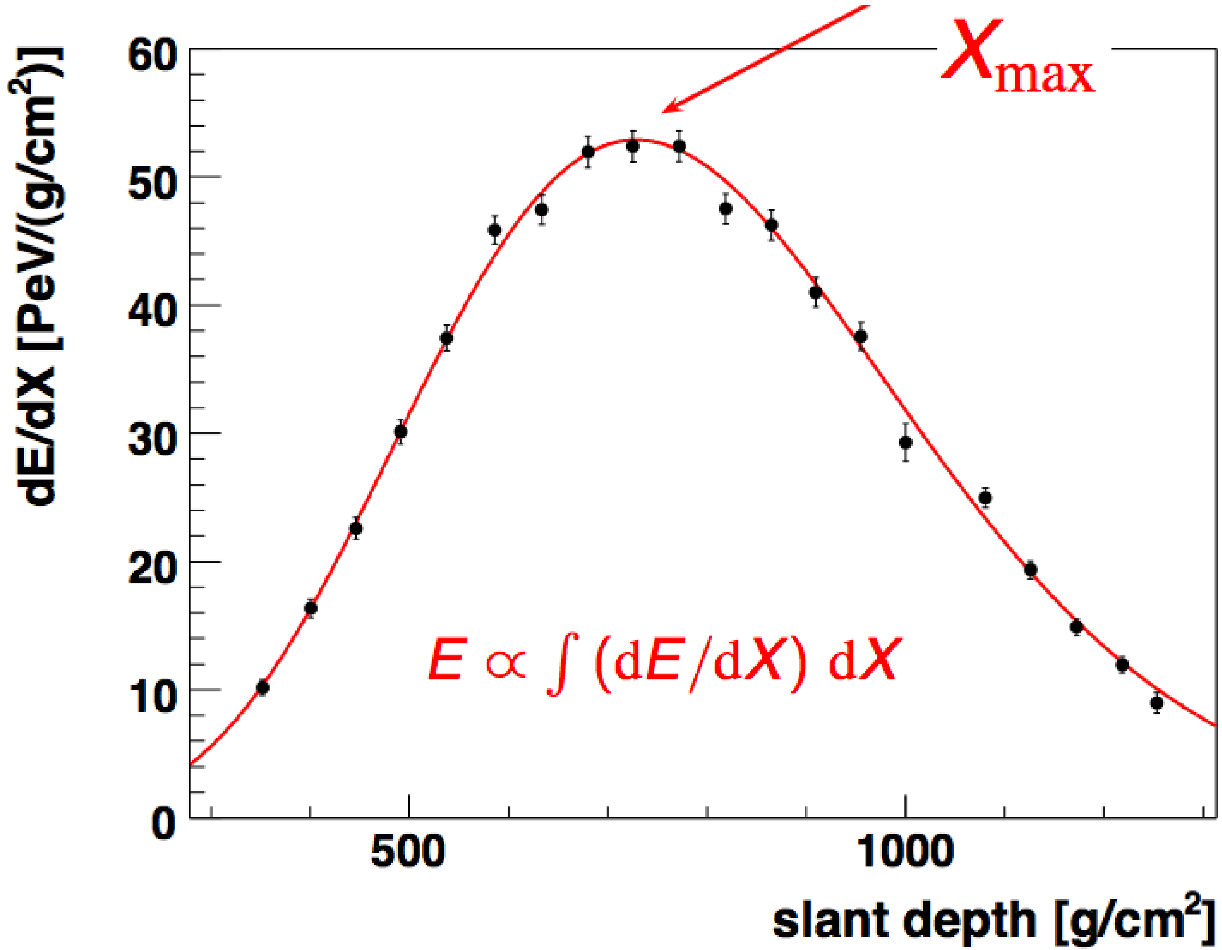
Energy dependent parameters: Elongation rate changes

$$\frac{dX_{\max}(E)}{d\ln E} = \lambda_{\text{rad}} \left[1 - \frac{dm(E)}{d\ln E} - \frac{d\ln f(E)}{d\ln E} \left(\frac{1 - f(E)\,(1+k)}{1 - f(E)}\right)\right]$$

$$+ k\,\frac{d\Lambda(E)}{d\ln E}$$



Elongation Rate
For protons



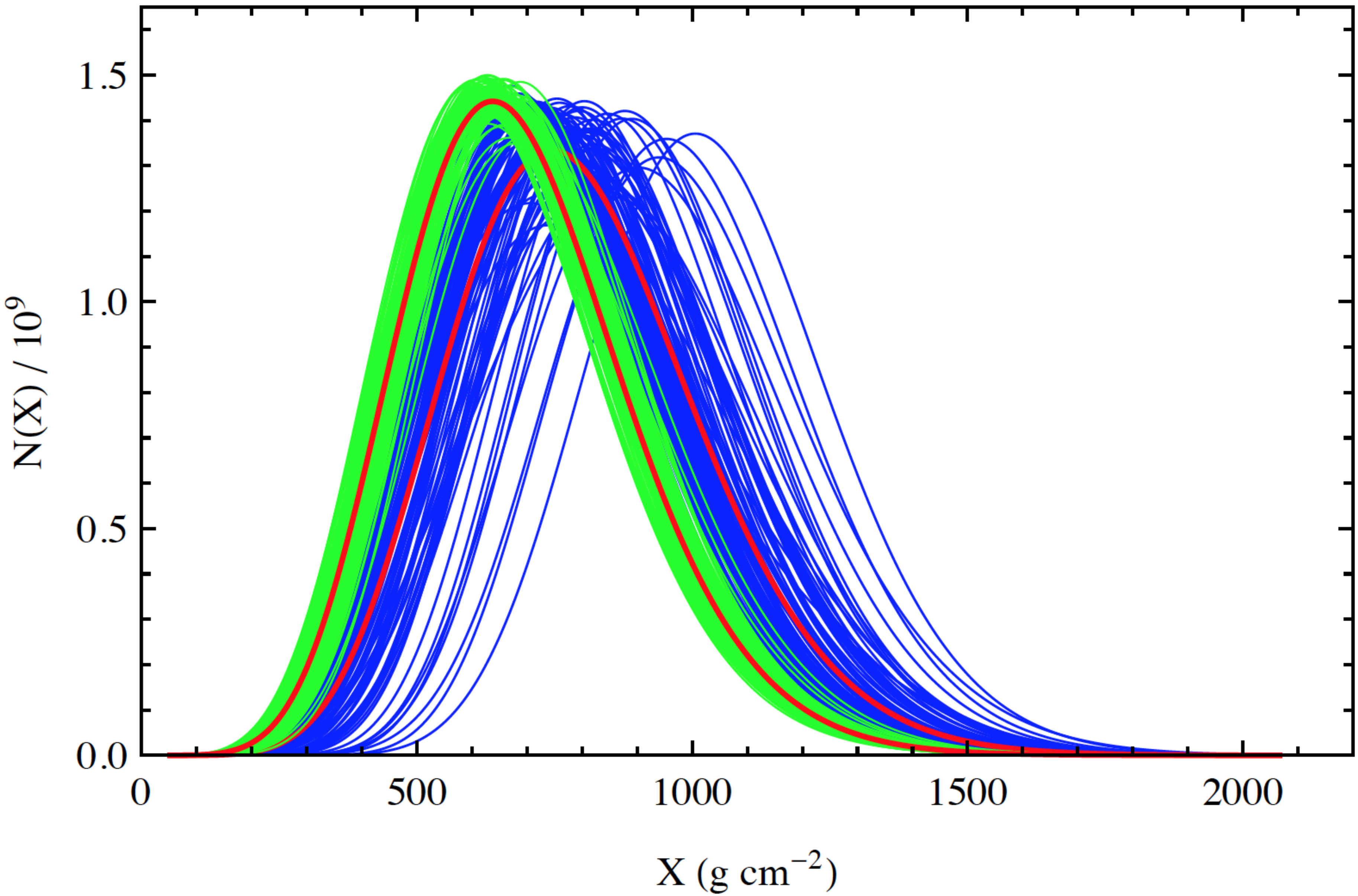
Area $\propto$ Energy

Shape depends on :

- Primary Identity
- Interaction Model

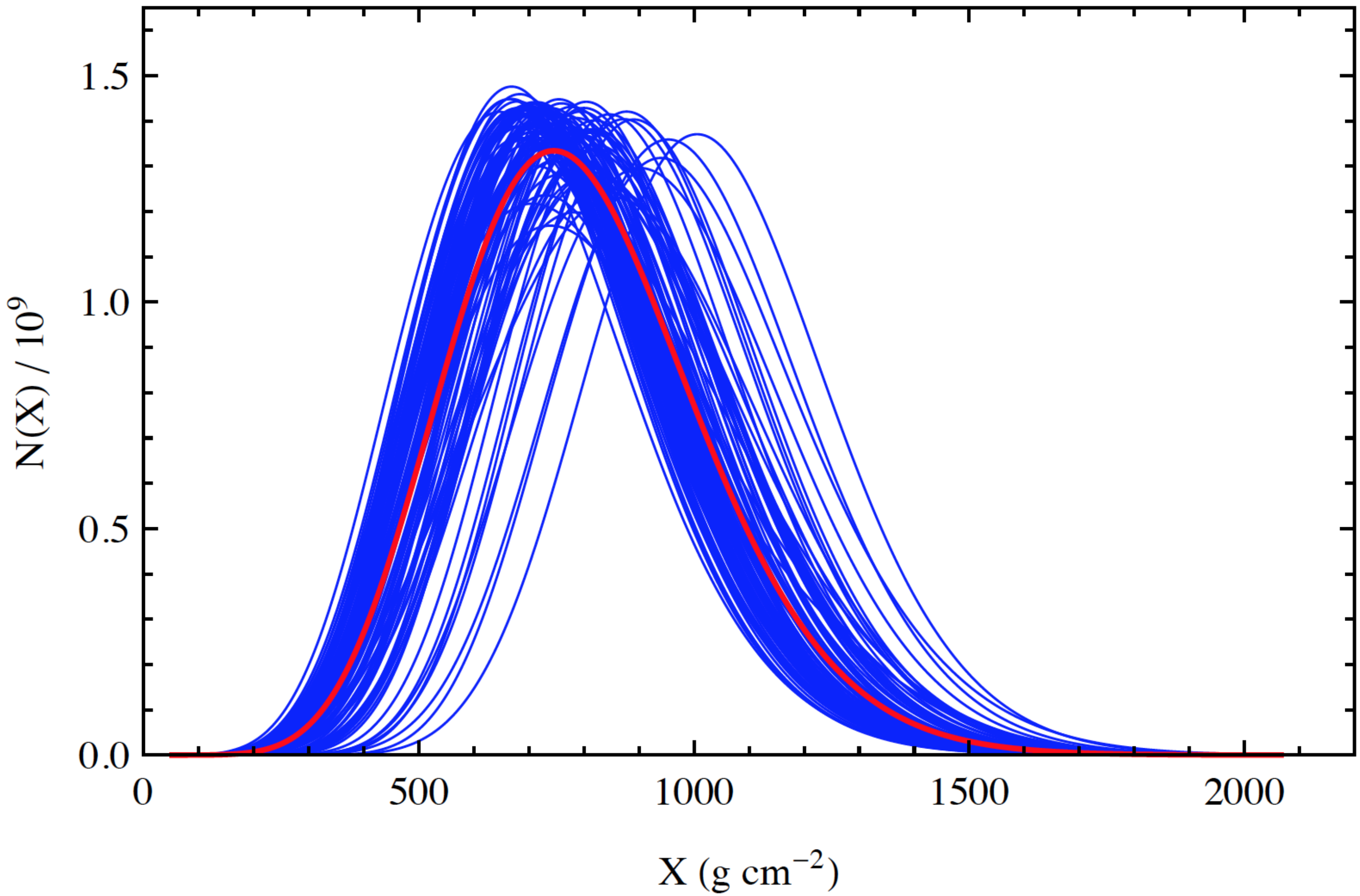
Compare proton and Iron nuclei showers

$$E_0 = 10^{18.25}\ \text{eV}$$



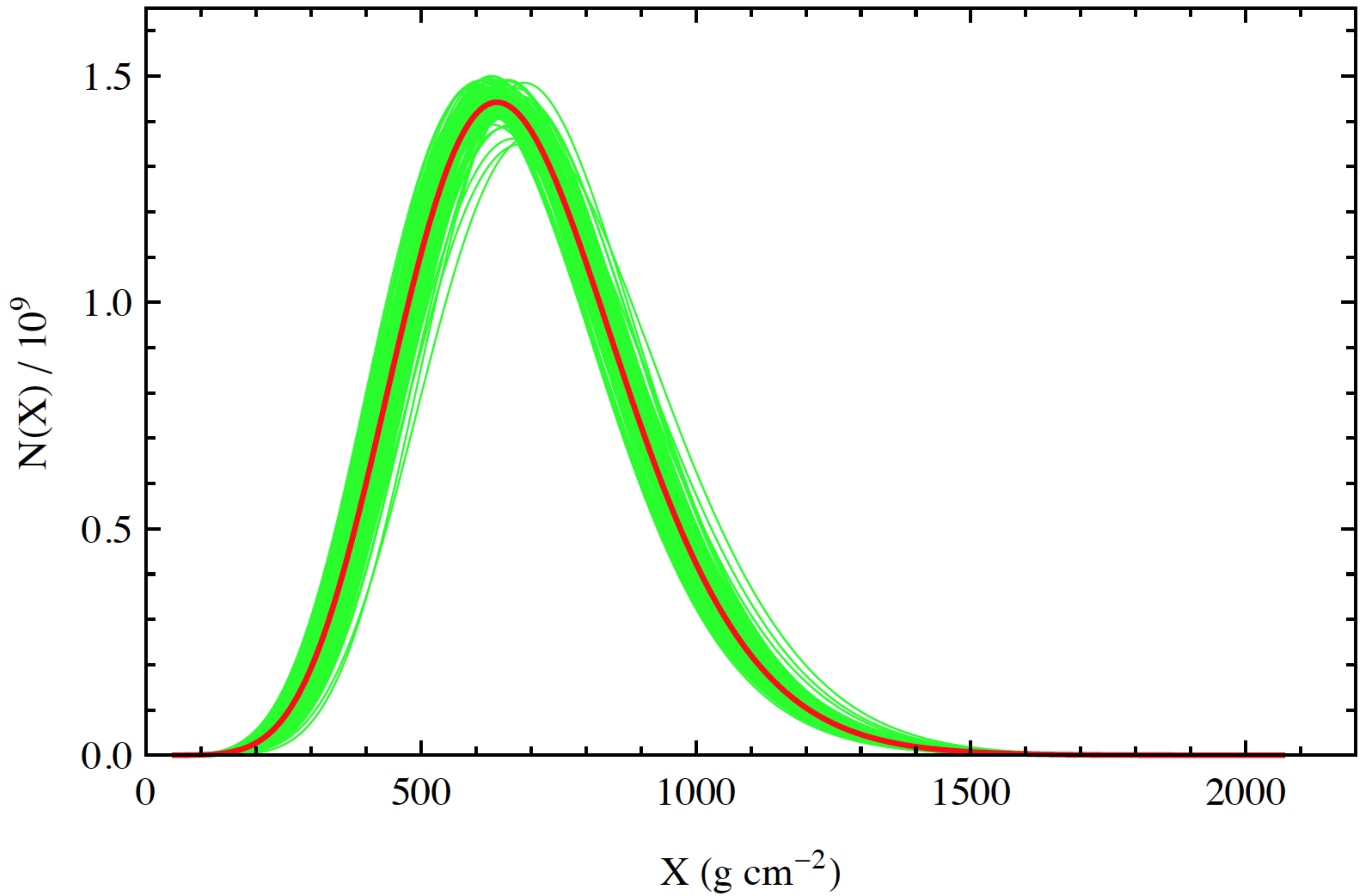
Montecarlo calculation of the development of individual proton showers

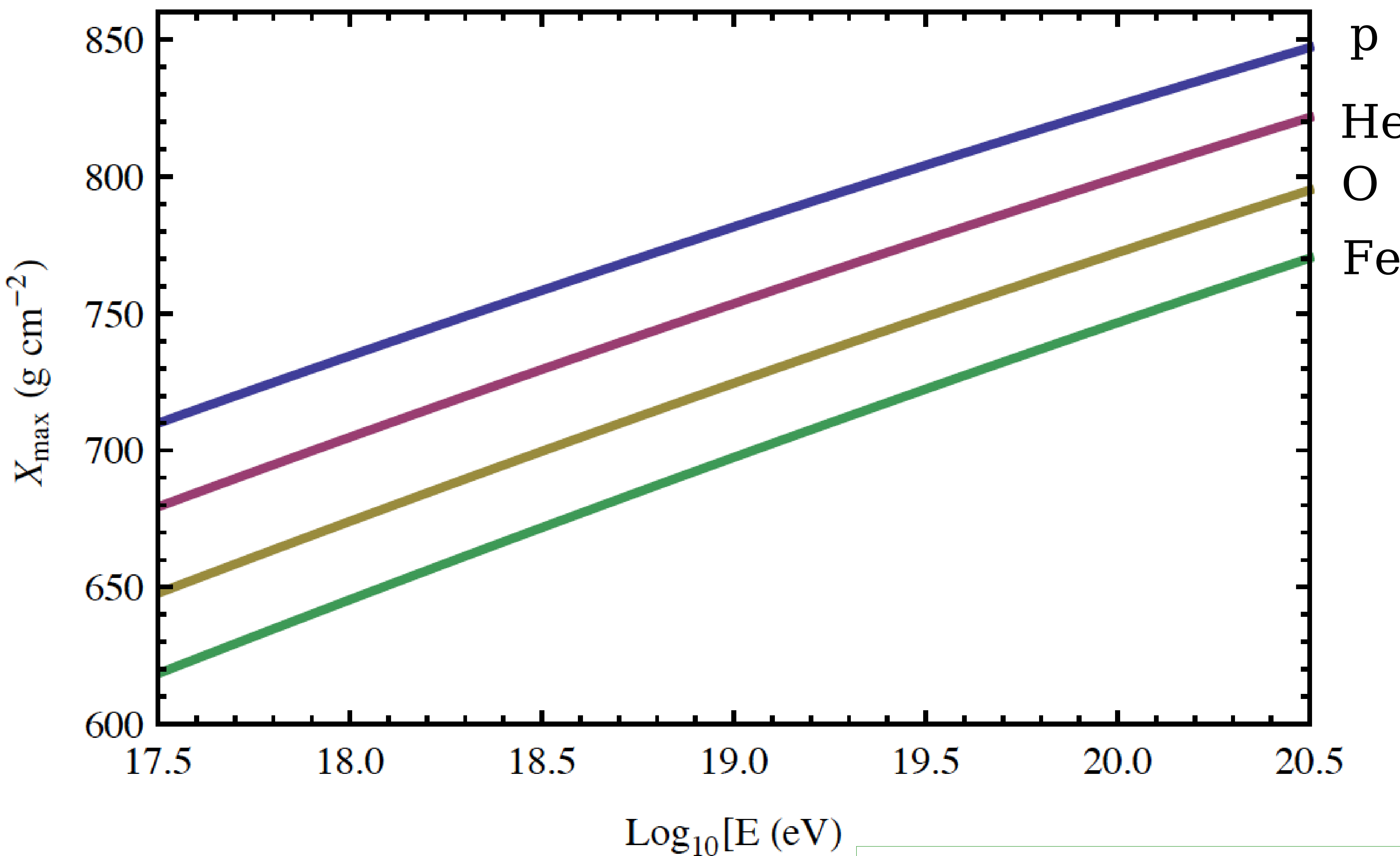
$$E_0 = 10^{18.25}\ \text{eV}$$



Montecarlo calculation of the development of individual iron nuclei showers

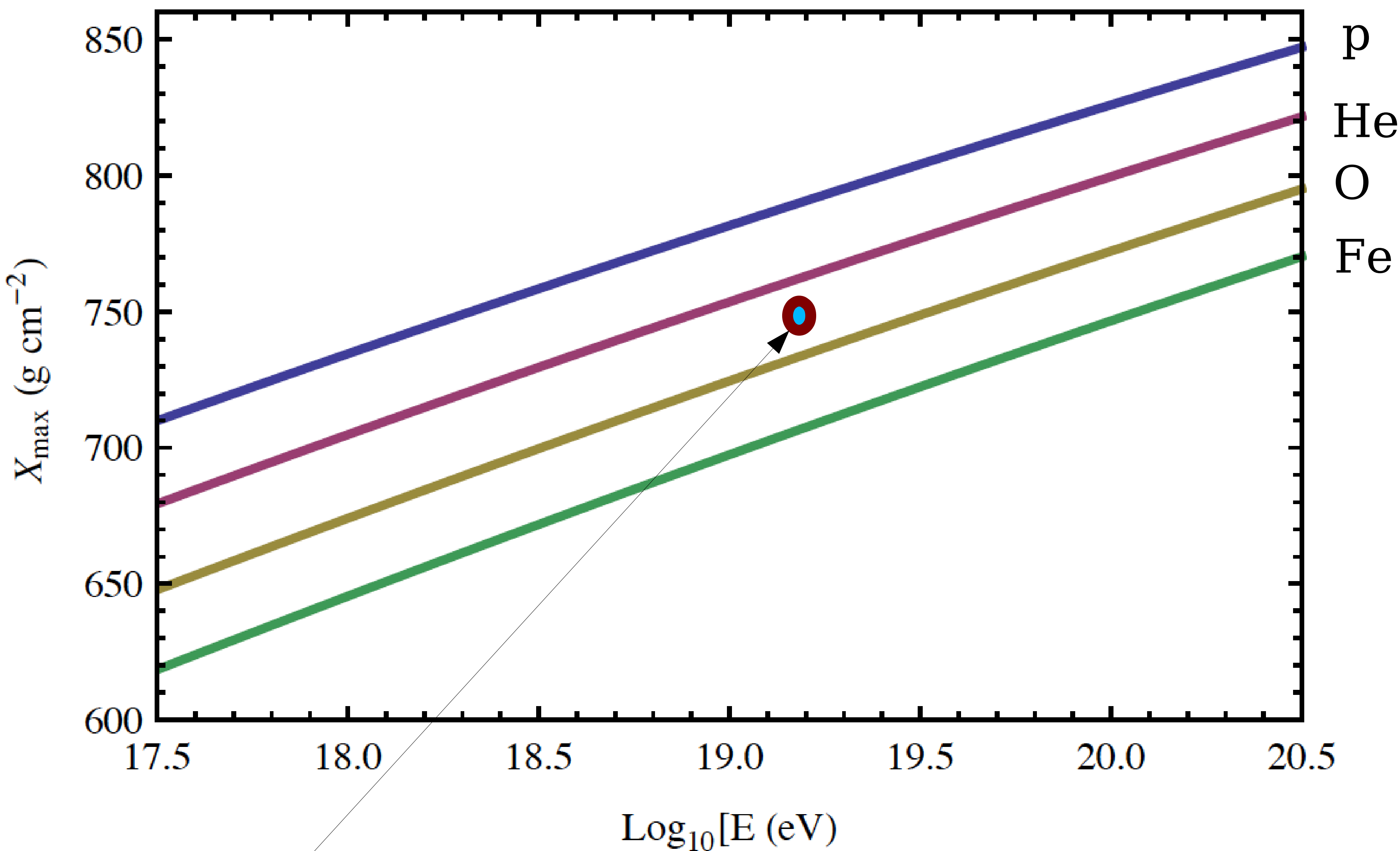
$$E_0 = 10^{18.25}\ \text{eV}$$





$$\boxed{\langle X_A \rangle \simeq \langle X_p \rangle - D_p \ \log_{10} A}$$

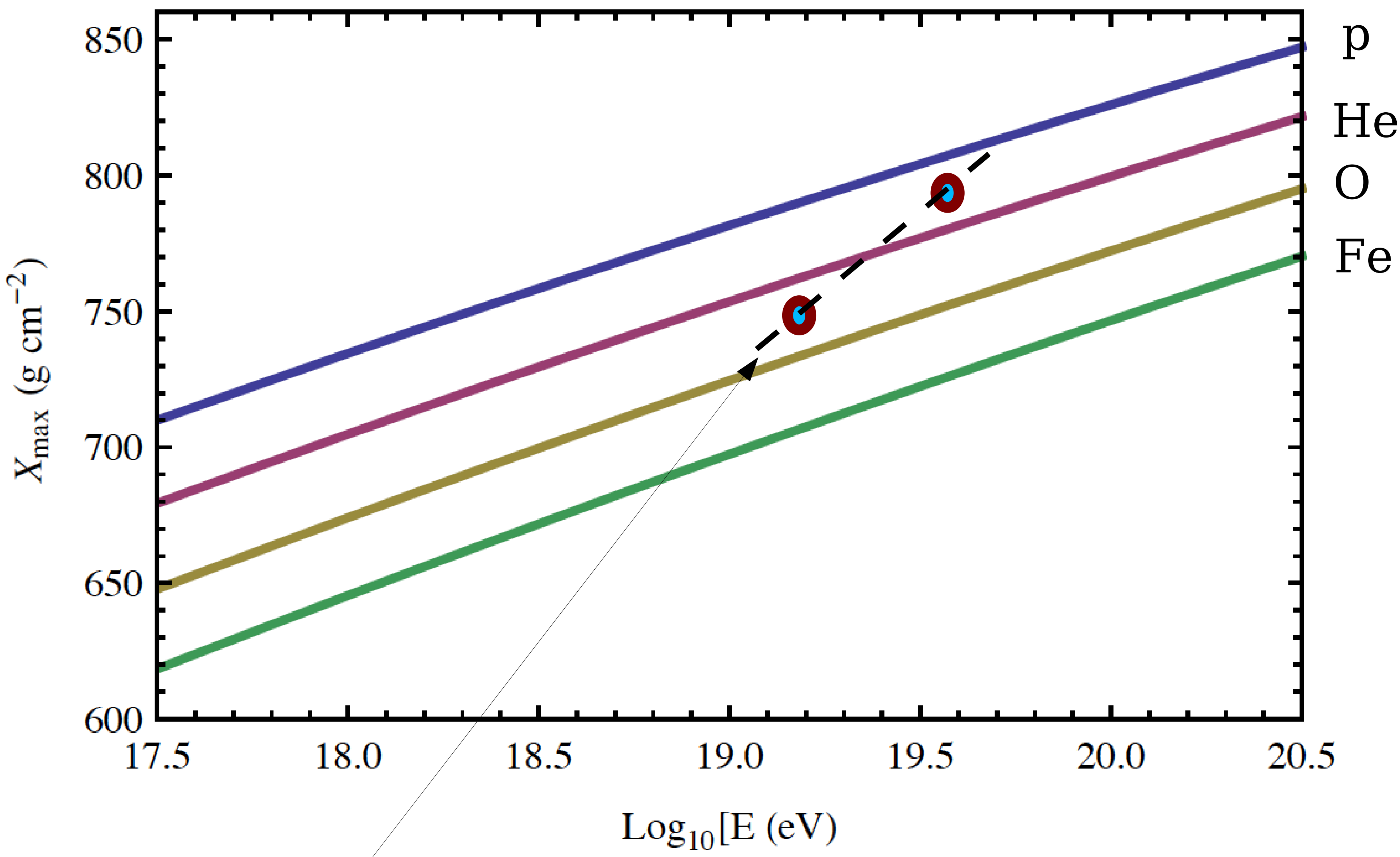
$$\langle X_{\mathrm{He}} \rangle \simeq \langle X_p \rangle - 30 \text{ g cm}^{-2}$$
$$\langle X_{\mathrm{O}} \rangle \simeq \langle X_p \rangle - 60 \text{ g cm}^{-2}$$
$$\langle X_{\mathrm{Fe}} \rangle \simeq \langle X_p \rangle - 90 \text{ g cm}^{-2}$$



Measurements of

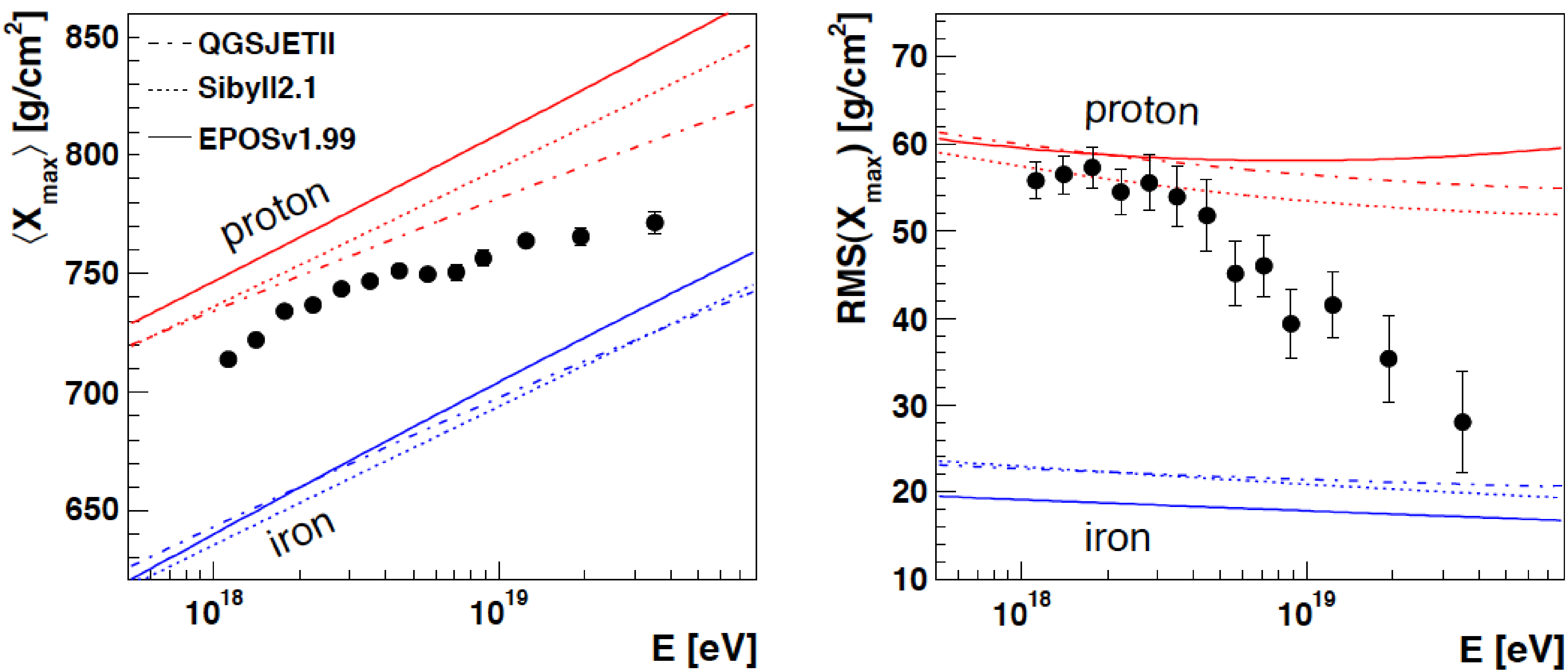
$$\langle \log A \rangle$$

$$\langle \ln A \rangle_E = \frac{\sum_A \phi_A(E) \; \ln A}{\sum_A \phi_A(E)}$$



Measurements of Composition evolution.

$\langle X_{max} \rangle$ and RMS



Compare DATA with predictions based on several assumptions for hadronic interactions....

With UHECR one studies at the same time

"Gigantic Astrophysical Beasts"
Millions of light years away
Length scale 10^{+24} cm

Microscopic
Partonic constituents of matter
Length scale 10^{-15} cm

Exciting

Difficult