Local Metric with Parameterized Evolution

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Stueckelberg-Horwitz-Piron (SHP) Formalism

Covariant canonical mechanics with parameterized evolution

8D unconstrained phase space $\implies$ $\tau \neq$ proper time

$$x^\mu(\tau),\ \dot{x}^\mu(\tau) \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad \lambda, \mu, \nu, \ldots = 0, 1, 2, 3$$

Canonical electrodynamics with scalar Hamiltonian ($K =$ total mass)

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e\, \dot{x}^\mu a_\mu(x, \tau) + e\, a_5(x, \tau) \qquad 0 = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}$$

$$K = \frac{\left[p - ea(x, \tau)\right]^2}{2M} - a_5(x, \tau) \qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$$

Want pseudo-5D metric that describes τ-evolution of spacetime

$$L = \frac{1}{2} M g_{\alpha\beta}(x, \tau) \dot{x}^\alpha \dot{x}^\beta \qquad \alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 5 \qquad x^5 = c_5 \tau$$

SHP — Geometry and Evolution

4D block universe $\mathcal{M}(\tau)$ at each τ

Physical event $x^\mu(\tau)$ in SHP

Irreversible occurrence **at** time τ

$$\tau_2 > \tau_1 \implies \begin{cases} x^\mu(\tau_2) \text{ occurs } \textbf{after } x^\mu(\tau_1) \\ x^\mu(\tau_2) \textbf{ cannot change } x^\mu(\tau_1) \\ \text{No grandfather paradox} \end{cases}$$

Evolution

4D block universe $\mathcal{M}(\tau)$ **occurs** at τ

Infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$ occurs at $\tau + d\tau$

$$\mathcal{M}(\tau) \xrightarrow[\text{Hamiltonian } K \text{ generates evolution in } \tau]{} \mathcal{M}(\tau + d\tau)$$

$$\left.\begin{array}{r} \text{scalar } K \\ \text{external } \tau \end{array}\right\} \implies \text{No conflict with general diffeomorphism invariance}$$

Geometry and Trajectory

Standard approach to motion in general relativity

Two neighboring events in spacetime manifold $\mathcal{M}$ (instantaneous displacement)

Interval $\delta x^2 = g_{\mu\nu} \delta x^\mu \delta x^\nu = (x_2 - x_1)^2$

Invariance of interval — geometrical statement about $\mathcal{M}$

Trajectory

Map arbitrary parameter ζ to sequence of events $x^\mu(\zeta)$

Timelike interval between any two events $\Rightarrow$ take $\zeta \to s$ (proper time)

4D block universe $\mathcal{M}$

Trajectory — sequence of instantaneous timelike displacements

"Motion" appears as displacements in $x^0(s)$

Path length $\longrightarrow$ Lagrangian

$$\delta x^2 = g_{\mu\nu} \delta x^\mu \delta x^\nu = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \delta s^2 \qquad\qquad g = (-,+,+,+)$$

$$L_{\text{constrained}} = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \qquad\qquad L_{\text{unconstrained}} = \frac{1}{2}\, g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

Geometry and Evolution

Pseudo-5D metric

Two neighboring events:

$$x^\mu(\tau) \in \mathcal{M}(\tau) \qquad\qquad \bar{x}^\mu(\tau+\delta\tau) \in \mathcal{M}(\tau+\delta\tau)$$

Distance

$$dx^\mu = \bar{x}^\mu(\tau+\delta\tau) - x^\mu(\tau) \simeq \bar{x}^\mu(\tau) + \dot{\bar{x}}^\mu(\tau)\delta\tau - x^\mu(\tau) = \delta x^\mu + \dot{\bar{x}}^\mu \delta\tau$$

Squared interval (referred to x coordinates)

$$dx^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu + g_{\mu\nu}\dot{\bar{x}}^\mu \delta x^\nu \delta\tau + g_{\mu\nu}\dot{\bar{x}}^\mu \dot{\bar{x}}^\nu \delta\tau^2 \simeq \underbrace{g_{\alpha\beta}(x,\tau)\,\delta x^\alpha \delta x^\beta}_{\alpha,\beta=0,1,2,3,5}$$

Contributions to interval

$g_{\mu\nu}\ \delta x^\mu \delta x^\nu$

Geometrical interval between two events at τ

Symmetries of spacetime manifold $\mathcal{M}$

$g_{55}\ \delta x^5 \delta x^5$

Dynamical interval between events in $\mathcal{M}(\tau)$ and $\mathcal{M}(\tau+\delta\tau)$

Symmetries of evolution generated by Hamiltonian K

Example in space

Particle in 2D space — expanding disk with radius $R(\tau) = \frac{1}{2}g\tau^2$

Points on expanding disk

$$\mathbf{q}_1 = R(\tau)(\cos\theta, \sin\theta) \qquad \mathbf{q}_2 = R_2(\tau_2)(\cos\theta_2, \sin\theta_2)$$

Distance

$$d\mathbf{q} = \mathbf{q}_2 - \mathbf{q}_1 \approx (\delta R + \delta_\tau R(\tau))\,\hat{\mathbf{R}} + R\delta\theta\ \hat{\boldsymbol{\theta}}$$

Geometrical distances: $\delta\theta = \theta_2 - \theta \qquad \delta R = R_2(\tau) - R(\tau)$

Dynamical distance: $\delta_\tau R(\tau) = R(\tau_2) - R(\tau) = g\tau\delta\tau$

Interval

$$d\mathbf{q}^2 = \delta R^2 + R^2\delta\theta^2 + g^2\tau^2\delta\tau^2 + 2g\tau\delta R\delta\tau = g_{ab}\delta\zeta^a\delta\zeta^b$$

Pseudo-3D metric

$$\delta\zeta = (\delta R, \delta\theta, \delta\tau) \qquad g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & R^2 & 0 \\ 1 & 0 & g\tau \end{pmatrix}$$

Example in space

Equations of motion

Lagrangian

$$L = \tfrac{1}{2} M g_{ab} \dot{\zeta}^a \dot{\zeta}^b = \tfrac{1}{2} M \left(\dot{R}^2 + R^2 \dot{\theta}^2 + 2 g \tau \dot{R} + g^2 \tau^2 \right)$$

$$\left.\begin{aligned} 0 &= \frac{d}{d\tau} M \left(\dot{R} + g\tau \right) - M R \dot{\theta}^2 \\ 0 &= \frac{d}{d\tau} \left(M R^2 \dot{\theta} \right) \longrightarrow M R^2 \dot{\theta} = \ell \end{aligned}\right\} \longrightarrow M \ddot{R} = \frac{\ell^2}{M R^3} - M g$$

Qualitative result

Particle at edge of disk sees force $F = \ell^2 / M R^3 - M g$

$M g > \ell^2 / M R^3 \implies$ Particle moves at edge of disk. As if attracted by gravitational force

$F_R = -M g \longrightarrow$ Appears as "external" force. Enters through evolution of circular geometry

Canonical Mechanics in General 5D Spacetime

Lagrangian

$$L = \frac{1}{2} M g_{\alpha\beta}(x^\mu, x^5) \dot{x}^\alpha \dot{x}^\beta \qquad \lambda, \mu, \nu = 0,1,2,3 \qquad \alpha, \beta, \gamma = 0,1,2,3,5$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\gamma}{D\tau} = \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\beta\alpha} \right)$$

Canonical momentum

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = M g_{\alpha\beta} \dot{x}^\beta \qquad \longrightarrow \qquad \dot{x}^\alpha = \frac{1}{M} g^{\alpha\beta} p_\beta$$

Conserved Hamiltonian

$$K = \dot{x}^\alpha p_\alpha - L = \frac{1}{2M} g^{\alpha\beta} p_\alpha p_\beta \qquad \frac{dK}{d\tau} = M g_{\alpha\beta} \dot{x}^\alpha \frac{D\dot{x}^\beta}{D\tau} = 0$$

Poisson bracket

$$\frac{dK}{d\tau} = \{K, K\} + \frac{\partial K}{\partial \tau} = \frac{1}{2M} p_\alpha p_\beta \, \frac{\partial g^{\alpha\beta}}{\partial \tau} = 0$$

Break 5D symmetry $\longrightarrow$ 4D+1

Fix non-dynamical scalar $x^5 \equiv c_5 \tau$

$$L = \frac{1}{2} M g_{\alpha\beta}(x,\tau) \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2} M g_{\mu\nu}(x,\tau)\ \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} M c_5^2\ g_{55}(x,\tau)$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\alpha}{D\tau} = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma \longrightarrow \begin{cases} \ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma} \dot{x}^\lambda \dot{x}^\sigma + 2 c_5 \Gamma^\mu_{5\sigma} \dot{x}^\sigma + c_5^2 \Gamma^\mu_{55} = 0 \\ \ddot{x}^5 = \dot{c}_5 \equiv 0 \end{cases}$$

Symmetry-broken connection

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta} \right) \qquad\qquad \Gamma^5_{\alpha\beta} \equiv 0$$

Hamiltonian

$$K = p_\mu \dot{x}^\mu - L = \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu - \frac{1}{2} M c_5^2\ g_{55}$$

$$\frac{dK}{d\tau} = \frac{\partial K}{\partial \tau} = -\frac{1}{2} M \dot{x}^\mu \dot{x}^\nu\ \partial_\tau g_{\mu\nu} - \frac{1}{2} M c_5^2\ \partial_\tau g_{55}$$

Matter

Non-thermodynamic dust

Number of events per spacetime volume $= n(x)$

5-component event current $= j^{\alpha}(x,\tau) = \rho(x)\dot{x}^{\alpha}(\tau) = Mn(x)\dot{x}^{\alpha}(\tau)$

Continuity equation

$$\nabla_{\alpha} j^{\alpha} = \nabla_{\mu} j^{\mu} + \partial_{\tau}\rho = \partial_{\mu} j^{\mu} + j^{\lambda}\Gamma^{\mu}_{\lambda\mu} + j^{5}\Gamma^{\mu}_{5\mu} + \partial_{\tau}\rho = 0$$

Mass-energy-momentum tensor

$$\nabla_{\beta} T^{\alpha\beta} = 0 \qquad\qquad T^{\alpha\beta} = \rho\dot{x}^{\alpha}\dot{x}^{\beta} \longrightarrow \begin{cases} T^{\mu\nu} = \rho\dot{x}^{\mu}\dot{x}^{\nu} \\ T^{5\beta} = c_5 j^{\beta} \end{cases}$$

Einstein equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}$$

$\nabla_{\beta} G^{\alpha\beta} = 0$ requires $R_{\alpha\beta}$ calculated with

$$\Gamma^{\mu}_{\alpha\beta} = \tfrac{1}{2}g^{\mu\nu}\left(\partial_{\beta}g_{\nu\alpha} + \partial_{\alpha}g_{\nu\beta} - \partial_{\nu}g_{\alpha\beta}\right) \qquad \Gamma^{5}_{\alpha\beta} \equiv 0$$

Ricci Tensor

Separate

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\beta\gamma} = R^{\lambda}_{\alpha\beta\lambda} + R^{5}_{\alpha\beta 5}$$

Discard $\Gamma^5_{\alpha\beta}$

$$R_{\mu\nu} = \left(R_{\mu\nu}\right)^{4D}$$

$$R_{\mu 5} = \frac{1}{c_5}\partial_\tau \Gamma^{\lambda}_{\mu\lambda} - \partial_\lambda \Gamma^{\lambda}_{\mu 5} + \Gamma^{\lambda}_{\sigma 5}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{\mu 5}$$

$$R_{55} = \frac{1}{c_5}\partial_\tau \Gamma^{\lambda}_{5\lambda} - \partial_\lambda \Gamma^{\lambda}_{55} + \Gamma^{\lambda}_{\sigma 5}\Gamma^{\sigma}_{5\lambda} - \Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{55}$$

Low energy velocity

$$\dot{x} = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) \simeq c\dot{t}\,(1, \mathbf{0})$$

Weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\partial_0 h_{\mu\nu} = 0$

$$\Gamma^\mu_{00} = -\tfrac{1}{2} g^{\mu\nu} \partial_\nu h_{00} \qquad \Gamma^\mu_{50} = \tfrac{1}{2c_5} g^{\mu 0} \partial_\tau h_{00} \qquad \Gamma^\mu_{55} = -\tfrac{1}{2} g^{\mu\nu} \partial_\nu h_{55}$$

Equations of motion

$$\ddot{t} = (\partial_\tau h_{00})\,\dot{t} \qquad \ddot{\mathbf{x}} = \tfrac{1}{2} c^2 \dot{t}^2 \; \nabla h_{00} + \tfrac{1}{2} c_5^2 \; \nabla h_{55}$$

$$\frac{d^2\mathbf{x}}{d\tau^2} = \dot{t}^2 \frac{d^2\mathbf{x}}{dt^2} + (\partial_\tau h_{00})\,\dot{t}\frac{d\mathbf{x}}{dt} \;\longrightarrow\; \frac{d^2\mathbf{x}}{dt^2} = \frac{1}{2} c^2 \nabla h_{00} + \frac{1}{\dot{t}^2} \left[\frac{1}{2} c_5^2 \nabla h_{55} - (\partial_\tau h_{00})\,\dot{t}\frac{d\mathbf{x}}{dt} \right]$$

'Unperturbed' system

$\nabla h_{55} = \partial_\tau h_{00} = 0$ Newtonian potential $h_{00} \xrightarrow[r\to\infty]{} 0$

$$g_{00} = -\left(1 - \frac{2GM}{rc^2} \right)$$

Newtonian Approximation

Toy model perturbation: τ-dependent mass $M = M(\tau)$

Small mass shift $\Delta M(\tau)$ over time $\Delta\tau$

$$\ddot{t} = (\partial_\tau h_{00})\,\dot{t} = \frac{2G\dot{M}}{rc^2}\dot{t} \;\longrightarrow\; \dot{t} = \exp\left(\frac{2G\;\Delta M(\tau)}{rc^2}\right) \approx 1 + \frac{2G\;\Delta M}{rc^2}$$

Equations of motion

$$\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{2G}{rc^2}\frac{dM}{dt} = -\frac{GM}{r^2}\hat{\mathbf{r}} + \frac{1}{2}c_5^2\left(1 - \frac{4G\;\Delta M}{rc^2}\right)\nabla h_{55}$$

Plane polar coordinates with $\nabla h_{55} = 0$

$$\ddot{r} - r\dot{\phi}^2 + \frac{2G}{rc^2}\frac{dM}{dt}\dot{r} = -\frac{GM}{r^2} \qquad\qquad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) + \frac{2G}{c^2}\frac{dM}{dt}\dot{\phi} = 0$$

$\ell = Mr^2\dot{\phi}$ not conserved

Neglecting $\dot{\phi}^2 \;\longrightarrow\; \ddot{r} + \dfrac{2G}{c^2}\dfrac{dM}{dt}\dfrac{d}{dt}(\ln r) = -\dfrac{GM}{r^2}$

Newtonian gravitation with dissipation

Extended Spherical Solution

Metric $g_{\alpha\beta}$

Line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

$$ds^2 = -c^2 B(r,\tau)\, dt^2 + A(r,\tau)\, dr^2 + r^2 d\theta^2 + \sin^2\theta d\phi^2 + c_5^2 Q(r,\tau)$$

Lagrangian (taking $\theta = \pi/2$)

$$L = \frac{1}{2}\left[-c^2 B(r,\tau)\, \dot{t}^2 + A(r,\tau)\, \dot{r}^2 + r^2\dot{\phi}^2 + c_5^2 Q(r,\tau)\right]$$

New non-zero components of $\Gamma^\mu_{\alpha\beta}$ and $R_{\alpha\beta}$: $\Gamma^0_{50}, \Gamma^1_{15}, \Gamma^1_{55}$, R_{05}, R_{15}, R_{55}

Extend Schwarzschild

$$B(r,\tau) = A^{-1}(r,\tau) = 1 - \frac{2GM(\tau)}{rc^2} \qquad\qquad Q = \frac{1}{r}\left(1 - \frac{GM(\tau)}{rc^2}\right)$$

Flow of mass $\longrightarrow$ non-zero Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \propto G\frac{dM}{d\tau} \xrightarrow[M \to \text{constant}]{} 0$$

Extended Spherical Solution

Equations of Motion

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \ddot{t} + \frac{\partial_r B}{B}\dot{r}\dot{t} + \frac{\partial_\tau B}{B}\dot{t} \;\longrightarrow\; \dot{t} = \frac{1}{B} \qquad\qquad 0 = \ddot{\phi} + 2\frac{1}{r}\dot{r}\dot{\phi} \;\longrightarrow\; \dot{\phi} = \frac{J}{r^2}$$

$$0 = \ddot{r} + \frac{1}{2}\frac{\partial_r A}{A}\dot{r}^2 + \frac{1}{2}c^2\frac{\partial_r B}{A}\dot{t}^2 - \frac{1}{A}r\dot{\phi}^2 + \frac{\partial_\tau A}{A}\dot{r} - c_5^2\frac{1}{2}\frac{\partial_r Q}{A}$$

Combining

$$\frac{d}{d\tau}\underbrace{\frac{1}{2}\left(-c^2\frac{1}{B} + A\dot{r}^2 + \frac{J^2}{r^2} - c_5^2 Q\right)}_{\text{Hamiltonian } K} = -\frac{1}{2}\dot{r}^2\partial_\tau A - \frac{1}{2}c_5^2\partial_\tau Q$$

$\dot{K} \neq 0 \;\longrightarrow\;$ particle mass exchanged with $g_{\alpha\beta}$ through $M = M(\tau)$

Static $g_{\alpha\beta}$

$\partial_\tau A = \partial_\tau Q = 0 \;\Rightarrow\; M =$ constant $\;\Rightarrow\; \dot{K} = 0$

Radial solution $\qquad \tau = \displaystyle\int \frac{dr}{\sqrt{c^2 + \left(K - J^2/r^2 + c_5^2 Q(r)\right) B(r)}}$

Thank You
For Your
Patience

Slides and preprints: http://cs.hac.ac.il/staff/martin