Divergence-type hyperbolic theories for ultra-relativistic fluids

Marcelo Rubio

Instituto de Astronomía Teórica y Experimental
CONICET
Observatorio Astronómico de Córdoba
Córdoba, Argentina

STARS2019, La Habana, Cuba

May 7th, 2019

Introduction and Motivations

- Relativistic hydrodynamics is nowadays one of the coarsest areas of General Relativity, since it allows a formal framework for the thermodynamic description and dynamical evolution of matter and energy.
  - Accounts both for fluid systems with high Lorentz factor ($\gamma \gg 1$) and those with intense gravitational field (background or the one generated by the fluid itself).
- Applications in small and large scale phenomena:
  - Nuclear and Plasma Physics
    - Magneto-acoustic shock waves ($v \sim 4 \times 10^8\ cm/s$);
    - Heavy-ion collisions (quarks and gluons), proton-proton scattering;
  - Astrophysics and Cosmology
    - Stellar collapse of massive stars (prominent sources of gravitational waves), accretion on compact objects in binary systems, AGN and jets formation processes;
    - theories of galaxy-formation, evolution of density perturbations in the cosmic environment, among others.

Fluid approximation and dynamical equations

- The fluid approximation assumes the system as a continuum medium, such that any point represents an ensemble of microscopic components.
- The state of the fluid is generally well described by the velocity field $\vec{v}(t, \vec{x})$, and two thermodynamic quantities (like temperature and pressure).
- The dynamics is governed by the local conservation of particle number density, energy and momentum.
- Non-relativistic ideal fluids:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + (\vec{v} \cdot \nabla)(\rho\vec{v}) + \nabla p = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{v}(E + p)) = 0$$

Hyperbolicity issues: The Initial-Value problem

- The previous set of first-order differential equations is of hyperbolic type if the equation of state $p = p(\rho, \varepsilon)$ is such that the speed of sound is real:

$$c_s = \sqrt{\frac{\partial p}{\partial \rho} + \frac{p}{\rho^2}\frac{\partial p}{\partial \varepsilon}}$$

- Hyperbolicity of the evolution equations is essential in order to guarantee a well-posed initial-value formulation of the theory [Hadamard, 1908]:
  - existence and uniqueness of solution;
  - Continuous dependence of the evolution with respect to the initial data:

    $$\|u(t, \vec{x})\| \leq K e^{\alpha t} \, \|u(0, \vec{x})\|,$$

    where K, α are real constants.
- Including dissipation in the non-relativistic regime leads to the Navier-Stokes equations, which turn out to be hyperbolic and causal, in a similar fashion to the ideal case.

Relativistic fluids

- Matter and energy in General Relativity is generally encoded in the energy-momentum tensor field T_{ab} as a source for solving Einstein's equations. An inertial observer t^a will measure, for instance:
  - Energy density: $e = T_{ab}t^a t^b$
  - Momentum density: $p^i = h^{ia}t^b T_{ab}$
  - Spatial stresses: $\perp^{ij} = h^{ia}h^{ib}T_{ab}$
- For an ideal fluid, such a tensor has a particular form:

  $$T_{ab} = (\rho + p)u_a u_b + pg_{ab}$$

  and the corresponding evolution equations are

  $$\nabla_a T^{ab} = 0\,, \qquad \nabla_a N^a = 0,$$

  where $N^a = nu^a$.
- Under the same physical assumptions as in the non-relativistic case, ideal fluid system of equations admits a well-posed initial-value formulation. ¿Relativistic dissipative fluids?

Divergence-type fluid theories (DTT)

- We consider fluid theories with the following three properties:
  1. Dynamical ("physical") variables: N^a and T^{ab}
  2. Evolution equations:

$$\begin{aligned} \nabla_a N^a &= 0 \\ \nabla_a T^{ab} &= 0 \\ \nabla_a A^{abc} &= I^{bc} \end{aligned}$$

  3. Existence of a local entropy current, S^a, such as, as a consequence of the evolution equations, satisfies

$$\nabla_a S^a = \sigma \geq 0$$

- The most general theory satisfying the three properties listed before is determined by a generating scalar function

$$\chi = \chi(\xi, \xi_a, \xi_{ab}),$$

such that

$$N^a = \frac{\partial^2 \chi}{\partial \xi \partial \xi_a}, \quad T^{ab} = \frac{\partial^2 \chi}{\partial \xi_a \partial \xi_b}, \quad A^{abc} = \frac{\partial^2 \chi}{\partial \xi_a \partial \xi_{bc}}, \quad \sigma = -I^{ab} \xi_{ab}$$

Ultra-relativistic fluids

- We are interested in the description of ultra-relativistic fluids ($\gamma \gg 1$) at high temperatures.
- Both kinetic and thermal energy are much larger than the rest energy.
- This property suggests that there is no an intrinsic spatial scale associated to the theory.
- Thus, we will consider theories that are invariant under a re-scaling of the backgroud metric; i.e.,

$$\widehat{g} = \Omega^2 g\,, \quad \Omega \neq 0\,, \quad \partial_a \Omega \neq 0$$

- Conformal invariance imposes physical conditions for the equation of state of the fluid.
- We will also assume here that the particle number of the system is not conserved.

Conformally invariant DTT

- Dynamical variables: T^{ab}, A^{abc}
- Evolution equations:

$$\nabla_a T^{ab} = 0 \qquad \nabla_a A^{abc} = I^{bc}$$

- "Abstract" variables: ξ_a, ξ_{ab}, with $g^{ab}\xi_{ab} = 0$.

Problem

Find the most general scalar function $\chi(\xi_a, \xi_{ab})$ up to second order dissipative contributions.

- Conformal invariance requirements:

$$g_{ab}T^{ab} = 0, \qquad \hat{n}_a A^{abc} - \hat{n}_a A^{(bc)a} = 0, \quad \hat{n}_a = \frac{\partial_a \Omega}{\Omega}$$

Conformally invariant DTT

- Zeroth order ($A^{abc} \equiv 0$; $I^{ab} \equiv 0$)
  - Possible scalars: $\mu = \xi^a \xi_a$

$$\chi = \chi^o(\mu) \quad \to \quad T^{ab} = 4\chi^o_{\mu\mu}\xi^a\xi^b + 2\chi^o_\mu g^{ab}$$

  - By the identification $\xi_a \leftrightsquigarrow u_a$, we get

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab}$$

  - Conformal invariance: $g_{ab}T^{ab} = 0 \;\Rightarrow\; \chi^o(\mu) = \frac{\chi^o_o}{\mu}$, $\chi^o_o \in \mathbb{R}$
  - Temperature field: $T = \frac{1}{\sqrt{-\mu}}$
- First order
  - Possible scalars: $\mu = \xi^a \xi_a$, $\nu = \xi^{ab}\xi_a\xi_b$

$$\chi = \chi^o(\mu) + \chi^o(\mu)\nu$$

  - Orthonormal decomposition for the dissipative tensor:

$$\xi_{ab} = \frac{\nu}{\mu^2}\xi_a\xi_b + \frac{2}{\mu}\xi_{(a}r_{b)} + \tau_{ab}$$

Conformally invariant DTT

- First order
  - Conformal invariance $\rightsquigarrow$ $\chi^1(\mu) = \frac{\chi_o^1}{\mu^3}$, with $\chi_o^1 \in \mathbb{R}$.
  - Emergent energy-momentum tensor:

$$T^{ab} = \frac{4}{3}\rho\left(u^a u^b + \frac{g^{ab}}{4}\right) + 2u^{(a}q^{b)} + \tau_\perp^{ab}$$

  - By imposing the most general source tensor at this order,

$$I_{ab} = \frac{\tau}{\lambda\mu}\xi_a\xi_b - \frac{2}{\kappa}\xi_{(a}r_{b)} - \frac{1}{\lambda}\tau_{ab}$$

    we obtain the following constitutive relations:

$$q^a = -K\left(D^a T + \frac{\sqrt{-\mu}}{\mu^2}h^{ab}\dot{\xi}_b\right) \qquad \text{Fourier's Law}$$

$$\tau_\perp^{ab} = -2\lambda\chi^1\sqrt{-\mu}\,\sigma^{ab} \qquad \text{Shear viscosity}$$

$$\nabla_a S^a = \frac{\tau^2}{\lambda} + \frac{2r^a r_a}{\kappa} + \frac{\tau^{ab}\tau_{ab}}{\lambda} \qquad \text{Entropy prod.}$$

Second order contribution and hyperbolicity

- Second order scalars: $\psi^1 = \xi^{ab}\xi_{ab}$, $\psi^2 = \xi^{ab}\xi_b\xi_{ac}\xi^c$, $\psi^3 = \nu^2$
- Second order generating function:

$$\chi = \chi^0(\mu) + \chi^1(\mu)\,\nu + \sum_{i=0}^{3} \chi_i^2(\mu)\,\psi^i$$

- Integration of the second-order coefficients:

$$\chi_1^2 = \frac{\chi_o^2}{\mu^3}, \quad \chi_2^2 = -\frac{12}{\mu}\chi_1^2, \quad \chi_3^2 = \frac{24}{\mu^2}\chi_1^2$$

- Hyperbolicity near equilibrium solutions

$$\mathcal{K}_{AB}^{a}\nabla_a\xi^B = I_A, \quad K_{AB}^{a} = \begin{pmatrix} \frac{\partial^3\chi^o}{\partial\xi_a\partial\xi^b\partial\xi^c} & \frac{\partial^3\chi^1}{\partial\xi_a\partial\xi^b\partial\xi^{cd}} \\ \frac{\partial^3\chi^1}{\partial\xi_a\partial\xi^b\partial\xi^{cd}} & \frac{\partial^3\chi^2}{\partial\xi_a\partial\xi^{bc}\partial\xi^{de}} \end{pmatrix}$$

Theorem (MR, Lehner, Reula ['18])

Let $(\mathcal{M}, g_{ab})$ and ξ^a any equilibirum solution. Then, it is always possible to find a symmetric-hyperbolic second-order dissipative theory in some neighbourhood of ξ^a.

Numerical implementation of the full theory

- System of conservation laws:

$$
\begin{aligned}
\partial_t T^{oo} &= -\partial_i T^{oi} \\
\partial_t T^{oi} &= -\partial_j T^{ij} \\
\partial_t A^{ooi} &= -\partial_j A^{joi} + I^{oi} \\
\partial_t A^{oij} &= -\partial_k A^{kij} + I^{ij}
\end{aligned}
$$

- Dynamical variables $\rightsquigarrow \{T^{oo}, T^{oi}, A^{ooi}, A^{oij}\}$
- Fluxes $\rightsquigarrow \{T^{ij}, A^{ioo}, A^{joi}, A^{kij}\}$

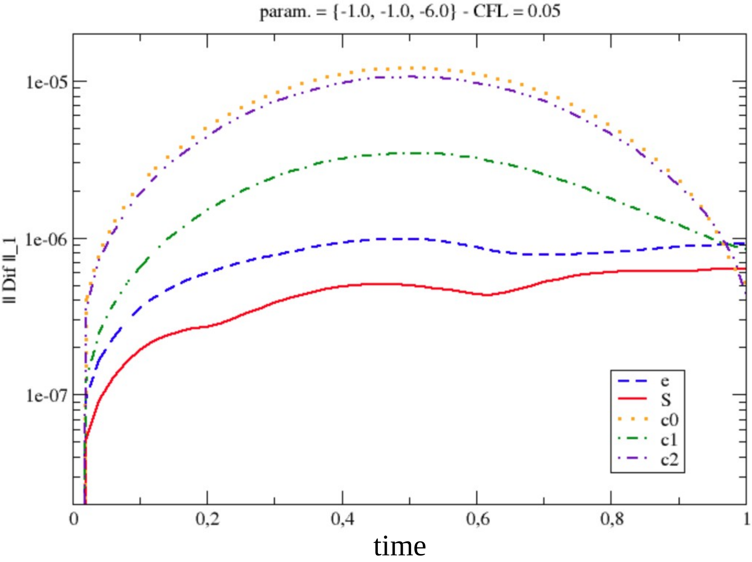
Problem

Find expressions for the fluxes in terms of the dynamical variables.
$\rightsquigarrow$ ¿Does conformal invariance help to address this issue in a simpler way? **No!** Iterative methods need to be implemented in order to get

$$\text{Physical Variables} \quad \mapsto \quad \text{Abstract variables}$$

$\rightsquigarrow$ 1-D version (rotational symmetry in the planes $\perp$ to v)

Convergence and preliminary results



Convergence and preliminary results

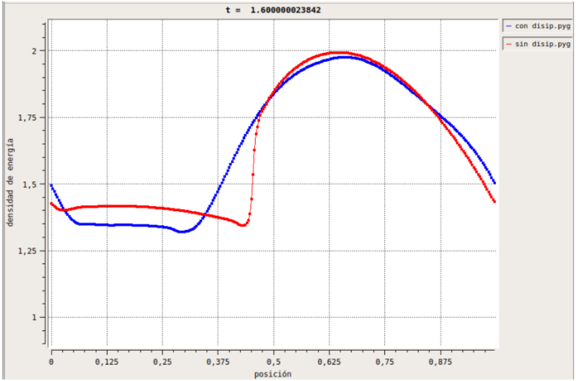


Figure 1: Red line: perfect fluid case; blue line: dissipative fluid case

Conclusions and future perspectives

- We have developed a well-posed theory in order to describe and have a better understanding about the dynamics of ultra-relativistic dissipative fluids on high-temperature regimes.
- A similar treatment can be developed for a general second-order relativistic fluid theory (i.e., with an arbitrary equation of state and including particle-number conservation).
- A numerical code for evolving these fluids propagating in one direction has been developed. We plan to use this toy model in order to understand different properties and effects that are not currently well understood (scaling relations, Riemann problem, turbulence effects, etc).
- We are working on a 3D code in order to address more "real" problems regarding astrophysical plasmas around compact objects (among other projects).