Non-linear Electrodynamics for astrophysical plasmas

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Introduction and Motivations

- Non-linear Electrodynamics (NLED) turns out to be relevant in several contexts:
  - QCD: Light-light scattering leads to non-linear effects due to vacuum polarization (Euler-Heisemberg Lagrangian);
  - Condensed matter: Interaction between the molecules of some dielectrics and crystals and external electromagnetic fields (typically observed at very high light intensities);
  - Cosmology and Astrophysics: non-linearities may play important roles in the description of the dark sector of the Universe, avoidance of singularities, physics of charged black holes, ...
- In a covariant framework, NLED theories are described from a Lagrangian density

$$\mathcal{L}(F, G)$$

  where $F = F^{ab}F_{ab}$ and $G = {}^*F^{ab}F_{ab}$ are the electromagnetic invariants.
- Maxwell's theory $\rightsquigarrow$ $\mathcal{L} = -F/4$, which leads to linear EOM.

Well-posedness of NLED theories?

- A crucial aspect to address when considering extensions of well-known theories is the corresponding Initial-Value problem. Why?
  - Predictabiity power of the theory;
  - Tell us about the propagation speeds of info;
  - Numerical implementation: avoidance of mathematical instabilities;
  - energy estimates
- Well-posedness of a theory implies:
  - Existence (at least for a short time interval);
  - Uniqueness (predictability power);
  - Stability (continuous dependence on the initial data):

$$\|u(t,x)\| \leq Ke^{\alpha t}\,\|u(0,x)\|$$

- Hyperbolicity analysis of the EOM: set of algebraic conditions that the principal symbol of the system must satisfy in order to guarantee a well-posed theory [Hadamard, Friedrichs, Kreiss, Geroch, ...]
  - NLED has constraint equations, and thus the evolution equations are generally not uniquely defined. The distinction of an evolution system over the others generally implies a choice of coordinates, and covariance breaks down.

Force-Free Electrodynamics

- Evolution of the electromagnetic field in the presence of a magnetically dominated plasma, which are believed to play a key role in the physics of pulsars and AGN's.
- In those regimes,the electromagnetic field dominates over the matter interactions and its dynamics decouples from the matter degrees of freedom, obeying a modified (non-linear) version of Maxwell's equations.
- Conditions for the FF approximation:
  - Justify the presence of the plasma on the surroundings of the central object;
  - the plasma mass density is much lower (by orders of magnitude) than the electromagnetic field energy density.

Force-Free Electrodynamics

- **Goldreich-Julian:** *"gravitationally induced electric field along magnetic lines of a spinning NS with a dipolar magnetic field is strong enough as to being capable of pulling charges into the surrounding space, thus generating a plasma"* (1969)
- **Wald** $\rightsquigarrow$ exact solution for vacuum electrodynamics on a stationary and axisymmetric spacetime "immersed" on a uniform magnetic field aligned with the rotation axis (1974)
  - It also possesses non-zero $\vec{E} \cdot \vec{B}$ near the black hole horizon!
- **Blandford-Znajek**
  - vacuum solutions are unstable to a pair production cascade under typical astrophysical situations;
  - a force-free magnetosphere (with a magnetized accretion disk) would be produced near a rotating black hole (1977)

Force-Free Electrodynamics

- Plasma dynamics is described by the laws of (ideal) Hydrodynamics:

$$\nabla_a N^a = 0\,, \qquad \nabla_a T^{ab} = 0$$

where $T^{ab} = T^{ab}_{EM} + T^{ab}_{MAT}$ and

$$T^{ab}_{EM} = \frac{1}{4\pi}\left(F^a{}_c F^{bc} - \frac{1}{4} g^{ab} F_{cd} F^{cd}\right),$$

coupled to (linear) Maxwell's equations

$$\nabla_a F^{ab} = 4\pi J^b\,, \qquad \nabla_{[a} F_{bc]} = 0$$

- Force-Free approximation neglects the matter contribution with respect to the electromagnetic counterpart:

$$T^{ab} \simeq T^{ab}_{EM},$$

which implies, from the local conservation,

$$0 = \nabla_a T^{ab} = \nabla_a T^{ab}_{EM} \propto F^b{}_c J^c \quad \rightsquigarrow \quad \boxed{F^a{}_b J^b = 0}$$

Force-Free Electrodynamics

- From the degeneracy condition $F^a{}_b\, J^b = 0$ and Maxwell's equations

$$\nabla_a F^{ab} = 4\pi J^b\,, \qquad \nabla_{[a} F_{bc]} = 0,$$

  we obtain the Force-Free equations

$$\boxed{F^a{}_b \nabla_c F^{bc} = 0\,, \qquad \nabla_{[a} F_{bc]} = 0}$$

- Algebraic consequences of $F^a{}_b\, J^b = 0$.
  - F^{ab} is degenerate $\rightsquigarrow G = 0$
  - $F > 0 \rightsquigarrow$ Magnetically dominated system!
  - $\mathrm{Ker}(F^a{}_b)$ is a 2-dimensional integrable surface.
  - F_{ab} is algebraically simple $\rightsquigarrow F_{ab} = 2\ell^1_{[a}\ell^2_{b]}$.
- Well-posedness (F. Carrasco, O. Reula [16']) of FF equations is guaranteed when considering F^{ab} as the dynamical variables.
  - Numerical simulations of truly stationary jets;
  - Both the aligned and misaligned cases exhibit a collimated Poynting flux along the direction of the asymptotic magnetic field.

Alternative formulation: Euler Potentials

- Degeneracy of F^{ab} and integrability of $\mathrm{Ker}(F^a{}_b)$ implies the existence of two scalar functions ϕ_1, ϕ_2 such that

$$F_{ab} = \nabla_a \phi_1 \nabla_b \phi_2 - \nabla_b \phi_1 \nabla_a \phi_2$$

- ϕ_i are called "Euler Potentials". Using them as dynamical variables, the non-trivial Force-Free equations reduce to

$$\begin{cases} \nabla_a \phi_1 \nabla_c \left(\nabla^a \phi_1 \nabla^c \phi_2 - \nabla^a \phi_2 \nabla^c \phi_1 \right) & = & 0 \\ \nabla_a \phi_2 \nabla_c \left(\nabla^a \phi_1 \nabla^c \phi_2 - \nabla^a \phi_2 \nabla^c \phi_1 \right) & = & 0 \end{cases}$$

- Advantages:
  - There are only two scalar dynamical variables!
  - Suitable for a numerical implementation

Issue:

Does FFE in Euler potentials admite a well-posed initial-value formulation?

- Strategy: analyze the hyperbolicity of an equivalent first-order reduction.

First-order systems

Constant–coefficient case: Consider the system

$$(*)\begin{cases} u_t &= A^i \partial_i u =: P(\partial)\, u \\ u(x,0) &= f(x) \end{cases}$$

$u = u(x,t) \in \mathbb{R}^s$; $x = (x_1, \cdots, x_n)$ coordinates; $A^i \in \mathbb{R}^{s\times s}$.

- If the initial profile has the form

  $$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\omega\cdot x} \hat{f}(\omega)\, d^n\omega, \quad \hat{f} \text{ comp. supp.}$$

  there is a unique solution, given by

  $$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\omega\cdot x} e^{P(i\omega)t} \hat{f}(\omega)\, d^n\omega,$$

  where $P(i\omega)$ is obtained by identifying $\partial/\partial x_j \leftrightarrow i\omega_j$.

Definition

System (*) is **well posed** if there are constants $K > 0$ and α such that, for $t \geq 0$,

$$|e^{P(i\omega)t}| \leq K e^{\alpha t}, \quad \forall\, \omega \in \mathbb{R}^s.$$

First-order systems

$$\text{W-P} \;\leftrightarrow\; |e^{P(i\omega)t}| \leq K e^{\alpha t} \;\leftrightarrow\; |e^{(P(i\omega)-\alpha I)t}| \leq K$$

- **Issue:** Characterize all matrices A such that $|e^{At}| \leq K$, with K independent of A!
- **Theorem** (Kreiss, '89). If F is a set of matrices $A \in \mathbb{C}^{n\times n}$,

$$\begin{array}{c} \exists\ K_1 \text{ such that} \\ |e^{At}| \leq K_1\ \forall\ A \in F \end{array} \quad \Leftrightarrow \quad \begin{array}{l} \forall\ A \in F,\ s \in \mathbb{C},\ \text{Re}\ s > 0, \\ A - sI \text{ is non singular and} \\ \exists\ K_2 \text{ such that} \\ |(A - sI)^{-1}| \leq \dfrac{K_2}{\text{Re}\ s} \end{array}$$

- For general first-order systems, the algebraic characterization for hyperbolicity requires smoothness in the coefficients.
- **Theorem** (Strang, '66). If the system

$$u_t = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u,$$

is well-posed, then the system that results by evaluating A_α on x_0 is also well-posed.

Failure of Kreiss criterion

- Fourier transformed equations

$$\boxed{-\varepsilon^{ij} G_{kj} k_0^2 \varphi_i + \varepsilon^{ij} \left[|\vec{k}|^2 G_{kj} - (\vec{\ell}_k \cdot \vec{k})(\vec{\ell}_j \cdot \vec{k}) \right] \varphi_i = 0}$$

- Defining now the variables

$$u_i = k_0 \varphi_i \; ; \quad v_i = |\vec{k}| \varphi_i$$

the system is equivalent to

$$\underbrace{\begin{pmatrix} k_0 & 0 & -\frac{k_1^2+k_3^2}{|\vec{k}|} & -\frac{(\vec{\ell}_1\cdot\vec{k})(\vec{\ell}_2\cdot\vec{k})}{|\vec{k}|G_{22}} \\ 0 & k_0 & -\frac{(\vec{\ell}_1\cdot\vec{k})(\vec{\ell}_2\cdot\vec{k})}{|\vec{k}|G_{11}} & -\frac{k_2^2+k_3^2}{|\vec{k}|} \\ |\vec{k}| & 0 & -k_0 & 0 \\ 0 & |\vec{k}| & 0 & -k_0 \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = 0.$$

- $\mathbb{A}$ is **not diagonalizable** if $k_0 = k_3 = 0$!

Failure of Kreiss criterion

- For $k_0 = k_3 = 0$, choose $k_1 = k_2 = \sqrt{2}\kappa$, $0 < \kappa \in \mathbb{R}$, y $s \in \mathbb{R}$. Then

$$\mathbb{A} - sI = \begin{pmatrix} -s & 0 & -\kappa & -\alpha\kappa \\ 0 & -s & -\kappa/\alpha & -\kappa \\ 2\kappa & 0 & -s & 0 \\ 0 & 2\kappa & 0 & -s \end{pmatrix}, \quad \alpha := \frac{|\ell_1|}{|\ell_2|} = \text{const.} > 0 ,$$

- The inverse was found to be

$$(\mathbb{A} - sI)^{-1} = \frac{1}{s^2\left(s^2 + 4\kappa^2\right)} \times$$

$$\begin{pmatrix} -s\left(s^2+2\kappa^2\right) & -2\alpha\kappa^2 s & \kappa s^2 & -\alpha\kappa s^2 \\ \frac{2\kappa^2 s}{\alpha} & s\left(s^2+2\kappa^2\right) & \frac{\kappa s^2}{\alpha} & -\kappa s^2 \\ -2\kappa\left(s^2+2\kappa^2\right) & -4\alpha\kappa^3 & -s\left(s^2+2\kappa^2\right) & -2\alpha\kappa^2 \\ \frac{4\kappa^3}{\alpha} & 2\kappa\left(2\kappa^2 - s^2\right) & \frac{2\kappa^2 s}{\alpha} & s\left(s^2+2\kappa^2\right) \end{pmatrix}$$

- There are matrix elements such that do not satisfy Kreiss's bound for well-posedness!
- Force-Free Electrodynamics in Euler Potentials does not have a well-posed initial value formulation.

Final remarks

- We studied in detail the initial value problem of Force-Free Electrodynamics using Euler Potentials as the unique dynamical variables of the theory.
- Unlike the case in which F^{ab} is evolved, this alternative formulation is weakly-hyperbolic, meaning that there are physical modes that grow in a way that cannot be bounded by the corresponding initial data.
- We conclude that this alternative formulation is not convenient in order to implement the Force-Free equations numerically.
- As future work, there is the intention to see how does the instabilities of the FFE with Euler Potentials behave, in order to have a better understanding of it dynamical evolution.
- Using the primitive FFE's, it is possible to simulate astrophysical jets emerging from compact objects, as black holes, neutron stars, pulsars and magnetars.