Mathematical description of the influence of the Universe expansion on the black hole metrics.

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The solution of Schwarzschild is obtained from a general ansatz with the form

$$ds^2 = e^{2A(r)}dt^2 - e^{2B(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Where $A(r)$ and $B(r)$ are two functional forms that must be determined

To obtain the solution of the Einstein field equations considering the effect of the expansion, the general Ansatz of the solution must be disturbed by multiplying the scale factor to the spatial terms.

$$ds^2 = e^{2A(r)}c^2dt^2 - a(t)^2 e^{2B(r)}dr^2 - a(t)^2 r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The non-trivial symbols of Christoffel are

$$\left\{\begin{array}{lll}
\Gamma^r_{tt} = e^{2(A-B)}\frac{c^2}{a^2}A', & \Gamma^\theta_{r\theta} = r^{-1}, & \Gamma^\varphi_{\theta\varphi} = \cot\theta, \\
\Gamma^t_{tr} = A', & \Gamma^\varphi_{r\varphi} = r^{-1}, & \Gamma^r_{\varphi\varphi} = -r\sin^2\theta e^{-2B}, \\
\Gamma^r_{rr} = B', & \Gamma^r_{\theta\theta} = -re^{-2B}, & \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \\
\Gamma^\theta_{t\theta} = \frac{\dot{a}}{a}, & \Gamma^r_{tr} = \frac{\dot{a}}{a}, & \Gamma^t_{rr} = e^{2(B-A)}\frac{a\dot{a}}{c^2}, \\
\Gamma^t_{\varphi\varphi} = e^{-2A}r^2\sin^2\theta\frac{a\dot{a}}{c^2}, & \Gamma^t_{\theta\theta} = e^{-2A}r^2\frac{a\dot{a}}{c^2}, & \Gamma^\varphi_{t\varphi} = \frac{\dot{a}}{a},
\end{array}\right\}$$

So that the non-zero components of the Ricci tensor are

$$R_{tt} = -\frac{c^2}{a^2}A''e^{2A-2B} - \frac{c^2}{a^2}A'^{\,2}e^{2A-2B} + \frac{3\ddot{a}}{a} + \frac{c^2}{a^2}A'B'e^{2A-2B} - \frac{2}{r}\frac{c^2}{a^2}A'e^{2A-2B}$$

$$R_{rr} = A'' + A'^{\,2} - A'B' - \frac{2}{r}B' - \frac{a^2}{c^2}\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)e^{2B-2A},$$

$$R_{\theta\theta} = -\frac{r^2}{c^2}\ddot{a}ae^{-2A} - rB'e^{-2B} + rA'e^{-2B} - 2r^2\,\frac{\dot{a}^2}{c^2}e^{-2A} + e^{-2B} - 1,$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta},$$

where the prime denotes the derivative with respect to r, and the dot with respect to t

Disturbed system of equations

$$\left(A'' + 2A'^{\,2} + \frac{2}{r}A'^{\,2}\right)e^{2A-2B} - \Gamma\left(\frac{rA'}{2} + 1\right) = \Delta$$

$$\left(\frac{2A'}{r} + \frac{1}{r^2}\right)e^{2A-2B} - \frac{e^{2A}}{r^2} = \Delta + \frac{\Gamma}{2}.$$

Where

$\Gamma = \text{cte} = 2\dot{H}\frac{a^2}{c^2}.$, $\Delta = \left(\frac{\Gamma}{2} + 3H^2\frac{a^2}{c^2}\right)$ and H is the Hobble's Constant

These mathematical equations are used

$$H = \frac{\dot{a}}{a}, \qquad \dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2, \qquad \dot{H} + H^2 = \frac{\ddot{a}}{a},$$

	Results of V.V.Kiselev, *Quintessence and black hole*. 2003	Our results
General Solution	$e^{2A(r)} = 1 - \frac{r_s}{r} - \frac{C}{r^{3\omega_q+1}}.$	$e^{2A} \approx \left(1 - \frac{r_s}{r} \pm \sqrt{\frac{K}{2}}\, r\right), \quad K = \Delta + \Gamma/2$ $e^{2A} = \left(1 - \frac{r_s}{r} + \delta\right) = e^{-2B}, \quad \delta(r) = \frac{C}{r^{\alpha}}, \quad \delta \approx \pm\sqrt{\frac{K}{2}}\, r,$
For free quintessence $r \gg r_s,\ r_s/r \approx 0$	$g_{tt} = -\frac{1}{g_{rr}} = 1 - \frac{C}{r^{3\,(-\frac{2}{3})+1}} = \left[1 - \frac{r}{C}\right]$	$e^{2A} \approx \left(1 \pm \sqrt{\frac{K}{2}}\, r\right),$ $\alpha = (3\omega + 1), \quad \omega = -2/3, \quad \alpha = -1$