Van Kampen Modes for Bunch Longitudinal Motion

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Main equations

$$H(z,p)=\frac{p^2}{2}+U(z)+V(z)$$ Hamiltonian

$$U(z)=U_{\mathrm{rf}}(z)-\int\lambda(z')W(z-z')dz'$$ Steady state potential

$$V(z)=-\int\rho(z')W(z-z')dz'$$ Perturbation of the potential

$$\lambda(z)=\int F(I)dp$$ Steady state linear density

$$\rho(z)=\int f(I,\phi)dp$$ Linear density perturbation

$$\frac{\partial f}{\partial t}+\Omega(I)\frac{\partial f}{\partial\phi}-\frac{\partial V}{\partial\phi}F'(I)=0$$ Vlasov equation

Steady state distribution

$$U(z) = U_{\text{rf}}(z) - \int \lambda(z') W(z - z') dz' \equiv U_{RHS}[\lambda]$$

$$\lambda(z) = \int_{U(z)}^{H_{\max}} \frac{F(I(H))}{\sqrt{2(H - U(z))}} dH$$

$$I(H) = \frac{1}{\pi} \int \sqrt{2(H - U(z))} dz$$

Numerical solution: use of an artificial time:

$$U(z) = U_{RHS}[\lambda] \quad \rightarrow \quad U(z, t + \Delta t) = U(z,t) - \varepsilon \left(U(z,t) - U_{RHS}[\lambda] \right)$$

$$\varepsilon \cong 0.05 - 0.25$$

$$U(z, t = 0) = U_{\text{rf}} \; ; \quad \lambda(z, t = 0) = \lambda_{\text{rf}}(z)$$ ← as if wake=0 at t=0

Takes 3-30 min with my Mathematica code

Vlasov Equation

$$\frac{\partial f}{\partial t} + \Omega(I)\frac{\partial f}{\partial \phi} - \frac{\partial V}{\partial \phi}F'(I) = 0$$

$$V(z) = -\int \rho(z')W(z-z')dz'$$

Following Oide & Yokoya (1990) :

$$f(I,\phi,t) = e^{-i\omega t}\sum_m \left[f_m(I)\cos m\phi + g_m(I)\sin m\phi \right]$$

$$\left[\omega^2 - m^2\Omega^2(I)\right] f_m(I) = -2m^2\Omega(I)F'(I)\int dI' \sum_n V_{mn}(I,I')f_n(I')$$

$$V_{mn}(I,I') = -\frac{2}{\pi}\int_0^\pi d\phi \int_0^\pi d\phi' \cos(m\phi)\cos(n\phi')W(z-z')$$

matrix elements

$$z(I,-\phi) = z(I,\phi)\,; \quad p(I,-\phi) = -p(I,\phi)$$

phase definition

$$z(I,\phi=0) = z_{\min}(I) \quad z(I,\phi=\pi) = z_{\max}(I)$$

Matrix Elements

$$V_{mn}(I,I') = -2\,\mathrm{Im}\int_0^{\infty} dq\,\frac{Z_{||}(q)}{q} G_m(q,I) G_n^*(q,I')$$

$$G_m(q,I) = \int_0^{\pi} \frac{d\phi}{\pi} \cos(m\phi) e^{iqz(I,\phi)}$$

Note: No bunch-to-bunch interaction here yet

These equations solve the problem.

CPU time ~ (number of azimuthal modes)^2.

If the wake is not strong compared with the RF, azimuthal mode coupling can be neglected, at least as a first step analysis.

Particle loss, emittance growth, or instability may happen because of:

- Finite bucket capacity. For a full RF bucket, and effectively repulsive wake, the threshold is 0.

- Azimuthal or radial mode coupling.
  - Azimuthal mode coupling requires rather high intensity
  - Radial mode coupling may happen either for non-monotonic distributions $F(I)$, or for significantly asymmetric distorted potential well.

- Loss of Landau damping.

- Bunch-to-bunch interaction and insufficient Landau damping.

- Improper tuned damper or feedback.

Integral Equation

Neglecting azimuthal mode coupling:

$$\left[\omega^2 - m^2\Omega^2(I)\right] f(I) = -m^2\Omega(I) F'(I) \int V_m(I, I') f(I') dI'$$

$$V_m(I, I') = -\frac{2}{\pi}\int_0^{\pi} d\phi \int_0^{\pi} d\phi' \cos(m\phi)\cos(m\phi') W(z - z') =$$

$$= -2\,\mathrm{Im}\int_0^{\infty} dq \frac{Z(q)}{q} G_m(q, I) G_m^*(q, I')$$

This is eigensystem problem for the Vlasov equation

Van Kampen modes for plasma

- What are the eigenfunctions and eigenvalues for the Vlasov equation?

$$\frac{\partial \tilde{f}}{\partial t} + p\frac{\partial \tilde{f}}{\partial z} - \alpha\frac{\partial \tilde{\lambda}}{\partial z}F' = 0 \ ; \quad \tilde{\lambda} \equiv \int \tilde{f}dp\, ; \quad \alpha = u_p^2 = \omega_p^2 / k^2$$

- With $\tilde{f} = \exp(ikz - i\omega t)\tilde{f}_\omega(p)$, it leads to

$$(v - p)\tilde{f}_v(p) = -\alpha F'(p)\int \tilde{f}_v(p')dp'\, ; \quad v = \omega / k$$

- Eigenfunctions $\tilde{f}_\omega(p)$ can be normalized as $\int \tilde{f}_\omega(p)dp = 1$. With that, an infinite set of solutions follows:

$$\tilde{f}_v(p) = -\text{P.V.}\frac{\alpha F'(p)}{v - p} + A(v)\delta(v - p)\, ; \quad A(v) = 1 + \alpha\,\text{P.V.}\int \frac{F'(p')dp'}{v - p'}$$

Van Kampen modes for plasma (2)

$$\tilde{f}_v(p) = -\mathrm{P.V.}\frac{\alpha F'(p)}{v-p} + A(v)\delta(v-p)\,; \quad A(v) = 1 + \alpha\,\mathrm{P.V.}\int \frac{F'(p')dp'}{v-p'}$$

Here v is an arbitrary real number. The eigenfrequency ◆$=kv$.
This infinite set of eigenmodes was found for plasma oscillations in 1955 by a Dutch physicist N. G. van Kampen.

Generally, the spectrum consists of continuous and discrete parts.
Continuous modes are singular and stable, $v=p$.
Discrete modes are smooth, $A(v)=0$. They may be unstable (not necessarily). In that case, growing and decaying modes appear by pairs:

$$\mathrm{Im}(\omega_\mu) = \pm\Gamma_\mu$$

If the discrete set is empty, any smooth initial condition can be expanded over the continuous basis. The Landau damping results as phase mixing of the singular continuous modes. Thus, stable discrete mode describes loss of Landau damping (LLD)

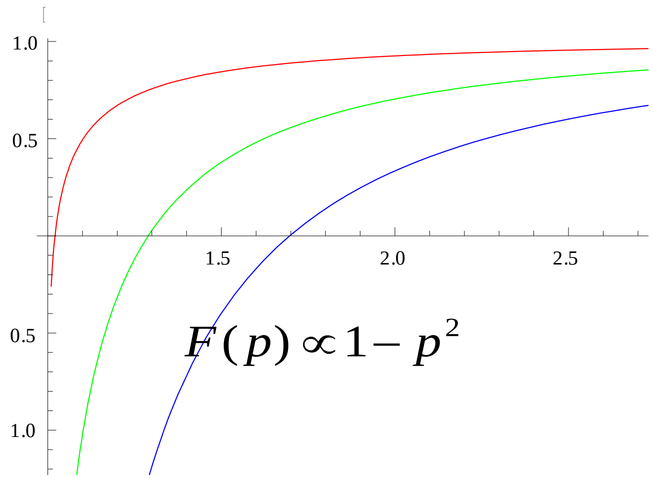
If the distribution function is monotonic, $pF'(p) \le 0$, there are no unstable modes.

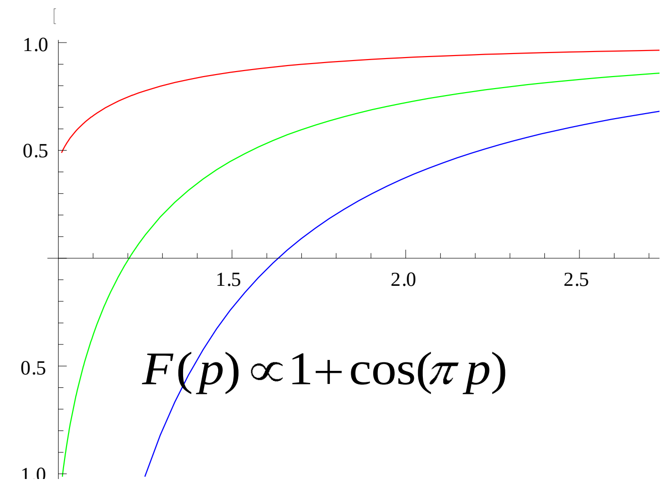
Van Kampen modes normally appear as numerical solutions of Vlasov equation.

Loss of Landau damping for van Kampen modes

For finite width of the distribution, $F(|p| > p_{\max}) = 0$, a discrete undamped mode can appear outside the continuous spectrum. The condition for that is

$$\varepsilon(v) = 1 + u_p^2 \int \frac{F'(p)dp}{v-p} = 0 \,, \quad u_p = \omega_p / k$$





$u_p =$ 0.5, 1.0, 1.5

For the hard-edge distribution (left), the discrete mode appears at any interaction parameter.

For the soft-edge case (right), there is a threshold for LLD.

Van Kampen modes for a bunch

$$\left[\omega^2 - m^2\Omega^2(I)\right] f(I) = -m^2\Omega(I)F'(I)\int V_m(I,I')f(I')dI'$$

This spectral equation is similar to a classical plasma case (van Kampen).

Generally, it gives continuous and discrete spectrum.

The modes of continuous spectrum are singular, their frequencies coincides with the particle spectrum in the distorted potential well, and their decoherence describes Landau damping:

$$f_\omega(I) = A(I)\,\mathrm{P.V.}\frac{1}{m\Omega(I)-\omega} + B(I)\delta(m\Omega(I)-\omega) + C(I)$$

Instead, the discrete modes are smooth regular functions. They are either unstable, or without Landau damping. The discrete spectrum not necessarily exists.

At zero current limit, there are either no discrete modes, or they go infinitesimally close to the continuous spectrum.

Parabolic RF: instability w/o threshold

- For parabolic RF potential, there is always at least a single mode of the discrete spectrum: bunch motion as a whole. There are no resonant particles there, since all the incoherent frequencies are either suppressed or elevated by the potential well distortion.

- Although $\mathrm{Im}(\omega_{11}) = 0$, a slightest bunch-to-bunch talk would make this mode unstable. This is an example of loss of Landau damping (LLD instability).

- Check for this mode is a good tool to verify the code.

- In general, there is no that mode for other RF shape.

RF: SH, BS and BL

$$U_{\rm rf}(z) = (1-\cos z) + \frac{\alpha_2}{4}(1-\cos 2z)$$

$$\alpha_2 = \begin{cases} 0 & \Rightarrow \text{SH} \\ 1 & \Rightarrow \text{BS} \\ -1 & \Rightarrow \text{BL} \end{cases}$$

- SH: single RF harmonic
- BS: bunch shortening 2nd RF harmonic
- BL: bunch lengthening 2nd RF harmonic

Conventional eVs for the action are obtained from its dimensionless value by $\dfrac{E_0}{\omega_{\rm rf}} \dfrac{\nu_{s0}}{\eta h_{\rm rf}}$

For the energy offset this factor is given by $\delta E / E_0 = -p \dfrac{\nu_{s0}}{h_{\rm rf}\eta}$

With ν_{s0} as the zero-amplitude bare synchrotron tune in SH mode.

RW Interaction constant

Below, the results are shown for a pure resistive wake, and $m = 1$

$$W(s) = -k_W / \sqrt{-s}$$

$$Z(q) = k_W (1 - i\,\mathrm{sgn}\,q)\sqrt{\pi q / 2}$$

For these conventional units, the dimensionless interaction constant k_W is:

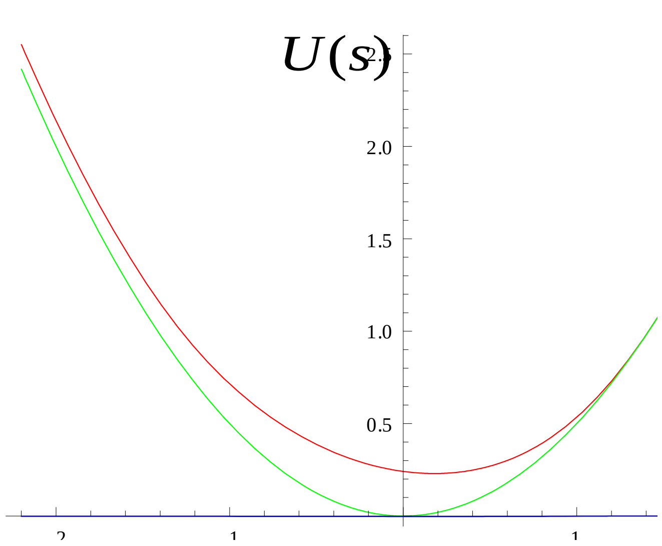
$$k_W = \frac{N r_0 \eta h_{RF}^2}{\gamma b C \nu_{s0}^2} \sqrt{\frac{2 C h_{RF} c}{\pi \sigma}}$$

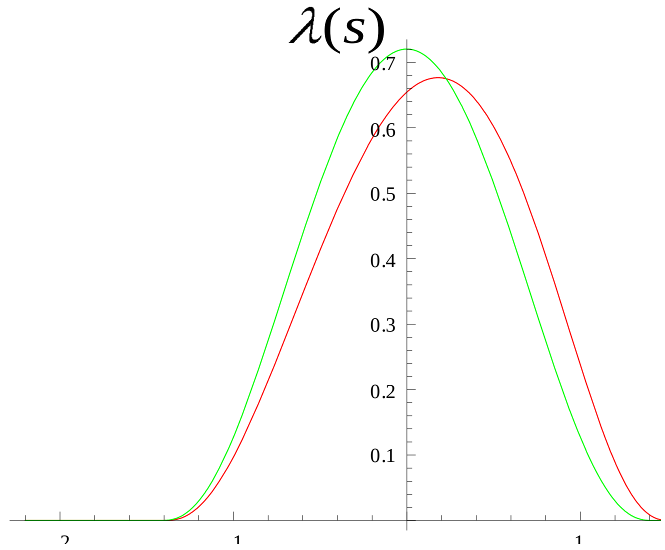
Note that k_W does not depend on the beam energy.

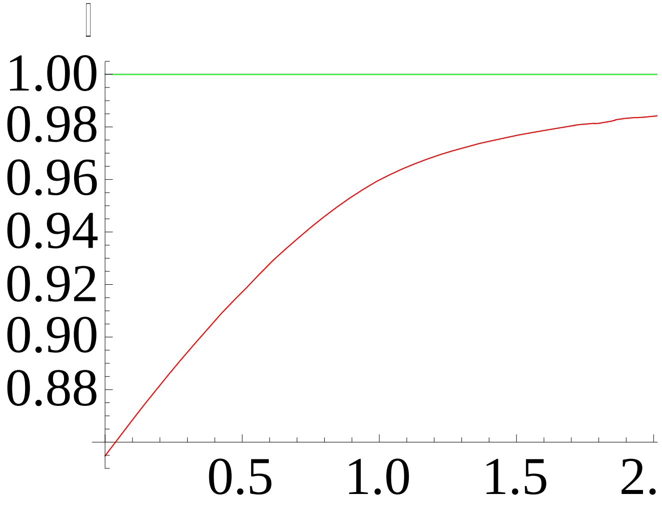
Why Landau damping may be lost

- Particle interaction always acts stronger on the incoherent frequencies (potential well distortion), than on the mode frequencies.
- An example: parabolic potential, where the first mode ω_{11} does not depend on the impedance at all.
- Thus, when all the particle frequencies are wake-suppressed, the highest-frequency mode jumps out of the continuous spectrum, and becomes discrete. For the SH RF, it means that lowest-amplitude particles are mostly excited by this mode.

Parabolic Potential Well Distortion





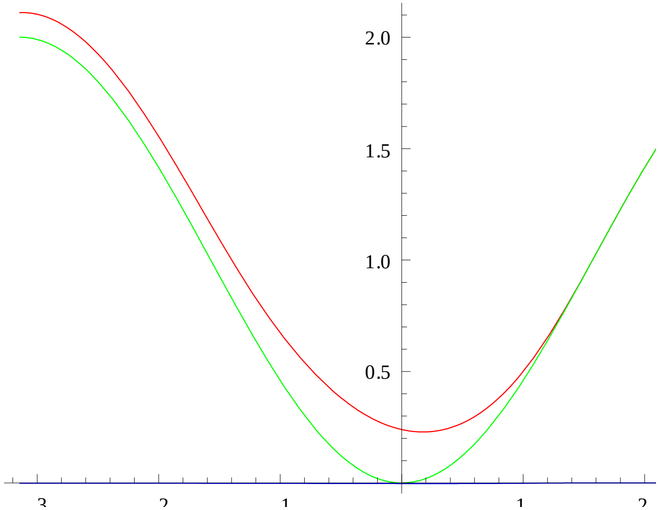


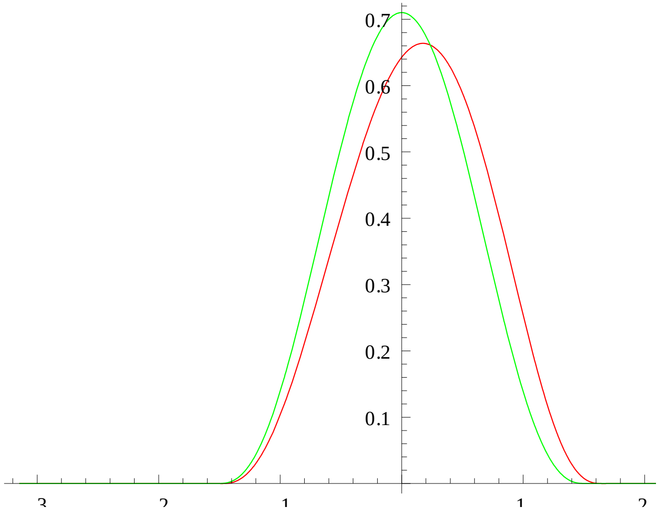
$$k_W = 0.2$$

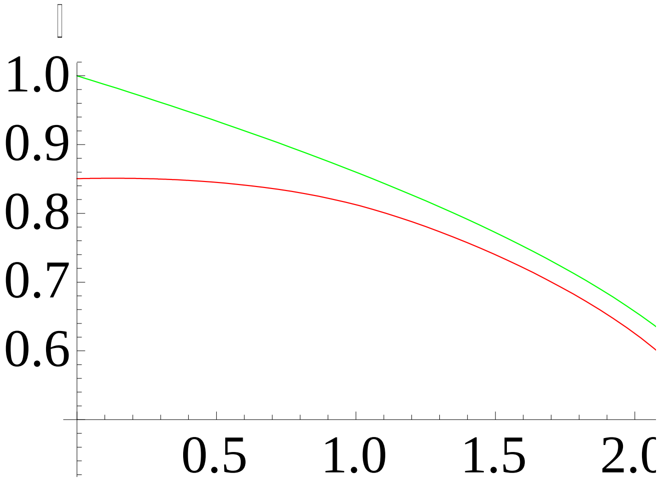
$$F(J) \propto (1 - J / J_{\text{lim}})^2$$

$$J_{\text{lim}} = 1.0$$

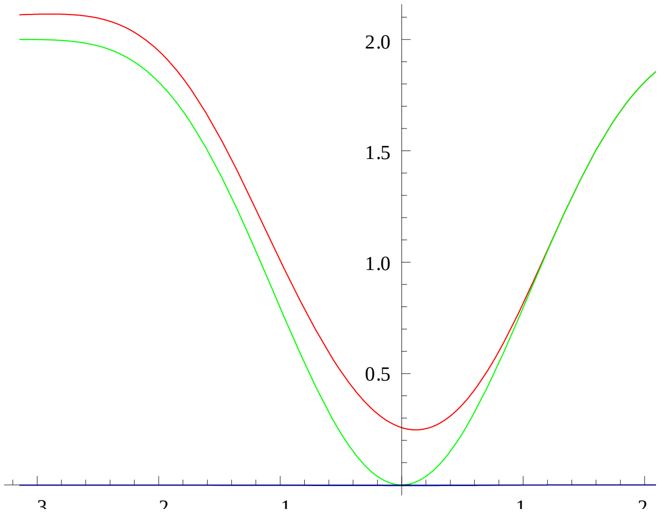
Same for SH

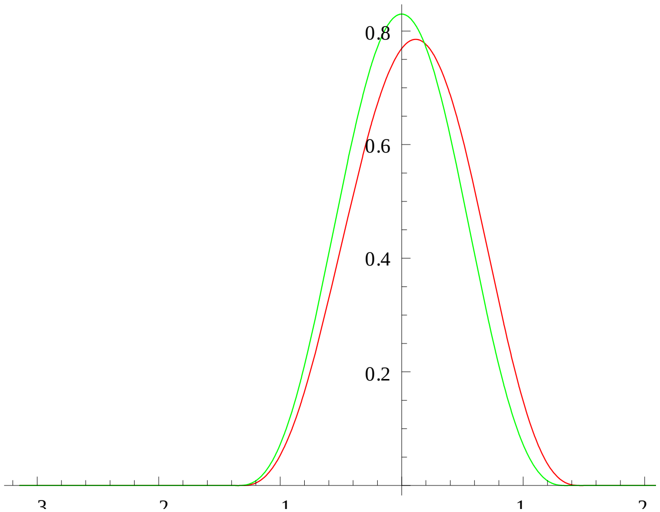


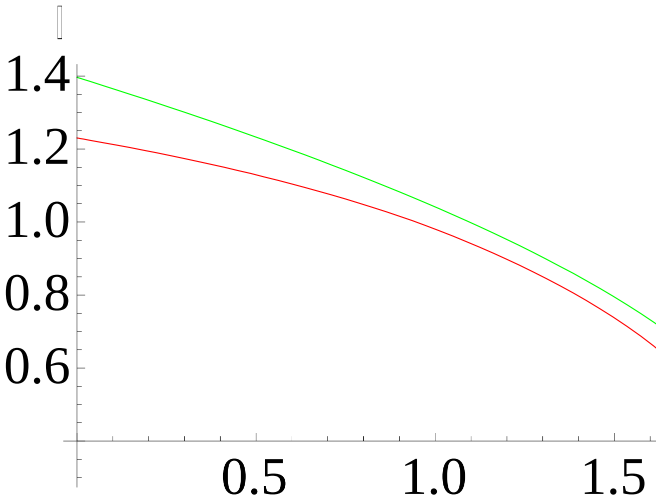




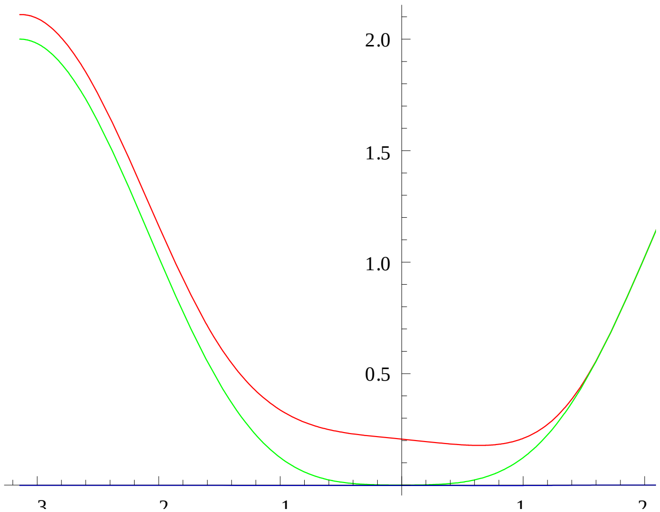
Same for BS

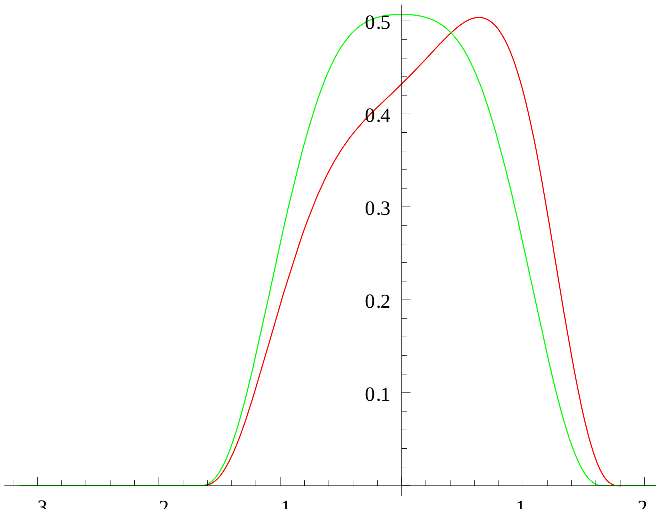


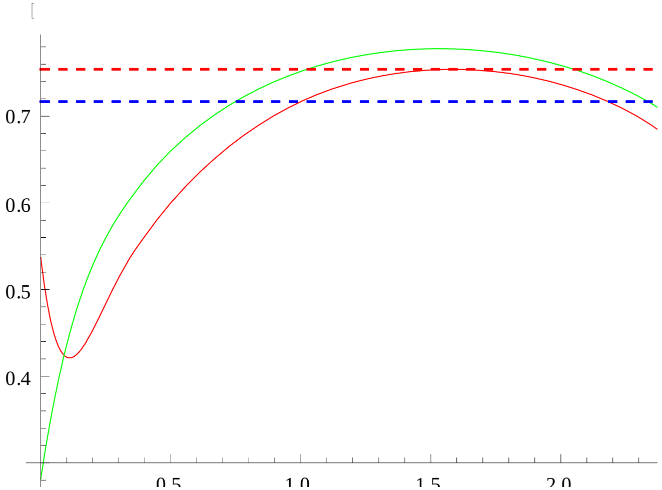




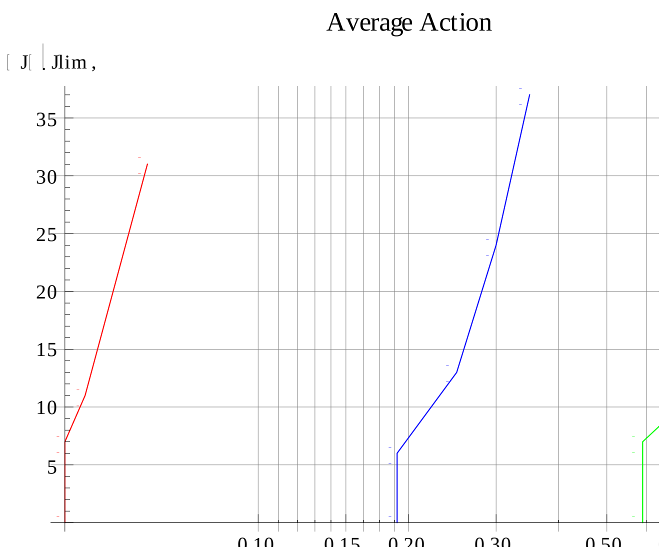
Same for BL

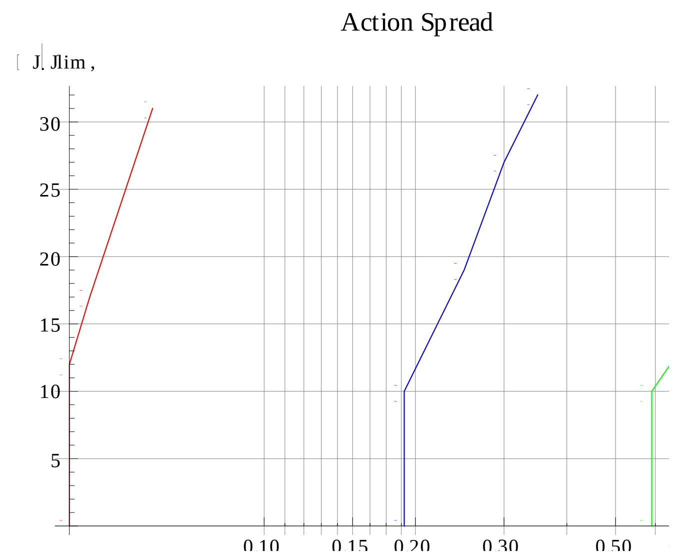




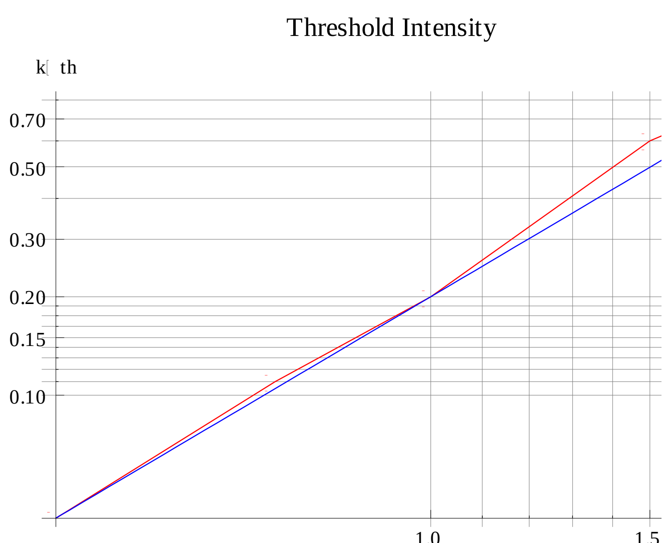


SH case, for the phase space density $F(J) \propto \sqrt{1 - J / J_{\text{lim}}}$





Relative average action and action spread for $J_{\text{lim}} =$ 0.5 , 1.0 , 1.5

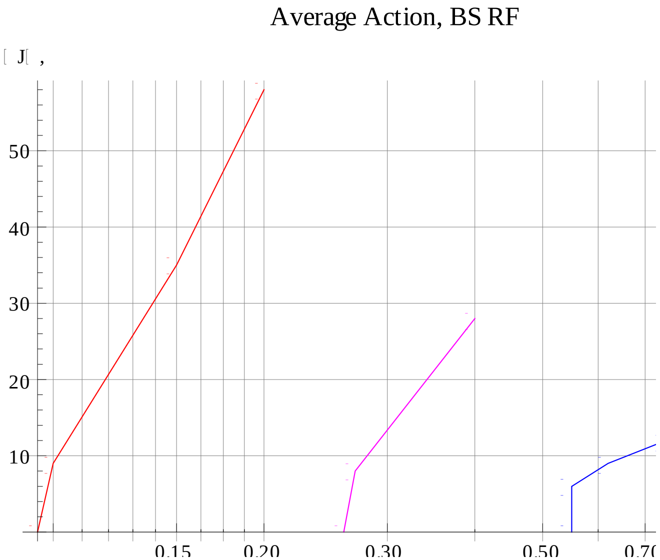


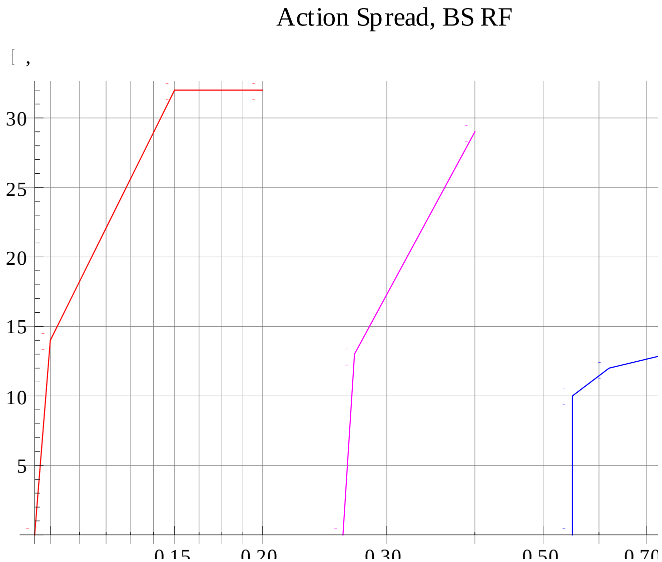
Threshold intensity versus emittance.

Blue line – fit $k_{\text{th}} = 0.2 J_{\text{lim}}^{9/4}$

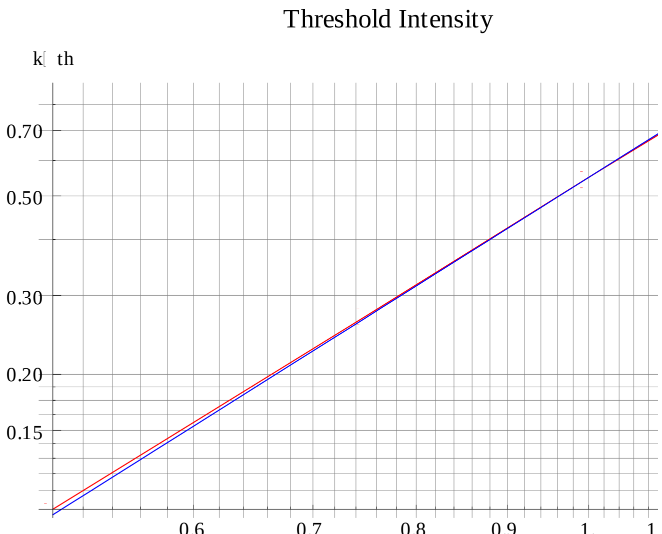
Power 9/4 agrees with naïve rigid-bunch model.

BS RF mode





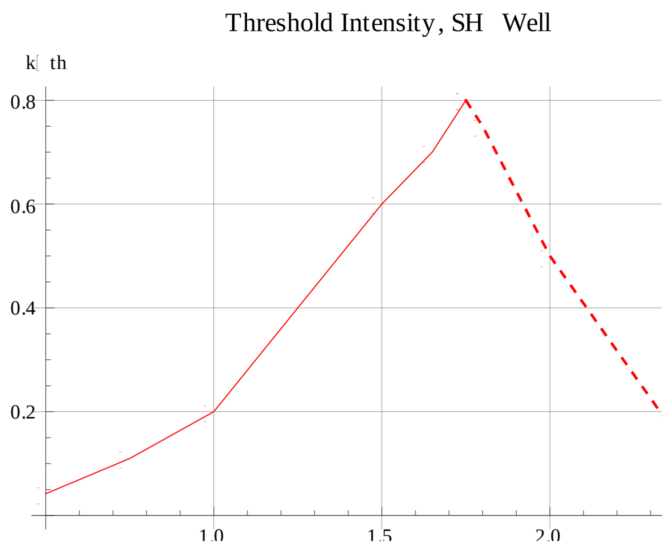
Relative average action and action spread for $J_{\text{lim}} =$ 0.5 , 0.75 , 1.0

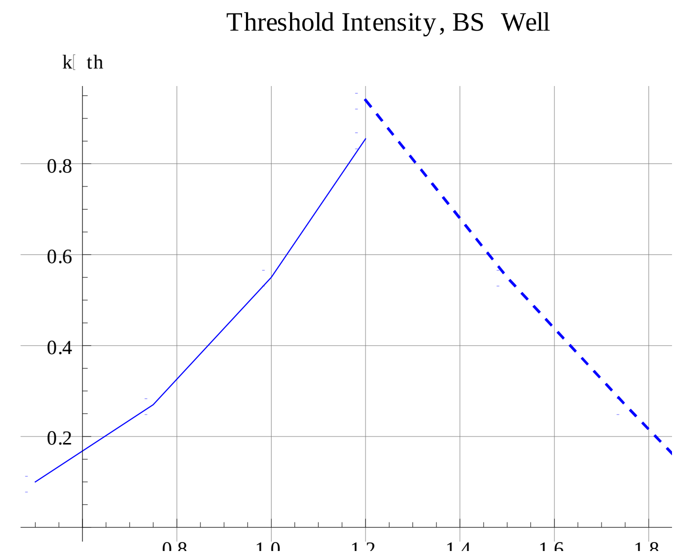


Threshold intensity versus emittance.

Blue line – fit $k_{\text{th}} = 0.55 J_{\text{lim}}^{5/2}$

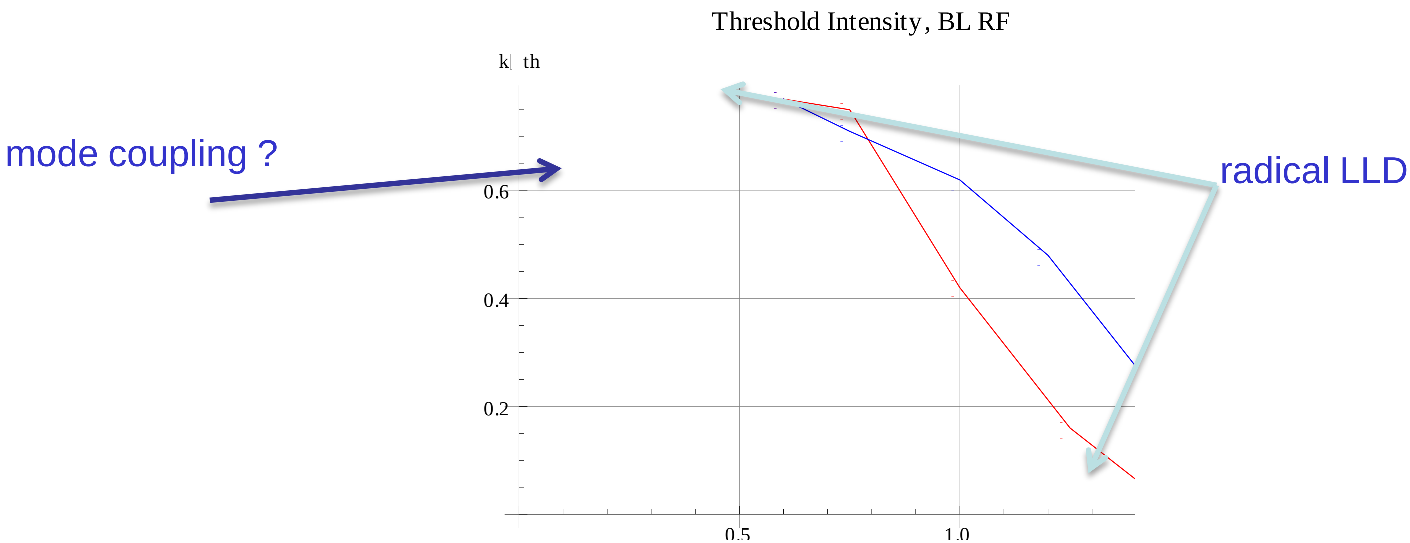
SH and BS, bucket capacity is taken into account





For the Tevatron, with $\mathrm{Re}(Z/n) = 0.8$ at 53 MHz (Ng, Run II HB), the resistive impedance gives $k_w = 0.08$. For $J_{\lim} \approx 0.9$ at the top energy, this corresponds to ~ 15 times below the red-line threshold. The inductive impedance is calculated ~ 2-3 times higher, so the gap may be expected to reduce about that factor. Also, the real distribution drops faster than HP. All that together may yield the gap ~ 2-4. The remaining discrepancy is further reduced by HOMs and fundamental mode taken into account.

BL RF mode

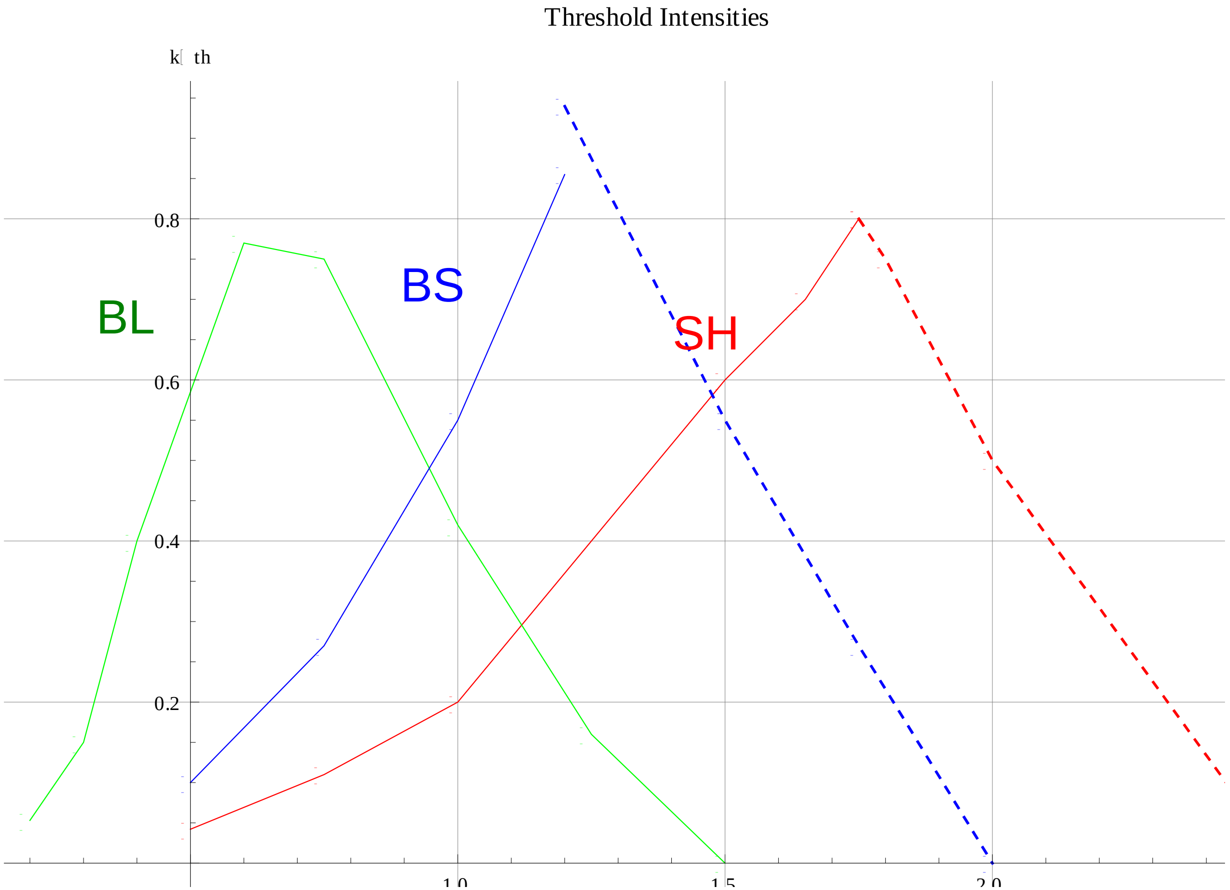


blue - $F(J) \propto (1 - J / J_{\lim})^2$ red - $F(J) \propto \sqrt{1 - J / J_{\lim}}$

For BL RF, the discrete mode excites mostly the tail particles.

NB: Low-current formal bucket capacity $J_{\max} = 3.0$.
LLD reduces is twice.

Stability areas for all the 3 RF modes



Next steps

- The space charge block (poor convergence is an issue). Comparison with rigid-bunch approximation (O. Boine-Frankenheim and colleges).
- Azimuthal mode coupling. Requires optimization of integral computation. Parallel computation?
- Multiple bunches and over-revolution wakes.
- Dampers and feedbacks.
- Specific applications for various machines.