The axial anomaly in hydrodynamics

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THE AXIAL ANOMALY AT FINITE TEMPERATURE*

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The axial anomaly and its constraints are examined at finite temperature. The axial anomaly is shown to be the same at $T \neq 0$ and $T = 0$. When $T \ll \Lambda$ we are able to argue the existence of a narrow resonance in the QCD medium corresponding to pion propagation. However, when T is comparable to Λ we are unable to obtain any useful information from the axial anomaly. In particular, we are unable to see how $T_{\text{chiral}} \geqslant T_{\text{conf}}$ might follow from the anomaly constraint.

Plan of the talk

- Hydrodynamics as a low-energy effective theory
- Relativistic hydrodynamics
- Triangle anomaly: a new hydrodynamic effect

A low-energy effective theory

Consider a thermal system: $T \neq 0$

Dynamics at large distances $\ell \gg \lambda_{\rm mfp}$ governed by a simple effective theory:

Hydrodynamics

Relativistic hydrodynamics

Conservation laws: $\partial_\mu T^{\mu\nu} = 0$

$\partial_\mu j^\mu = 0$ (if ∃ conserved charge)

Constitutive equations: local thermal equilibrium

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu}$$

$$j^\mu = n u^\mu$$

Total: 5 equations, 5 unknowns

Relativistic hydrodynamics

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$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$$

$$j^\mu = n u^\mu + \nu^\mu$$

Total: 5 equations, 5 unknowns

Dissipative terms

$$\tau^{ij} = -\eta(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\vec{\nabla}\cdot\vec{u}) - \zeta\delta^{ij}\vec{\nabla}\cdot\vec{u} \qquad \nu^i = -\sigma T \partial^i \left(\frac{\mu}{T}\right)$$

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shear viscosity (η)

bulk viscosity (ζ)

conductivity (diffusion) (σ)

Parity-odd effects?

- QFT: may have *chiral fermions*
  - example: QCD with massless quarks
- Even when the theory respects parity: there might by parity-odd kinetic coefficients:

$$j^\mu = nu^\mu + \xi\omega^\mu \qquad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$$

vorticity

- The same order in derivatives as dissipative terms
- NB: we consider only conserved chiral currents, i.e., $\bar{q}\gamma^\mu\gamma^5 t^3 q$

Landau-Lifshitz frame

- We can also have correction to the stress-energy tensor

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \xi'(u^\mu \omega^\nu + \omega^\mu u^\nu)$$

- Can be eliminated by redefinition of u^μ

$$u^\mu \to u^\mu - \frac{\xi'}{\epsilon + P}\omega^\mu$$

Only a linear combination $\xi - \frac{n}{\epsilon + P}\xi'$ has physical meaning

We will set $\xi' = 0$

New effect: chiral separation

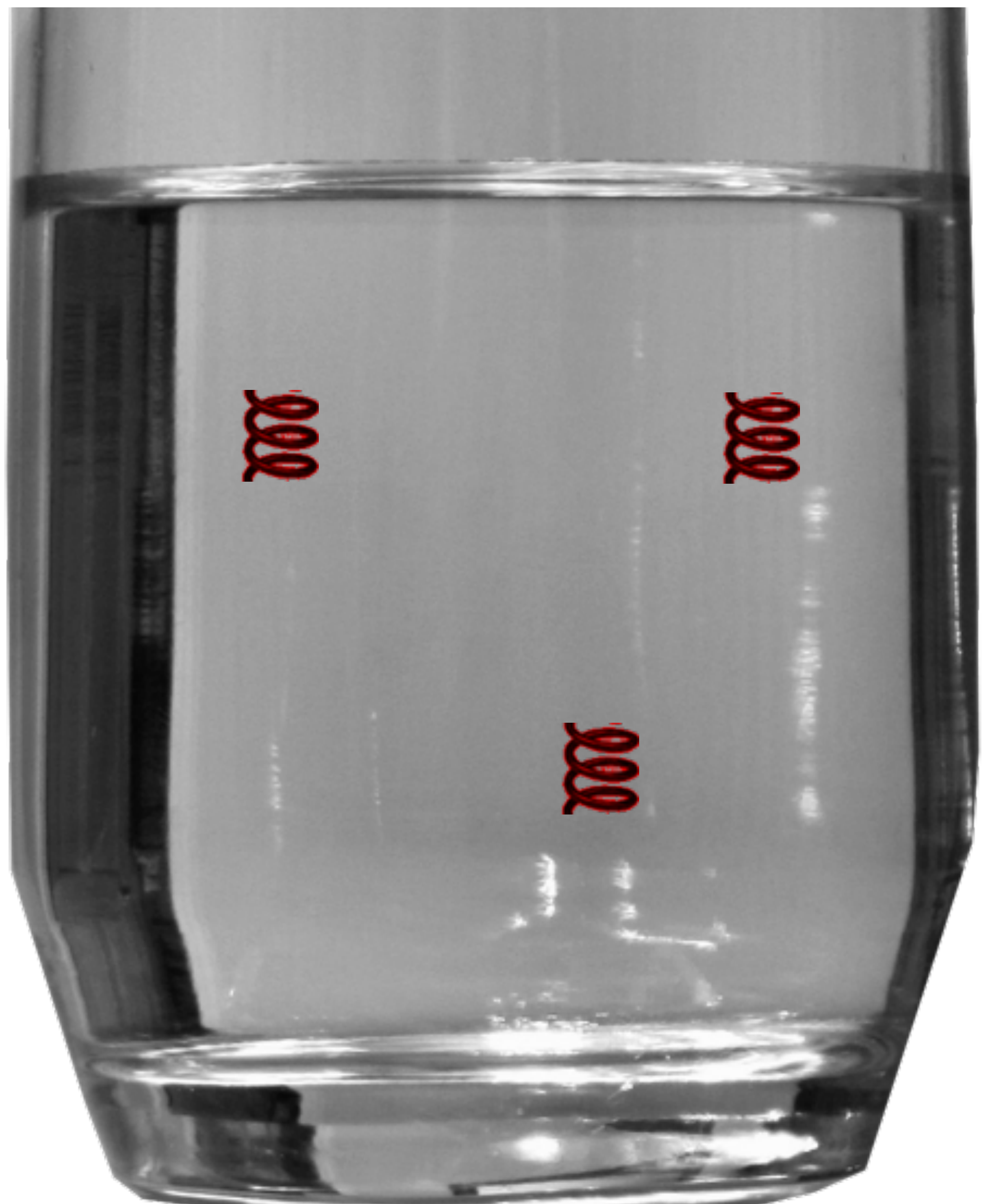
- Rotating piece of quark matter
- Initially only vector charge density $\neq 0$
- Rotation: lead to j^5: chiral charge density develops
- Can be thought of as chiral separation: left- and right-handed quarks move differently in rotation fluid
- Similar effect in nonrelativistic fluids?

Chiral separation by rotation

Chiral separation by rotation

Chiral separation by rotation

Chiral separation by rotation



Chiral separation by rotation

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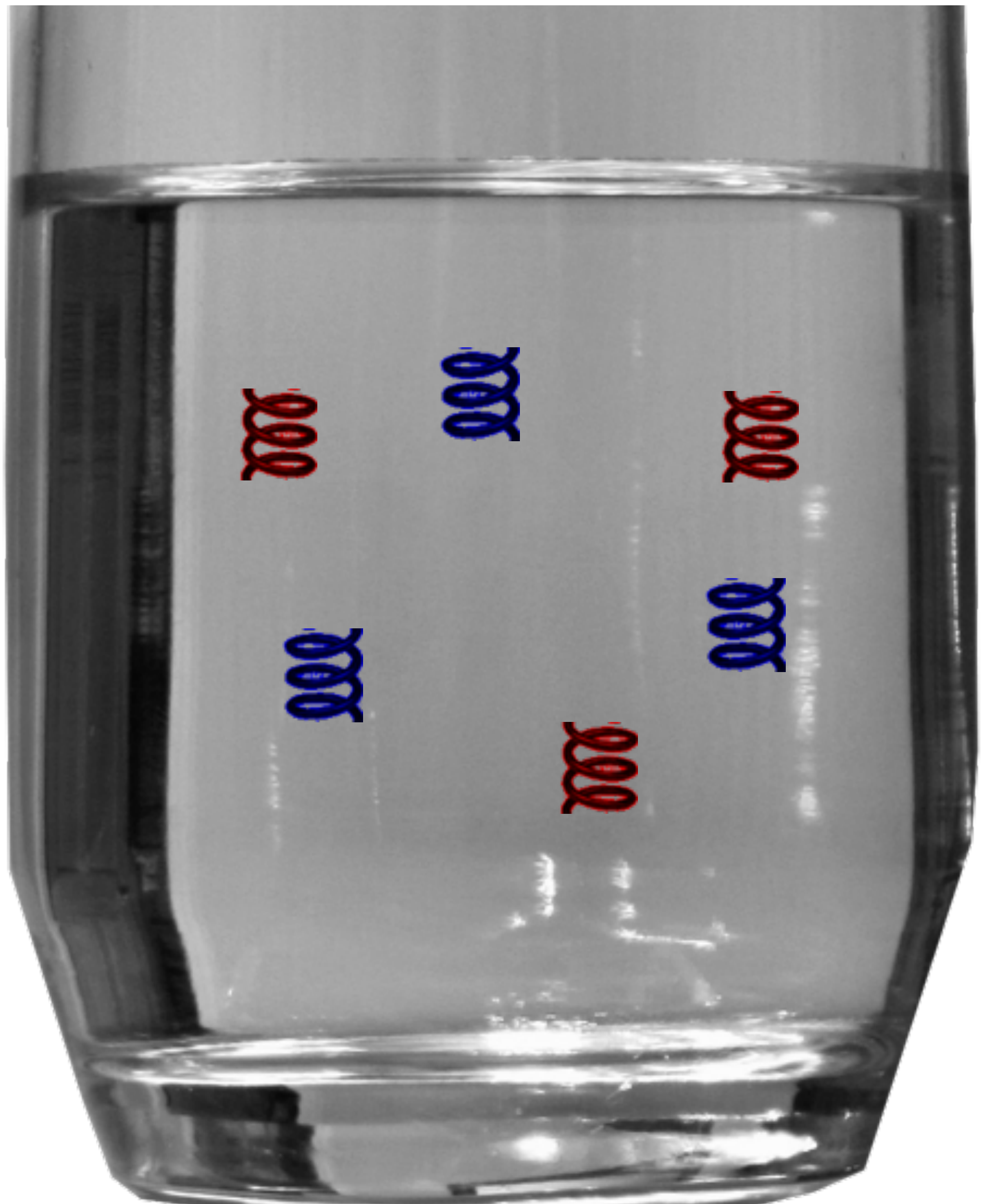
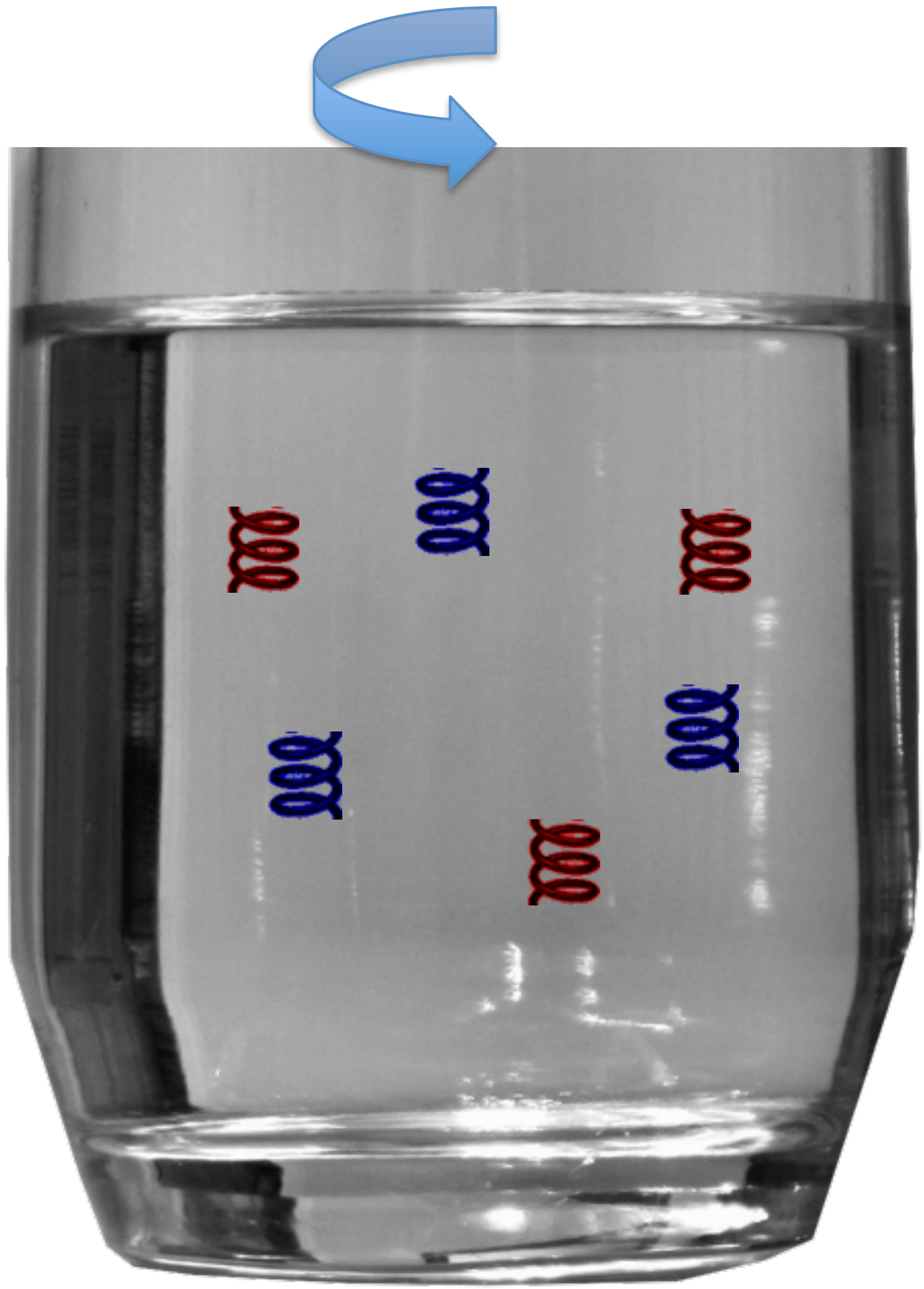
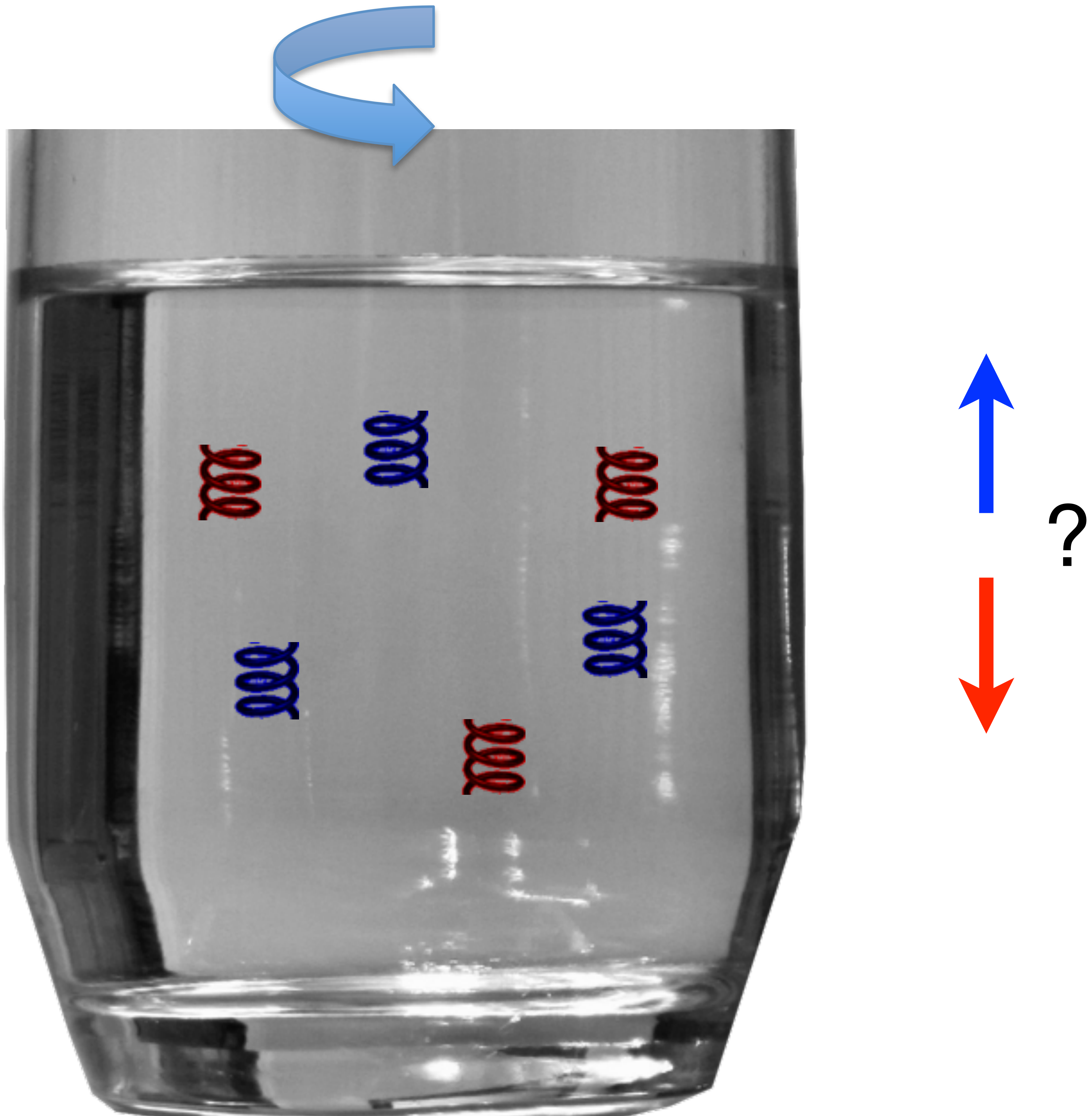


Chiral separation by rotation



Chiral separation by rotation



Can chiral separation occur in rigid rotation?

- If a chiral molecule rotates with respect to water, it will moves (propeller effect)
- In rigid rotation, molecules rotate with water
- $\Rightarrow$ no current in rigid rotation.
- Chiral separation occurs at higher orders in derivative expansion Andreev DTS Spivak

$$j_i^{\mathrm{chiral}} \sim (\partial_i v_j + \partial_j v_i)\omega_j + \cdots$$

but NOT ~~$\mathbf{j}^{\mathrm{chiral}} \sim \omega = \nabla \times \mathbf{v}$~~

Relativistic theories are different

- There can be current ~ vorticity
- It is related to triangle anomalies
    - currents are conserved, anomalies in 3-point functions
- The kinetic coefficient ξ is determined completely by anomalies and EOS

Missing from Landau-Lifshitz?

- Terms with epsilon tensor do not appear in the standard Landau-Lifshitz treatment of hydrodynamics
- Apparent reason: second law of thermodynamics

Dissipative terms

Landau-Lifshitz

$$\partial_\mu[(\epsilon + P)u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0$$

$$\partial_\mu(n u^\mu) + \partial_\mu \nu^\mu = 0$$

Dissipative terms

Landau-Lifshitz

$$\partial_\mu[(Ts+\mu n)u^\mu u^\nu]+\partial^\nu P+\partial_\mu\tau^{\mu\nu}=0$$

$$\partial_\mu(nu^\mu)+\partial_\mu\nu^\mu=0$$

Dissipative terms

Landau-Lifshitz

$$-\frac{u_\nu}{T}\times \partial_\mu[(Ts+\mu n)u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0$$

$$-\frac{\mu}{T}\times \partial_\mu(nu^\mu) + \partial_\mu \nu^\mu = 0$$

Dissipative terms

Landau-Lifshitz

$$-\frac{u_\nu}{T}\times \partial_\mu[(Ts+\mu n)u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0$$

$+$

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$$+$$

$$-\frac{\mu}{T}\times\ \partial_\mu(nu^\mu)+\partial_\mu\nu^\mu=0$$

$$\partial_\mu(su^\mu) \qquad = \qquad \frac{\mu}{T}\partial_\mu\nu^\mu+\frac{1}{T}\quad u_\nu\,\partial_\mu\tau^{\mu\nu}$$

Dissipative terms

Landau-Lifshitz

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$$+$$

$$-\frac{\mu}{T}\times\ \partial_\mu(nu^\mu)+\partial_\mu\nu^\mu=0$$

$$\partial_\mu(su^\mu-\frac{\mu}{T}\nu^\mu)=\qquad\frac{\mu}{T}\partial_\mu\nu^\mu+\frac{1}{T}\quad u_\nu\,\partial_\mu\tau^{\mu\nu}$$

Dissipative terms

Landau-Lifshitz

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$$\partial_\mu(su^\mu-\frac{\mu}{T}\nu^\mu)=-\partial_\mu\frac{\mu}{T}\quad\nu^\mu-\frac{1}{T}\partial_\mu u_\nu\quad\tau^{\mu\nu}$$

Dissipative terms

Landau-Lifshitz

$$-\frac{u_\nu}{T}\times \partial_\mu[(Ts+\mu n)u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0$$

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$$-\frac{\mu}{T}\times \partial_\mu(n u^\mu) + \partial_\mu \nu^\mu = 0$$

$$\partial_\mu(s u^\mu - \frac{\mu}{T}\nu^\mu) = -\partial_\mu \frac{\mu}{T} \quad \nu^\mu - \frac{1}{T}\partial_\mu u_\nu \quad \tau^{\mu\nu}$$

↑ entropy current

Dissipative terms

Landau-Lifshitz

$$-\frac{u_\nu}{T}\times\partial_\mu[(Ts+\mu n)u^\mu u^\nu]+\partial^\nu P+\partial_\mu\tau^{\mu\nu}=0$$

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↑ entropy current

Positivity of entropy production constraints the dissipation terms: only three kinetic coefficients η, ζ, and σ.

New kinetic coeff?

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) = -\frac{1}{T} \tau^{\mu\nu} \partial_\mu u_\nu - \nu^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

Can we add to the current: $\nu^\mu = \cdots + \xi \omega^\mu$?

Problem: Extra term in current would lead to violation of 2nd law:

$$\partial_\mu s^\mu = \cdots - \xi \omega^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

This can have either sign, and can overwhelm other terms

Forbidden by 2nd law of thermodynamics?

Surprise from fluid-gravity correspondence

Erdmenger et al. arXiv:0809.2488

Banerjee et al. arXiv:0809.2596

considered N=4 super Yang Mills at strong coupling

finite T and μ

discovered that there is a current ~ vorticity

Found the kinetic coefficient ξ(T,μ)

Effect coming from 5D CS term encoding anomaly

Back to hydrodynamics

- How can the argument based on 2nd law of thermodynamics fail?
  - 2nd law not valid? unlikely...
  - Maybe we were not careful enough?

$$\partial_\mu s^\mu = \cdots - \xi \omega^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

Can this be a total derivative?

If yes, then all we need to to is to modify s^μ

$$s^\mu \to s^\mu + D(T, \mu) \omega^\mu$$

so our task is to find D so that

$$\partial_\mu [D(T,\mu)\omega^\mu] = \xi(T,\mu)\omega^\mu \partial_\mu \left(\frac{\mu}{T}\right)$$

This equation has to be valid for all solutions to hydrodynamic equations

Not all ξ(T,μ) are allowed: only a special class

$$\xi = T^2 d' \left(\frac{\mu}{T}\right) - \frac{2nT^3}{\epsilon + P} d \left(\frac{\mu}{T}\right)$$

Instead of an arbitrary function of 2 variables: depends on a function of 1 variable

Questions

But this raises a number of questions:

- What determines the function d(μ/T)?
- Does anomaly play a role?
  - Holographic example suggests that it does
  - If d(μ/T) determined by anomalies, then we can determine ξ for massless QCD
- But how to see anomalies in hydrodymamics?

Turning on external fields

- To see where anomalies enter, we turn on external background U(1) field A_μ
- Now the energy-momentum and charge are not conserved

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda$$

$$\partial_\mu j^\mu = -\frac{C}{8}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$$ Itoyama-Mueller

- Power counting: A~1, F~O(p): right hand side has to be taken into account

Anomalous hydrodynamics

- These equations have to be supplemented by the constitutive relations:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$ +viscosities

$$j^\mu = n u^\mu + \xi \omega^\mu + \xi_B B^\mu \qquad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$$

+diffusion+Ohmic current

- We demand that there exist an entropy current with positive derivative: $\partial_\mu s_\mu \geq 0$

- The most general entropy current is

$$s^\mu = s u^\mu + \frac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu$$

Entropy production

- Positivity of entropy production completely fixes all functions ξ, ξ_B, D, D_B

$$\xi = T^2 d'\left(\frac{\mu}{T}\right) - \frac{2nT^3}{\epsilon + P} d\left(\frac{\mu}{T}\right)$$

$$= C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right)$$

$$d(x) = \frac{1}{2}Cx^2$$

anomaly coeff

Only need to know anomaly + EOS

$$\xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right)$$

$$j^\mu = \cdots + \xi\omega^\mu + \xi_B B^\mu$$

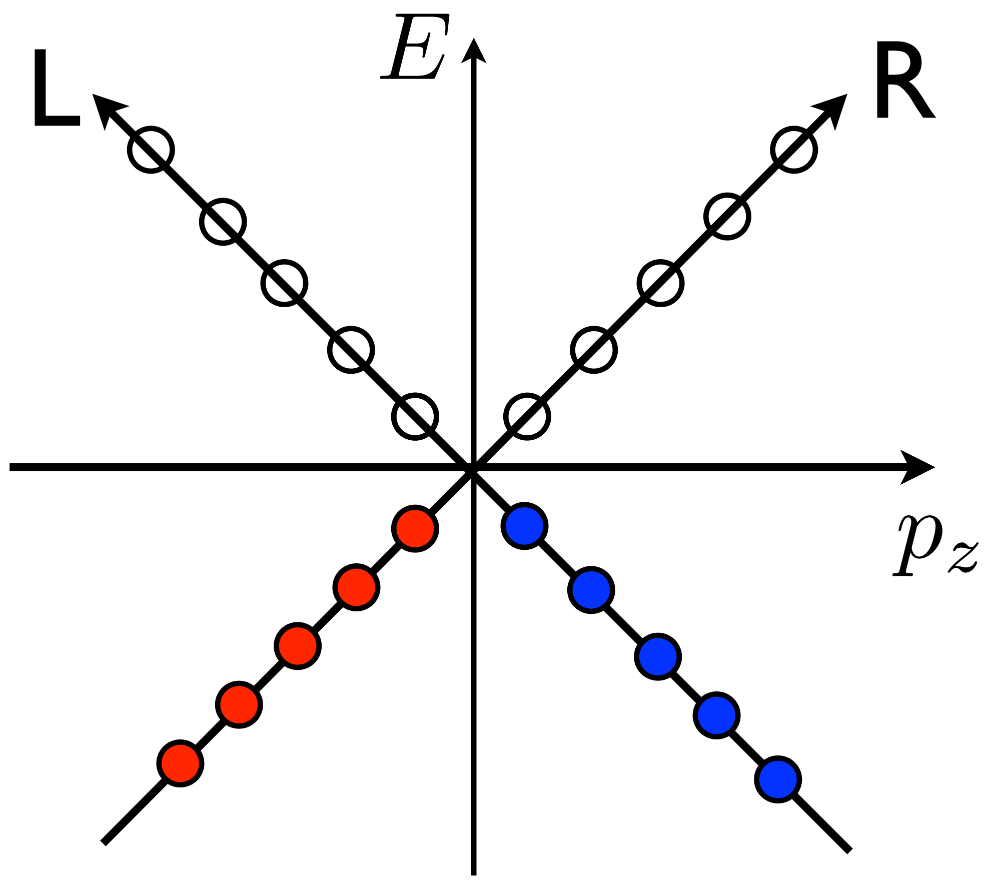
These expressions have been checked for N=4 SYM

Current induced by magnetic field

Spectrum of Dirac operator:

$$E^2 = 2nB + p_z^2$$

All states LR degenerate except for n=0



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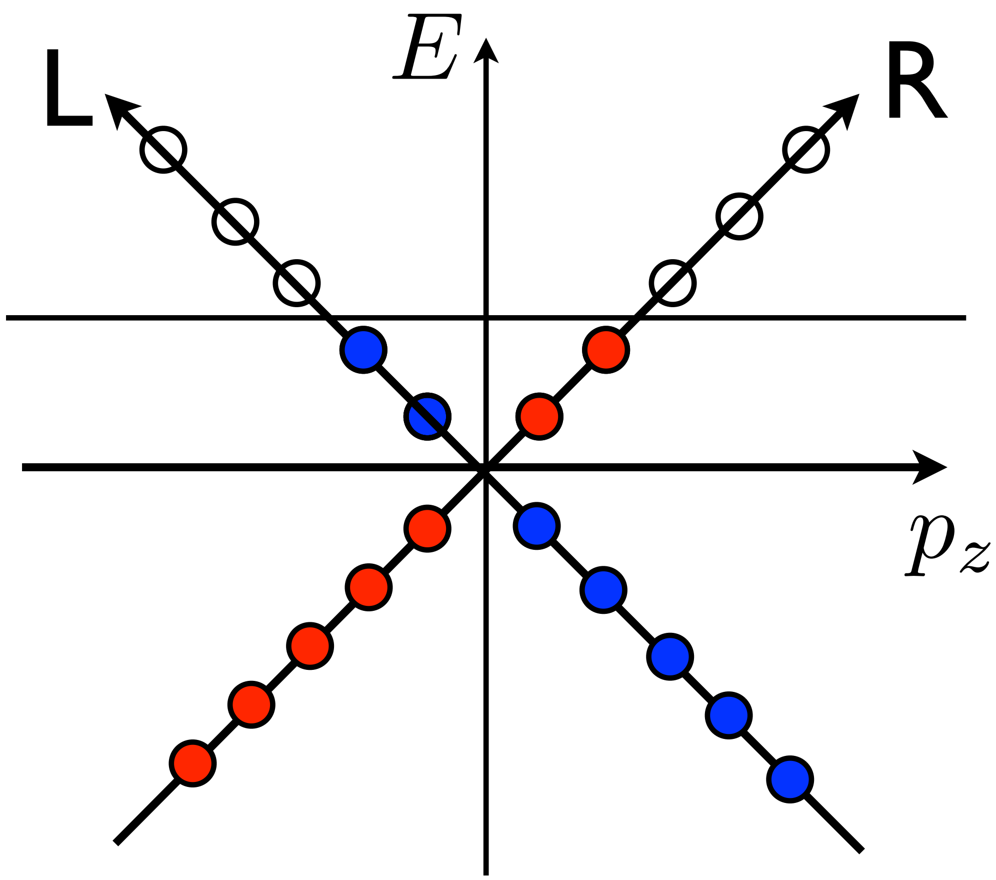


$$j_{\mathrm{L}} = -C\mu$$

$$j_{\mathrm{R}} = C\mu$$

If there is only right-handed fermions:

$$j^\mu = n u^\mu + C\mu B^\mu$$

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + \frac{C}{2}\mu^2 (u^\mu B^\nu + u^\nu B^\mu)$$

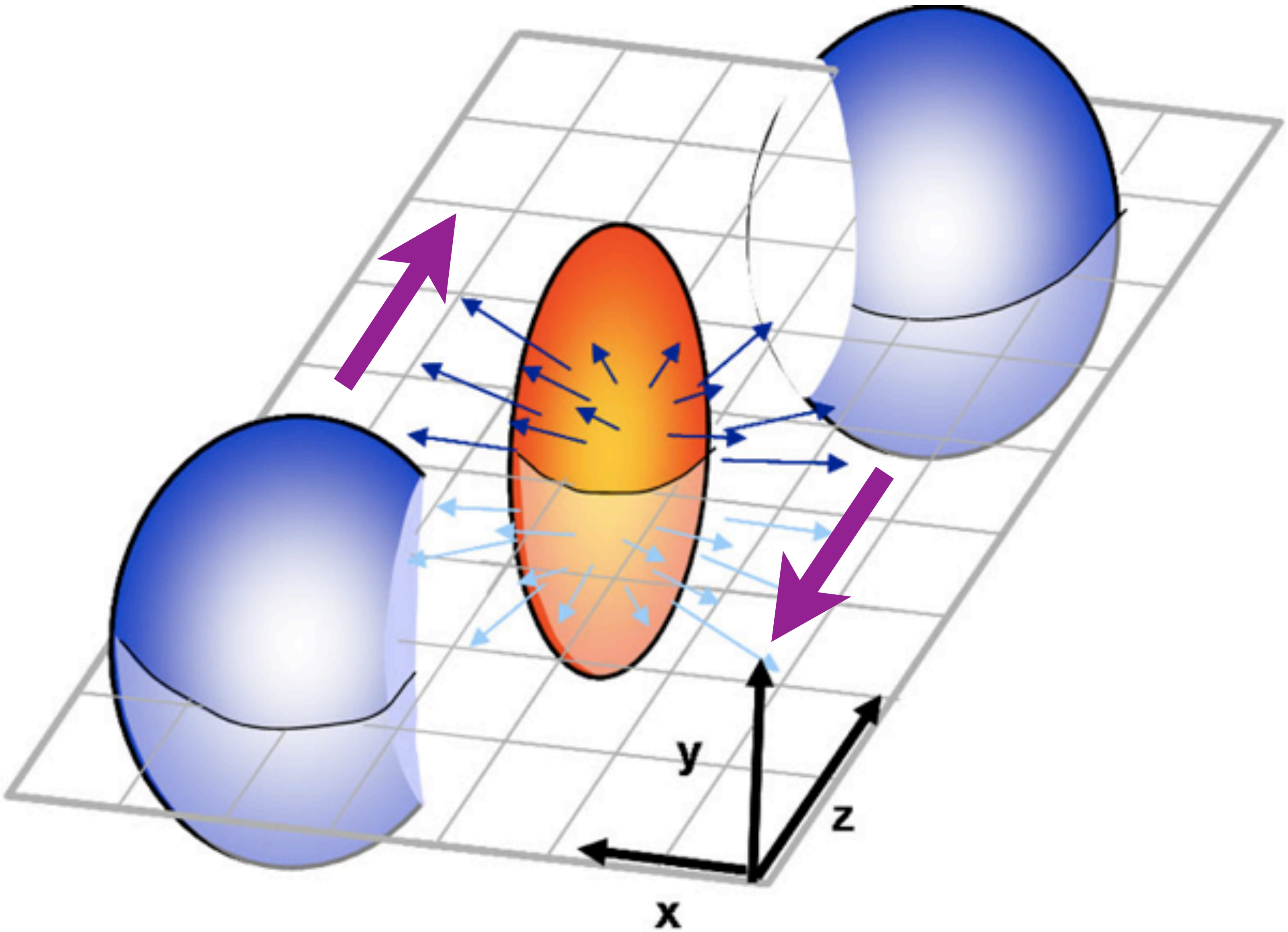
can be made disappear by redefining u^μ

Landau-Lifshitz frame: $u^\mu \to u^\mu - \frac{C}{2}\frac{\mu^2}{\epsilon + P} B^\mu$

$$j^\mu = n u^\mu + \xi_B B^\mu \qquad \xi_B = C\mu - \frac{C}{2}\frac{\mu^2 n}{\epsilon + P}$$

Simple understanding of ξ still lacking...

Observable effect on heavy-ion collsions?



Chiral charges accumulate at the poles: what happens when they decay?

A more speculative scenario

with Kharzeev

- Large axial chemical potential μ_5 for some reason
- B → vector current: charge separation

  chiral magnetic effect: Fukushima, Kharzeev, McLerran, Warringa

- Rotation → baryon current?
- Experimentally distinct signals?

From kinetic theory?

- The anomalous hydrodynamics current also exists in weakly coupled theories
- Should be derivable from kinetic theory, for example from Landau's Fermi liquid theories
- Comes from a correction to Landau's Fermi liquid theory?
- Related to Berry's curvature on the Fermi surface?

Conclusions

- A surprising finding: anomalies affect hydrodynamic behavior of relativistic fluids
- Confirmed by holographic models
- Further studies of experimental significance needed
- Anomalies in Landau's Fermi liquid theory?

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Happy birthday, Al!