Scattering amplitudes and BFKL Pomeron in QCD and in $N = 4$ SUSY

L. N. Lipatov
Petersburg Nuclear Physics Institute

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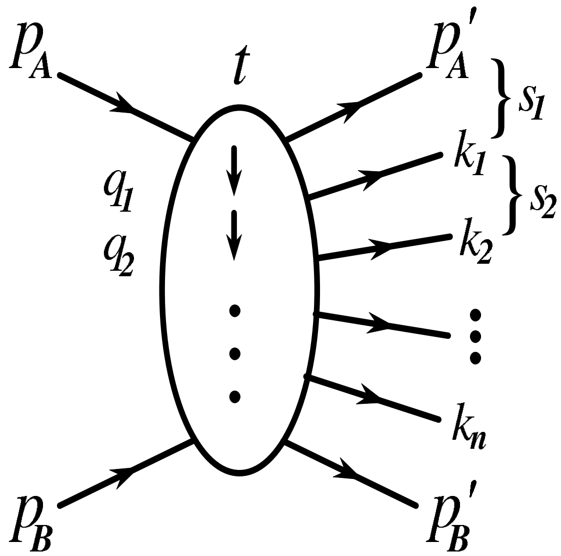
Collaborators: V. Fadin, A. Kotikov, V. Velizhanin, A. Rej, M. Staudacher, J. Bartels, A. Sabio Vera

1 Amplitudes in multi-Regge kinematics



$$M_{2\to 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1} c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi}\left(\ln\frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon}\right)\,,\ C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n \,|M_{2\to 2+n}|^2$$

2 Analyticity, unitarity and bootstrap

Steinmann relations for overlapping channels

$$\Delta_{s_r}\Delta_{s_{r+1}}M_{2\to 2+n}=0$$

Dispersion representation for $M_{2\to 3}$ in the Regge ansatz

$$M_{2\to 3}=c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)}+c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

Dispersion representation for $M_{2\to 4}$ in the Regge ansatz

$$M_{2\to 4}=d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2}+d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1}+d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$

$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,,\ j_r=j(t_r)$$

Bootstrap relation in LLA (BFKL (1975-1978))

$$\pi\omega(t_1)\,M_{2\to 2+n}=\sum_r \Im_{s_{0r}}M_{2\to 2+n}=\sum_t M_{2\to 2+t}M_{2+t\to 2+n}$$

3 BFKL and BKP equations

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2)=H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\;\sigma_t\sim s^{\Delta}\;,\;\;\Delta=-\frac{\alpha_s N_c}{2\pi}\,E$$

Bartels-Kwiecinski-Praszalowicz equation at large N_c

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)=H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\;,\;\;H=\frac{1}{2}\sum_{k=1}^{n}H_{k,k+1}=h+h^*\,,$$

$$\mathrm{H}_{12}=h_{12}+h_{12}^*\;,\;\;h_{12}=\ln(p_1p_2)+\textstyle\sum_{r=1,2}\frac{1}{p_r}\ln\rho_{12}\,p_r-2\gamma$$

Holomorphic factorization and Möbius invariance (L.L.)

$$\Psi(\vec{\rho}_1,\vec{\rho}_2,...,\vec{\rho}_n)=\sum_{r,s}a_{r,s}\,\Psi_r(\rho_1,...,\rho_n)\,\Psi_s(\rho_1^*,...,\rho_n^*)\,,$$

$$M_a^2\Psi_r=m(m-1)\Psi_r\,,\;\;M_a^{*2}\Psi_s=\widetilde{m}(\widetilde{m}-1)\Psi_s\,,\;\;m=\frac{1}{2}+i\nu+\frac{n}{2}$$

4 Integrability at $N_c \to \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \; T(u) = A(u) + D(u),$$

$$L_k = \begin{pmatrix} u + \rho_k \, p_k & p_k \\ -\rho_k^2 \, p_k & u - \rho_k \, p_k \end{pmatrix}, \; [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t_{r_1'}^{s_1}(u) \, t_{r_2'}^{s_2}(v) \, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u) \, t_{r_2}^{s_2'}(v) \, t_{r_1}^{s_1'}(u) \,, \; \hat{l} = u \, \hat{1} + i \, \hat{P}$$

Duality symmetry (L. (1999))

$$p_r \longrightarrow \rho_{r+1,r} \longrightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

5 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. and C.,C. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M)+\rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (2002) and integrability for γ (L (1997))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1)-\Psi(1)}{k+M}\right)$$

6 Pomeron and reggeized graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the energy-momentum conservation

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling asymptotics for Δ (KLOV, BPST)

$$j = 2 - \Delta\,,\ \Delta = \frac{1}{2\pi}\,\hat{a}^{-1/2}$$

7 Maximal helicity violation

BDS amplitudes in $N=4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,...,a_n} = \sum_{\{i_1,...,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} ... T^{a_{i_n}}\, f(p_{i_1}, p_{i_2}, ..., p_{i_n})\,, \;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right),$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon}\,, \;\; C^{(1)} = 0\,, \;\; C^{(2)} = -\zeta_2^2/2\,, \;\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k\, f_k^{(1)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}\,, \;\; f_1 = \beta(f) = -a\zeta_3/2 + a^2(2\zeta_5 + 5\zeta_2\zeta_3/3) + ...$$

8 Elastic BDS amplitude

Regge asymptotics at $s/t \to \infty$

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

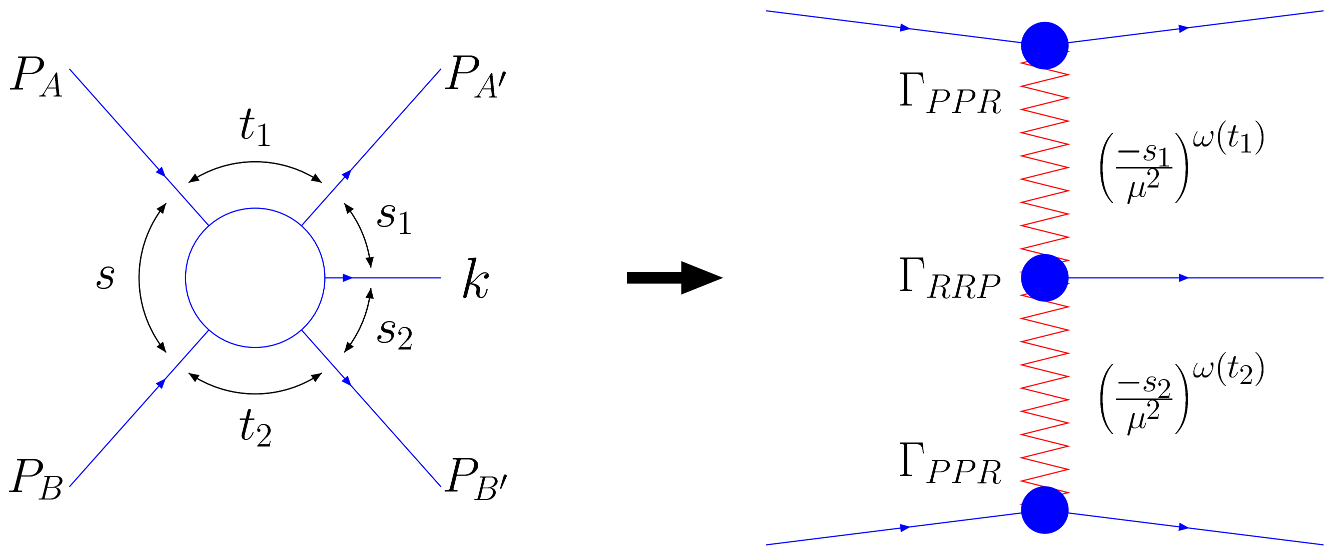
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right)$$

Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

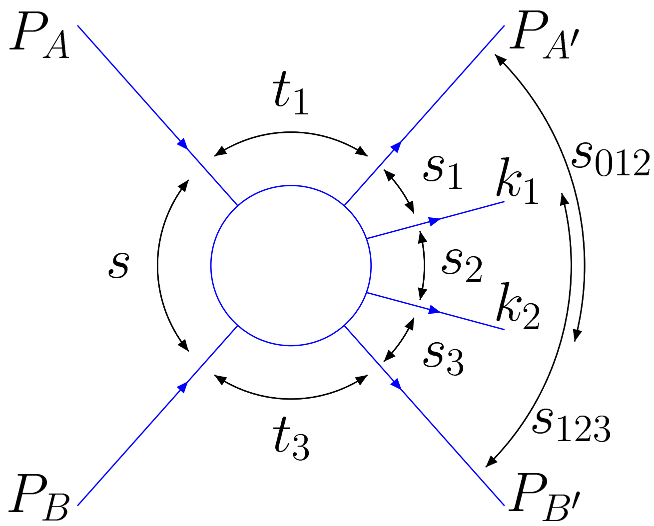
9 One particle production (BLS)



$$\ln\Gamma_{21} = -\frac{1}{2}\left(\omega(t_1)+\omega(t_2)-\int_0^a \frac{da'}{a'}\left(\frac{\gamma_K(a')}{4\epsilon}+\beta(a')\right)\right)\ln\frac{-k_\perp^2}{\mu^2}-$$

$$\frac{\gamma_K(a)}{16}\left(\ln^2\frac{-k_\perp^2}{\mu^2}-\ln^2\frac{-t_1}{-t_2}-\zeta_2\right)-\frac{1}{2}\int_0^a\frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma_K(a')}{4\epsilon^2}+\frac{\beta(a')}{\epsilon}+\delta(a')\right)$$

10 Regge factorization violation



$$M_{2\to 4}|_{s,s_2>0;\, s_1,s_3<0} = C\,\Gamma_1 \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma_{21} \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma_{32} \left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)} \Gamma_3 \,,$$

$$C = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi \left(\ln \frac{t_1 t_2}{(\vec{k}_1 + \vec{k}_2)^2 \mu^2} - \frac{1}{\epsilon}\right)\right]$$

11 Mandelstam cuts in j_2-plane

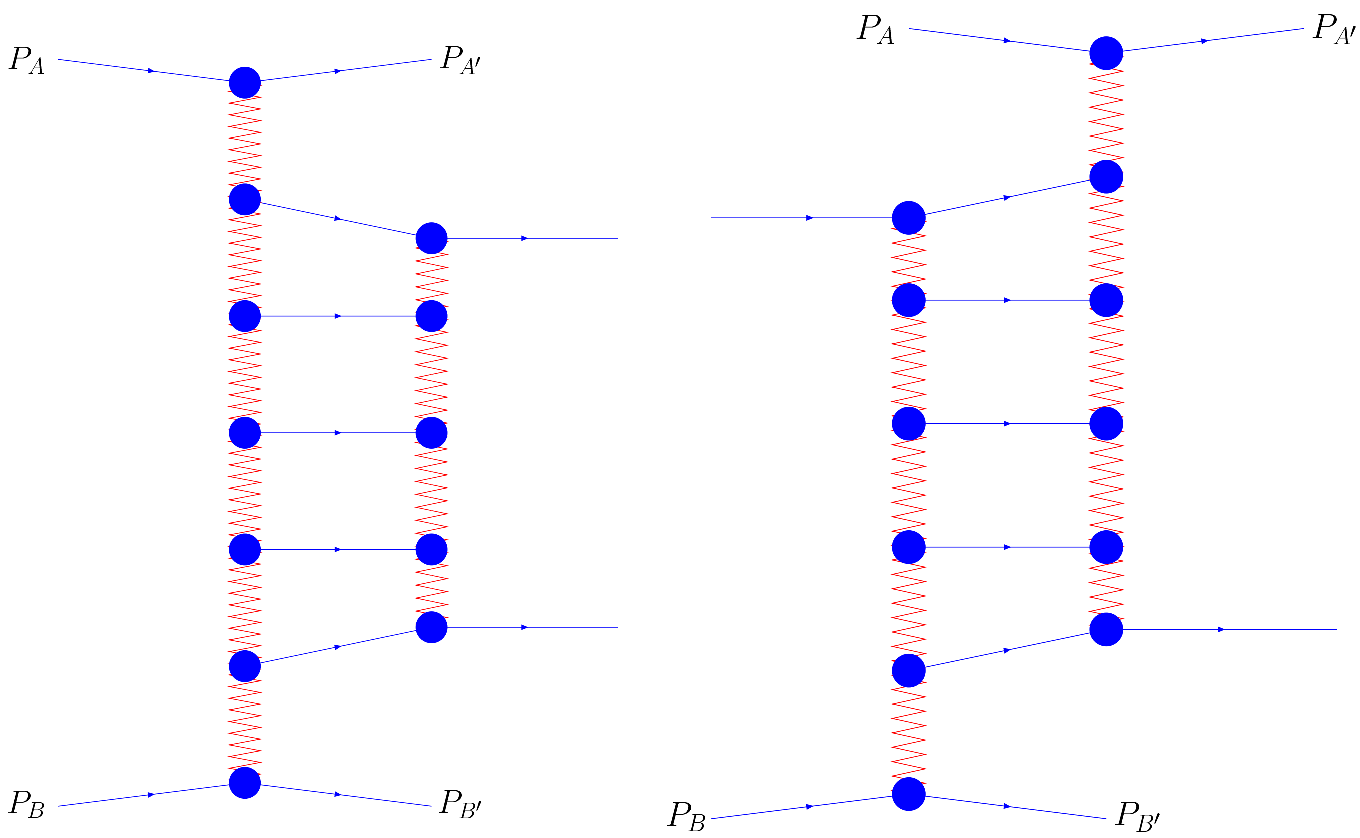


Figure 1: BFKL ladders in $M_{2\to4}$ and $M_{3\to3}$

12 BFKL equation for octets (BLS)

Regge singularity trajectories

$$\omega(t_2) = -a\left(E + \ln\frac{-t_2}{\mu^2} - \frac{1}{\epsilon}\right)$$

BFKL hamiltonian for partial waves f_{j_2}

$$H = \ln\frac{|p_1 p_2|^2}{|p_1+p_2|^2} + \frac{1}{2}\, p_1 p_2^* \ln|\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2}\, p_1^* p_2 \ln|\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2}\right)^{i\nu+n/2} \left(\frac{p_1^*}{p_2^*}\right)^{i\nu-n/2}, \; E_{n,\nu} = 2Re\,\psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

Factorization of infrared divergencies in LLA

$$M^{2\to4}_{|_{s,s_2>0;s_1,s_3<0}} = (1 + i\Delta_{2\to4})\; M^{BDS}_{2\to4}\,,$$

13 Möbius and conformal invariances

Analytic result in LLA in the region $a \ln s_2 \sim 1$

$$\Delta_{2\to 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(V^*\right)^{i\nu - \frac{n}{2}} V^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1\right)$$

Duality transformation to the Möbius representation

$$V = \frac{q_3 k_1}{k_2 q_1} \to \frac{z_{03} z_{0'1}}{z_{0'3} z_{01}}$$

Perturbation theory expansion

$$i\Delta_{2\to 4} = -2i\pi\, a^2 \ln s_2 \ln \frac{|k_1 + k_2||q_2|}{|k_2||q_1|} \ln \frac{|k_1 + k_2||q_2|}{|k_1||q_3|} + \ldots$$

Functions of 4-dimensional anharmonic ratios

$$i\Delta_{2\to 4} = \frac{a^2}{4} Li_2(\chi) \ln \frac{\chi t_2 s_{13}}{s_3 t_1} \ln \frac{\chi t_2 s_{02}}{t_3 s_1} + \ldots, \;\; \chi = 1 - \frac{s s_2}{s_{012} s_{123}}$$

14 Factorization and exponentiation

Factorization hypothesis

$$M_{2\to 4} = c\, M_{2\to 4}^{BDS} = M_{2\to 4}^{pole} + M_{2\to 4}^{cut}\,,\;\; c = 1 + i\Delta_{2\to 4}$$

BDS ansatz at $s, s_2 > 0, s_1, s_3 < 0$

$$M_{2\to 4}^{BDS} = |M_{2\to 4}^{BDS}|\, C\, e^{i\pi(\omega_a+\omega_b)}\,,\;\; \omega_1 \approx \frac{a}{2}\,\ln\frac{k_1^2\lambda^2}{q_1^2 q_2^2}\,,\;\; \omega_2 \approx \frac{a}{2}\,\ln\frac{k_2^2\lambda^2}{q_2^2 q_2^3}$$

Regge pole contribution

$$M_{2\to 4}^{pole} = |M_{2\to 4}^{BDS}|\left(e^{-i\pi\omega_2}2\sin\pi\omega_a\sin\pi\omega_b + e^{-i\pi\omega_2}e^{i\pi(\omega_a+\omega_b)}\right)$$

Mandelstam cut contribution

$$M_{2\to 4}^{cut} = i\int_{-i\infty}^{i\infty}\frac{d\omega_{2'}}{2\pi i}(-s_2)^{\omega_{2'}}\, f_{2\to 4}(\omega_{2'})$$

Is the exponentiation of $M_{2\to 4}$ compatible with the analyticity?

15 Multi-gluon states in octet channels

Channels with the Mandelstam cuts constructed from n gluons

$$s_1, s_2, ..., s_{n-1}, s_{n+1}, ..., s_{2n} < 0\,,\; s, s_n > 0$$

Schrödinger equation for octet composite states

$$H\Psi = E\Psi\,,\;\; \omega(t) = a\left(-\ln\frac{-t}{\mu^2} + \frac{1}{\epsilon}\right) - \frac{a}{2}E\,,\;\; a = \frac{g^2 N_c}{8\pi^2}$$

Holomorphic separability

$$H = h + h^*\,,\;\; h = \ln\frac{p_1 p_n}{q^2} + \sum_{r=1}^{n-1} h^t_{r,r+1}$$

Helpful operator relation

$$\ln\partial = -\ln x + \frac{1}{2}\left(\psi(x\partial + 1) + \psi(-x\partial)\right)$$

16 Möbius invariance in the p-space

Duality transformation

$$p_k = Z_{k-1,k}\,,\;\; \rho_{k,k+1} = i\frac{\partial}{\partial Z_k} = i\partial_k$$

Consequence of the Möbius invariance in Z-space

$$h = \ln(Z_1^2\partial_1) - 2\psi(1) + \ln\partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}\,,\;\; Z_0 = 0\,,\;\; Z_n = \infty\,,$$

$$h_{1,2} = \ln(Z_{12}^2\partial_1) + \ln(Z_{12}^2\partial_2) - 2\ln Z_{12} - 2\psi(1)$$

First integral of motion

$$A' = Z_1 \prod_{s=1}^{n-2} Z_{s,s+1} \prod_{r=1}^{n-1} \partial_r\,,\;\; [h, A'] = 0$$

17 Integrable Heisenberg spin chain

Helpful identity

$$[L_k(u)L_{k+1}(u), h_{k,k+1}] = -i\,(L_k(u) - L_{k+1}(u))$$

Integrals of motion: $[D, h] = 0$

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k\,,\ \ q'_k = -\sum_{0<r_1<\ldots<r_k<n} Z_{r_1} \prod_{s=1}^{k-1} Z_{r_s,r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t}$$

Sklyanin ansatz and Baxter equation

$$\Omega = \prod_k Q(\hat{u}_k)\,\Omega_0\,,\ \ \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|Z_l|^4}\,,$$

$$D(u)Q(u) = (u+i)^{n-1} Q(u+i)$$

18 Three-gluon composite state

Wave function in the coordinate representation

$$\Psi = Z_2^{a_1+a_2} (Z_2^*)^{\widetilde{a}_1+\widetilde{a}_2} \int \frac{d^2 y}{|y|^2}\, y^{-a_2}\, (y^*)^{\widetilde{a}_2} \left(\frac{y-1}{y-Z_2/Z_1}\right)^{a_1} \left(\frac{y^*-1}{y^*-Z_2^*/Z_1^*}\right)^{\widetilde{a}_1}$$

Fourier transformation

$$\Psi(\vec{Z}_1, \vec{Z}_2) = \int d^2 p_1\, d^2 p_2\, \exp(i\vec{p}_1 \vec{Z}_1)\, \exp(i\vec{p}_2 \vec{Z}_2)\, \Psi(\vec{p}_1, \vec{p}_2)\,, \;\; E = E(a_1) + E(a_2)$$

Baxter-Sklyanin representation

$$\Psi^t(\vec{p}_1, \vec{p}_2) = P^{-a_1-a_2}\, (P^*)^{-\widetilde{a}_1-\widetilde{a}_2} \int d^2 u\, u\, \widetilde{u}\, Q(u, \widetilde{u}) \left(\frac{p_1}{p_2}\right)^{u} \left(\frac{p_1^*}{p_2^*}\right)^{u^*}$$

Baxter function

$$Q(u, \widetilde{u}) = \frac{\Gamma(-u)\,\Gamma(-\widetilde{u})}{\Gamma(1+u)\,\Gamma(1+\widetilde{u})}\, \frac{\Gamma(u-a_1)\,\Gamma(u-a_2)}{\Gamma(1-\widetilde{u}+\widetilde{a}_1)\,\Gamma(1-\widetilde{u}+\widetilde{a}_2)}\,, \;\; \int d^2 u = \int d\nu \sum_n$$

19 Discussion

1. Steinmann relations and bootstrap.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in $N = 4$ SUSY.
4. BDS amplitudes in the multi-Regge kinematics.
5. Breakdown of the Regge factorization.
6. Mandelstam cuts in planar amplitudes at $n \geq 6$.
7. Integrable open spin chain for octet channels.