Small-x amplitudes in the next-to-leading order

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Outline

1. Small-x amplitudes in $\mathcal{N}$=4 SYM
   - Regge limit in the coordinate space.
   - High-energy scattering and Wilson lines.
   - Non-linear evolution equation at small x.
   - Evolution equation for composite conformal dipoles in the NLO
   - NLO amplitude in $\mathcal{N} = 4$ SYM.
2. Small-x evolution in QCD.
   - NLO evolution of color dipoles in QCD.
   - Argument of coupling constant in the evolution equation.
3. Conclusions.

Conformal four-point amplitude

$$A(x,y,x',y') = (x-y)^4(x'-y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

$\mathcal{O} = \mathrm{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator
In a conformal theory the amplitude is a function of two conformal ratios

$$A = F(R,R')$$
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}$$

At large N_c

$$A(x,y,x',y') = A(g^2N_c) \qquad g^2N_c = \lambda \ - \ \text{‘t Hooft coupling}$$

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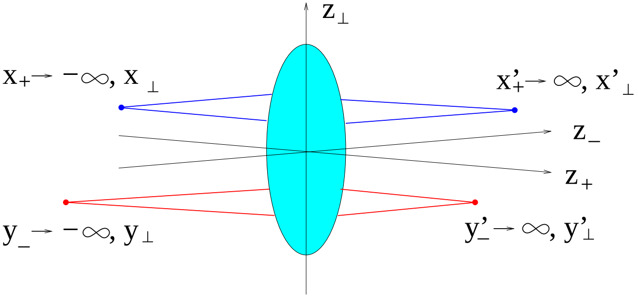
Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\mathrm{LO}} + c_n^{\mathrm{NLO}}\lambda)$$

Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho, \rho' \to \infty$



Full 4-dim conformal group: $A = F(R, r)$

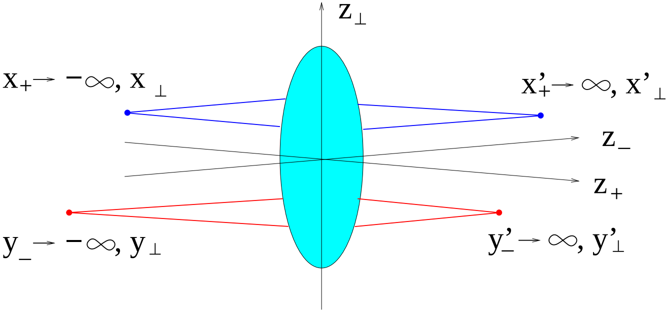
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \to \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

4-dim conformal group versus $SL(2,C)$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\rho, \rho' \to \infty$



Regge limit symmetry: 2-dim conformal group $SL(2,C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0,0,z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$\Omega(r,\nu)$ - solution of the eqn $(\Box_{H_3} + \nu^2 + 1)\Omega(r,\nu) = 0$.
Explicit form:

$$\Omega(r,\nu) \;=\; \frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{\kappa'^2}{(2\kappa'\cdot\zeta)^2}\Big)^{\frac{1}{2}-i\nu}$$

$$\zeta = p_1 + \frac{z_\perp^2}{s}p_2 + z_\perp, \qquad p_1^2 = p_2^2 = 0,\;\; 2(p_1,p_2) = s$$

$$\kappa \;=\; \frac{1}{2x_+}(p_1 - \frac{x^2}{s}p_2 + x_\perp) - \frac{1}{2y_+}(p_1 - \frac{y^2}{s}p_2 + y_\perp), \qquad \kappa^2\kappa'^2 \;=\; \frac{1}{R}$$

$$\kappa' \;=\; \frac{1}{2x'_-}(p_1 - \frac{x'^2}{s}p_2 + x'_\perp) - \frac{1}{2y'_-}(p_1 - \frac{y'^2}{s}p_2 + y'_\perp), \qquad 4(\kappa\cdot\kappa')^2 \;=\; \frac{r}{R}$$

The dynamics is described by $\omega(\lambda,\nu)$ and $F(\lambda,\nu)$.

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Pomeron intercept $\omega(\nu,\lambda)$ is known in two limits:

1. $\lambda \to 0$: $\omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$

$\chi(\nu) = 2\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

2. $\lambda \to \infty$: $AdS/CFT \Rightarrow \omega(\nu,\lambda) = 2 - \frac{\nu^2+4}{2\sqrt{\lambda}} + ...$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)

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The function $F(\nu,\lambda)$ in two limits:

1. $\lambda \to 0:$ $\quad F(\nu,\lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + ...$

$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

2. $\lambda \to \infty:$ $\quad AdS/CFT \quad \Rightarrow \quad F(\nu,\lambda) = \pi^3\nu^2 \frac{1+\nu^2}{\sinh^2 \pi\nu} + ...$

L.Cornalba (2007)

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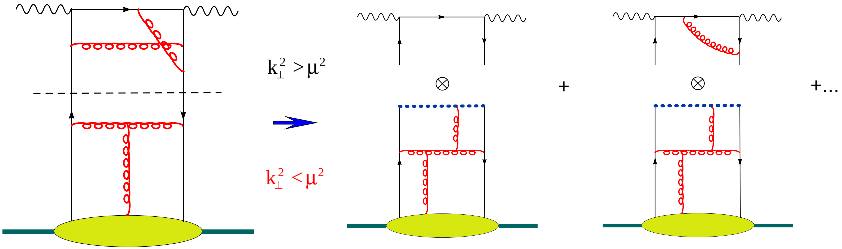
L.Cornalba (2007)

We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions
$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

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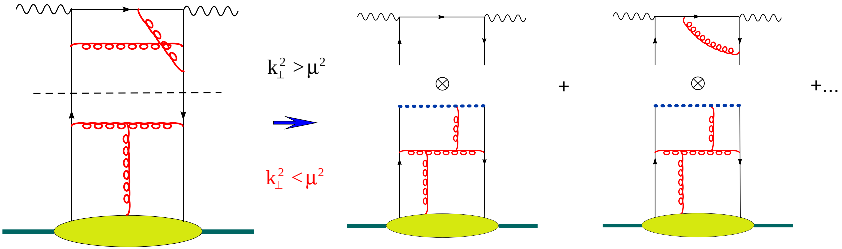

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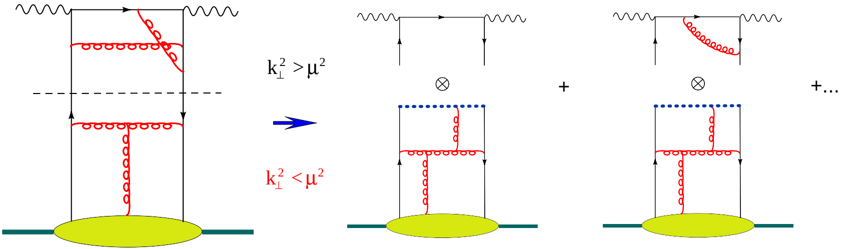
OPE in light-ray operators $\qquad (x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} \ = \ \frac{x_\xi}{2\pi^2 x^4}\Big[1 + \frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O(\frac{1}{x^2})$$

$[x,y] \ \equiv \ Pe^{ig\int_0^1 du \ (x-y)^\mu A_\mu(ux+(1-u)y)}$ - gauge link



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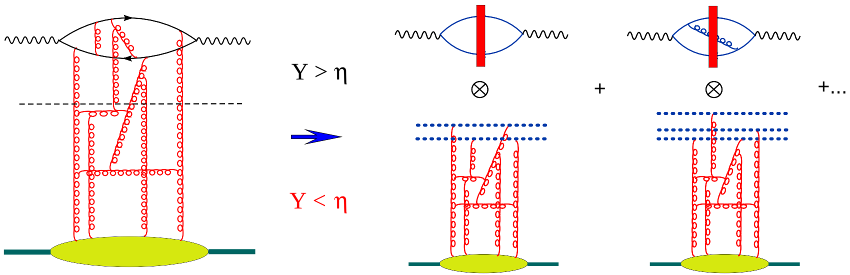
Renorm-group equation for light-ray operators ⇒ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \;=\; K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$



The high-energy operator expansion is

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2 \; I^{\mathrm{LO}}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

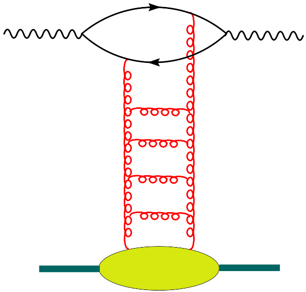
$$+ \int d^2z_1 d^2z_2 d^2z_3 \; I^{\mathrm{NLO}}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\mathrm{LO}} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad \mathit{CCP}, 2007$$

DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^*A \to \gamma^*A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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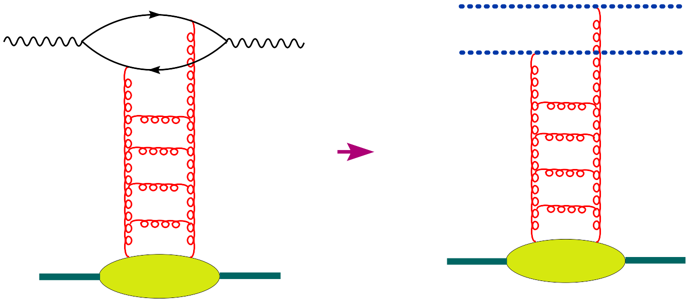

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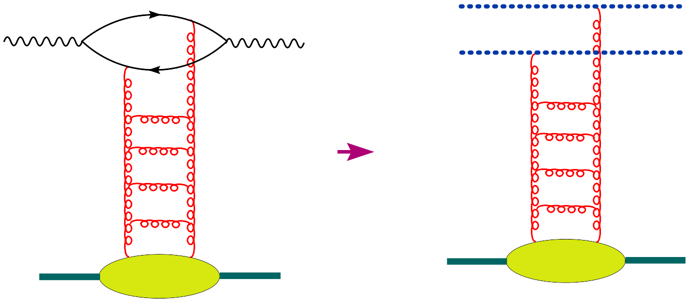


$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

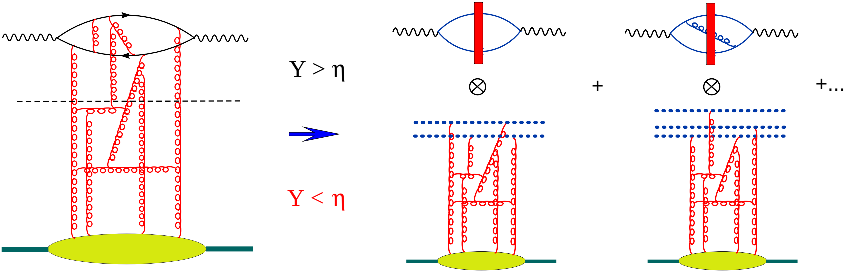
$$U(x_\perp) = Pe^{ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un+x_\perp)} \qquad \text{Wilson line}$$

Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

Expansion of the amplitude in color dipoles in the NLO



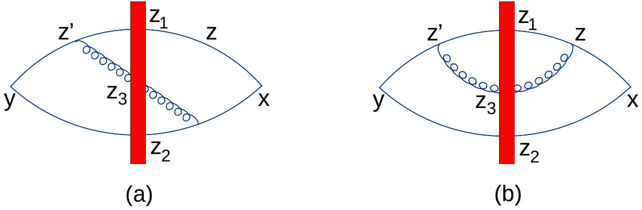
η - rapidity factorization scale

Rapidity Y > η - coefficient function (“impact factor”)
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

NLO impact factor



$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3-\frac{i\pi}{2}+C\Big]$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\big[{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf} \;=\; {\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\,\frac{\lambda}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}[{\rm Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}\;+\;O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = \ -\ I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ $\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

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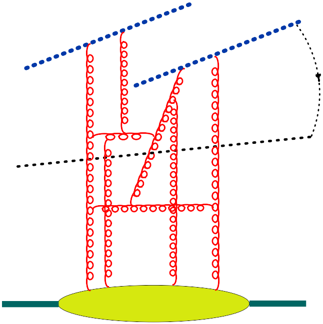

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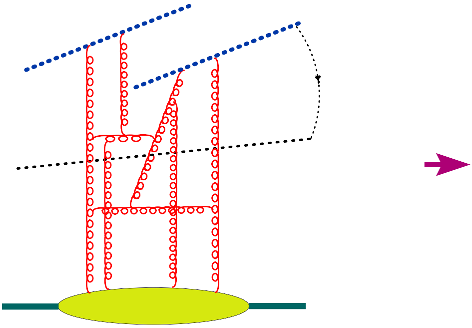

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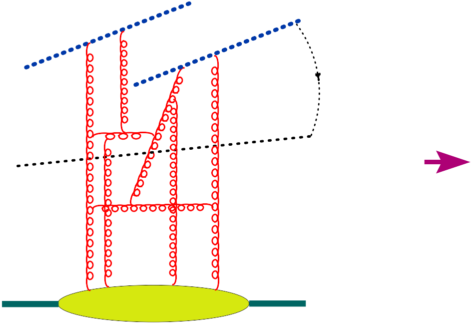
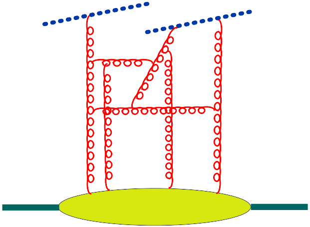
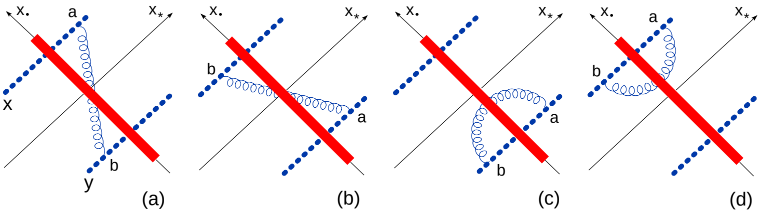


$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non-linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD ⇒ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD ⇒ non-linear eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

The story of the non-linear evolution at high energies

- L.V. Gribov, E.M. Levin, M.G. Ryskin (1983) - GLR equation suggested
- A.H. Mueller, J. Qiu (1986) - DLA limit of GLR equation proved
- A.H. Mueller + Nikolaev, Zakharov (1994) - dipole model for the high-energy scattering
- I.B. (1996) - NL evolution equation for Wilson-line operators
- Yu.Kovchegov (1999)- evolution equation for the structure functions of heavy nuclei
- JIMWLK (1997-2000) - RG equation for Color Glass Condensate

Conformal invariance of the non-linear equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \; = \; \text{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\frac{(x-y)^2 \not{z}^4}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

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SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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In the leading order - OK. In the NLO - ?

Non-linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_xU_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_xU_z^\dagger\}Tr\{U_zU_y^\dagger\} - N_c Tr\{U_zU_y^\dagger\}] +$$
$$\alpha_s^2\int d^2z d^2z'\left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle + O(\alpha_s^3)$$

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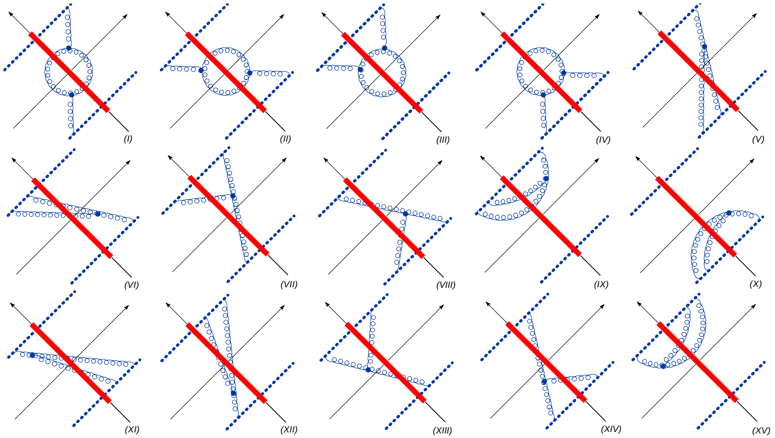
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

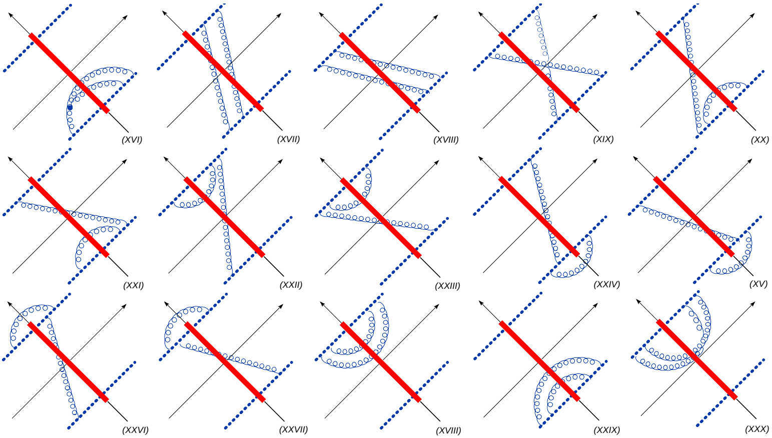
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

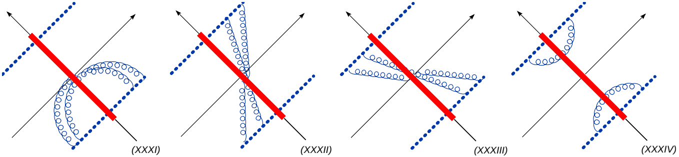
Gluon part of the NLO kernel: diagrams



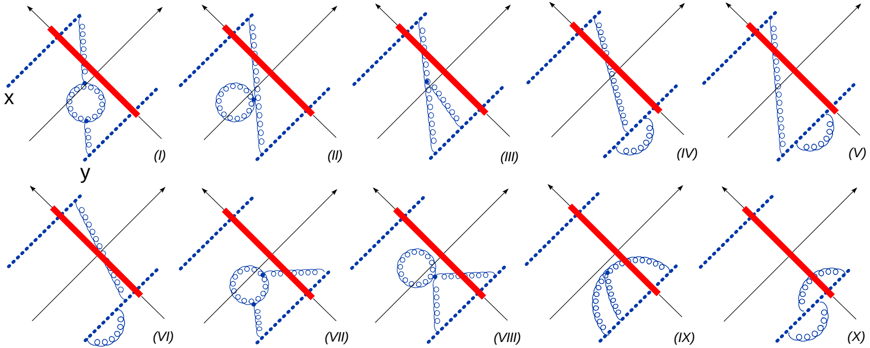
Diagrams for 1→3 dipoles transition



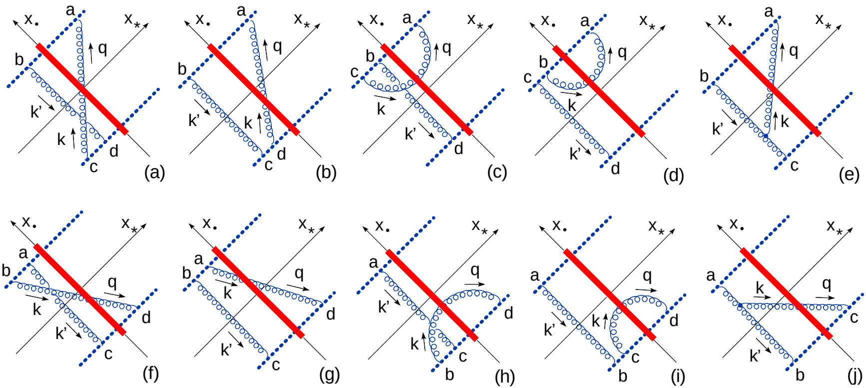
Diagrams for 1→3 dipoles transition



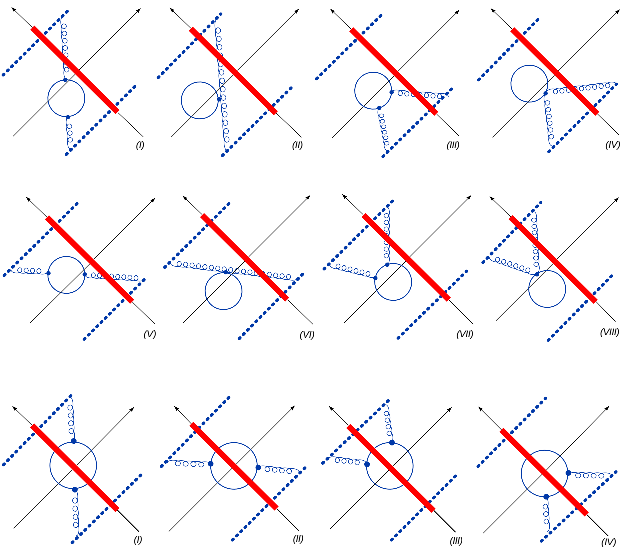
"Running coupling" diagrams



$1 \to 2$ dipole transition diagrams



$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}$$

$$\times\,[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$

$$\times\,\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

Evolution equation for color dipole in $\mathcal{N}=4$ (I.B. and G. Chirilli)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}$$
$$\times\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$-\ \frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$
\begin{aligned}
&\frac{d}{d\eta}\left[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\right]^{\mathrm{conf}} \\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\left[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\right]\left[\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\right]^{\mathrm{conf}} \\
&\quad -\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\left\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\left[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\right]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\right\} \\
&\quad \times\mathrm{Tr}\{[T^a,T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^\eta_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}[(\hat{U}^\eta_{z_3})^{aa'}(\hat{U}^\eta_{z_4})^{bb'}-(z_4\to z_3)]
\end{aligned}
$$

Now Möbius invariant!

NLO BFKL equation in $\mathcal{N}=4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x,y) \;=\; 1-\frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}^\eta_x\hat{U}^{\dagger\eta}_y\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1,z_2) \;=\; \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]$$

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Define

$$\begin{aligned}&\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)\\ &= \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{a e^{2\eta} z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]+\ldots\end{aligned}$$

such that $\frac{d}{d\eta}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) = 0$.

$\Rightarrow$The evolution can be rewritten in terms of a

NLO BFKL equation in $\mathcal{N} = 4$ SYM

NLO BFKL

$$
\begin{aligned}
& a\frac{d}{da}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \\
&= \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big][\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^a_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)] \\
&+\frac{\alpha_s^2N_c^2}{8\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\hat{\mathcal{U}}^a_{\rm conf}(z_3,z_4) \\
&\qquad + \frac{3\alpha_s^2N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)
\end{aligned}
$$

Eigenfunctions are determined by conformal invariance

$$E_{\nu,n}(z_{10},z_{20}) \;=\; \Big[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\Big]^{\frac{1}{2}+i\nu+\frac{n}{2}}\Big[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\Big]^{\frac{1}{2}+i\nu-\frac{n}{2}}$$

The expansion in eigenfunctions

$$\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \;=\; \sum_{n=0}^{\infty}\int d^2z_0\int d\nu\; E_{\nu,n}(z_{10},z_{20})\hat{\mathcal{U}}^a_{z_0,\nu,n} \quad\Rightarrow\quad a\frac{d}{da}\hat{\mathcal{U}}^a_{z_0,\nu,n} \;=\; \omega(n,\nu)\hat{\mathcal{U}}^a_{z_0,\nu,n}$$

$\omega(n,\nu) \equiv$ pomeron intercept = eigenvalue of the BFKL equation

Pomeron intercept = the eigenvalue of the BFKL equation

$$\omega(n,\nu) = \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu) + \frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],$$

$$\delta(n,\gamma) = 6\zeta(3) - \frac{\pi^2}{3}\chi(n,\gamma) - \chi''(n,\gamma) - 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)$$

where $\gamma = \frac{1}{2} + i\nu$ and

$$\chi(n,\gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1-\gamma+\frac{n}{2})$$

$$\Phi(n,\gamma) = \int_0^1 \frac{dt}{1+t}\, t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12} - \frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big) - \mathrm{Li}_2(t) - \mathrm{Li}_2(-t)$$
$$- \Big(\psi(n+1) - \psi(1) + \ln(1+t) + \sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t - \sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}$$

Pomeron intercept

Pomeron intercept = the eigenvalue of the BFKL equation

$$
\begin{aligned}
\omega(n,\nu) &= \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu)+\frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],\\
\delta(n,\gamma) &= 6\zeta(3)-\frac{\pi^2}{3}\chi(n,\gamma)-\chi''(n,\gamma)-\ 2\Phi(n,\gamma)-2\Phi(n,1-\gamma)
\end{aligned}
$$

where $\gamma=\frac{1}{2}+i\nu$ and

$$
\begin{aligned}
&\chi(n,\gamma)=2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})\\
&\Phi(n,\gamma)\ =\ \int_0^1\frac{dt}{1+t}\,t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)-\mathrm{Li}_2(t)-\mathrm{Li}_2(-t)\\
&-\Big(\psi(n+1)-\psi(1)+\ln(1+t)+\sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t-\sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}
\end{aligned}
$$

Coincides with Lipatov & Kotikov

Agrees with $j\to 1$ asymptotics of 3-loop splitting functions
Vogt, Moch, Vermaseren,(2003)

NLO impact factor

$$(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{1}{\pi^2}\int d^2z_1 d^2z_2\ \mathcal{R}^2 \Big\{\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)$$
$$-\frac{\lambda}{2\pi^2}\int \frac{d^2z_3\ z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{a z_{12}^2 \mathcal{Z}_3^2}{z_{13}^2 z_{23}^2} - i\pi\Big][\hat{\mathcal{U}}^{\rm conf}(z_1,z_3) + \hat{\mathcal{U}}^{\rm conf}(z_2,z_3) - \hat{\mathcal{U}}^{\rm conf}(z_1,z_2)]\Big\}$$

$\mathcal{R} \equiv \frac{(x-y)^2 z_{12}^2}{x_+ y_+ \mathcal{Z}_1 \mathcal{Z}_2}$ - conformal ratio

With two-gluon accuracy

$$(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{1}{\pi^2}\int d^2z_1 d^2z_2\ \mathcal{R}^2 \Big\{1$$
$$-\frac{\lambda}{4\pi^2}\Big[4{\rm Li}_2(1-\mathcal{R}) - \frac{2\pi^2}{3} + 2\Big(\ln\frac{1}{\mathcal{R}} + \frac{1}{\mathcal{R}} - 2\Big)\ln\frac{a\mathcal{Z}_1\mathcal{Z}_2}{z_{12}^2}\Big]\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)$$

The impact factor should not scale with energy $\Rightarrow\ a = \frac{x_+ y_+}{(x-y)^2}$ (analog of $\mu^2 = Q^2$ in DIS)

$$(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{1}{\pi^2}\int \frac{d^2z_1 d^2z_2}{z_{12}^4}\ \mathcal{R}^2\Big\{1 - \frac{\lambda}{4\pi^2}\Big[4{\rm Li}_2(1-\mathcal{R})$$
$$-\frac{2\pi^2}{3} + 2\Big(\ln\frac{1}{\mathcal{R}} + \frac{1}{\mathcal{R}} - 2\Big)\Big(\ln\frac{1}{\mathcal{R}} - i\pi - 4\ln 2 + 2C\Big)\Big]\Big\}\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)$$

NLO impact factors

The projection onto the conformal eigenfunctions $\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma$ ($\gamma = \frac{1}{2} + i\nu$) :

$$\int dz_1 dz_2 (x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma = \left(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\right)^\gamma [I_{\rm LO}^A(\gamma) + I_{\rm NLO}^A(\gamma)]\hat{\mathcal{U}}(z_0,\gamma),$$

$$\hat{\mathcal{U}}(z_0,\gamma) = \int d^2z_1 d^2z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma \hat{\mathcal{U}}(z_1,z_2)$$

$$I_{\rm LO}^A(\gamma) = \frac{\Gamma^2(1-\gamma)}{\Gamma(2-2\gamma)}\Gamma(1+\gamma)\Gamma(2-\gamma)$$

$$I_{\rm NLO}^A(\gamma) = -\frac{\lambda}{4\pi^2} I_{\rm LO}^A\Big(\psi'(\gamma) + \psi'(1-\gamma) + \frac{1-\chi(\gamma)}{\gamma(1-\gamma)} + \chi(\gamma)(-\frac{i\pi}{2} + C) - \frac{\pi^2}{3}\Big)$$

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Similarly

"normalization point" for the bottom IF is $b = \frac{x'_- y'_-}{(x'-y')^2}$

$$\int dz_1 dz_2 (x'-y')^4 T\{\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\}\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{1-\gamma} = \left(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\right)^{1-\gamma} [I^A_{\rm LO}(\gamma) + I^A_{\rm NLO}(\gamma)]\hat{\mathcal{V}}(z_0,\gamma),$$

$$\hat{\mathcal{V}}(z_0,\gamma) = \int d^2z_1 d^2z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma \hat{\mathcal{V}}(z_1,z_2)$$

$$I^B_{\rm LO}(\gamma) = \frac{\Gamma^2(1+\gamma)}{\Gamma(2+2\gamma)}\Gamma(1+\gamma)\Gamma(2-\gamma)$$

$$I^B_{\rm NLO}(\gamma) = -\frac{\lambda}{4\pi^2} I^A_{\rm LO}\Big(\psi'(\gamma) + \psi'(1-\gamma) + \frac{1-\chi(\gamma)}{\gamma(1-\gamma)} + \chi(\gamma)(-\frac{i\pi}{2} + C) - \frac{\pi^2}{3}\Big)$$

Assembling NLO $F(\nu)$

The last ingredient is the amplitude of scattering of two conformal dipoles ($\gamma \equiv \frac{1}{2} + i\nu$)

$$\langle \hat{\mathcal{U}}^a(z_0,\gamma)\hat{\mathcal{V}}^b(z_0',\gamma)\rangle = \delta(\nu-\nu')\delta(z_0-z_0')\ (ab)^{\frac{1}{2}\omega(\nu)}[A_{\rm LO}(\gamma)+A_{\rm NLO}(\gamma)]$$

$$A_{\rm LO}(\gamma) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}, \quad A_{\rm NLO}(\gamma) = -\frac{\lambda}{4\pi^2}A_{\rm LO}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)} - \frac{\pi^2}{3}\Big]$$

With our choice $a = \frac{x_+y_+}{(x-y)^2}$, $b = \frac{x'_-y'_-}{(x'-y')^2}$ $ab = R \Rightarrow$

$$\langle \hat{\mathcal{U}}(z_0,\gamma)\hat{\mathcal{V}}(z_0',\gamma)\rangle = \delta(\nu-\nu')\delta(z_0-z_0')\ R^{\frac{1}{2}\omega(\nu)}[A_{\rm LO}(\gamma)+A_{\rm NLO}(\gamma)]$$

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Now one can assemble $F(\nu)$ in the next-to-leading order

$$F(\nu) = F_{\rm LO}(\nu) + \lambda F_{\rm NLO}(\nu) + O(\lambda^2)$$

$$F_{\rm LO}(\nu) = I^A_{\rm LO}(\nu)A_{\rm LO}(\nu)I^B_{\rm LO}(\nu),$$

$$F_{\rm NLO}(\nu) = I^A_{\rm NLO}(\nu)A_{\rm LO}(\nu)I^B_{\rm LO} + I^A_{\rm LO}(\nu)A_{\rm NLO}(\nu)I^B_{\rm LO} + I^A_{\rm NLO}(\nu)A_{\rm LO}(\nu)I^B_{\rm NLO}(\nu)$$

DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Composite “conformal” dipole

$$[\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}_a$$
$$= \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} - \frac{\alpha_s}{4\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{ae^{2\eta}z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

Photon impact factor

$$(x-y)^6 T\{j_\mu(x)j_\nu(y)\} = \frac{1}{\pi^2}\int\frac{d^2z_1d^2z_2}{z_{12}^4}\,\mathcal{R}^3\big[1\ +O(\alpha_s)\big]$$
$$\times\,\hat{\mathcal{U}}^{\mathrm{conf}}(z_1,z_2)\frac{\partial^2}{\partial x^\mu\partial y^\nu}\big[-2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)+\kappa^2(\zeta_1\cdot\zeta_2)\big]$$

NLO impact factor: work in progress.

NLO evolution of composite "conformal" dipoles in QCD

$$
\begin{aligned}
a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}}_a \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}}_a \\
& \times \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3}\Big)\Big] \\
& + \frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{23}}^2z_{23}^2 + z_{24}^2z_{13}^2 - 4z_{12}^2z_{34}^2}{2({z_{23}}^2z_{23}^2 - z_{24}^2z_{13}^2)}\ln\frac{{z_{23}}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big] \\
& \times [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)] \\
& + \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{{z_{23}}^2z_{23}^2} + \Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2 - {z_{23}}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{{z_{23}}^2z_{23}^2}\Big] \\
& \times [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}
\end{aligned}
$$

$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

K_{NLO} = Running coupling part + Conformal "non-analytic" (in j) part
$+$ Conformal analytic ($\mathcal{N}=4$) part

Linearized K_{NLO} reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$
$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

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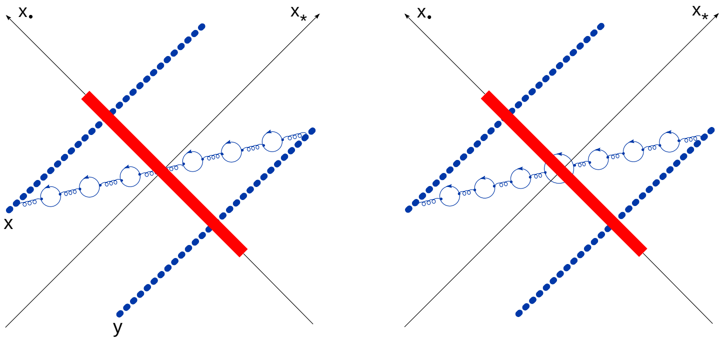
Renormalon-based approach: summation of quark bubbles



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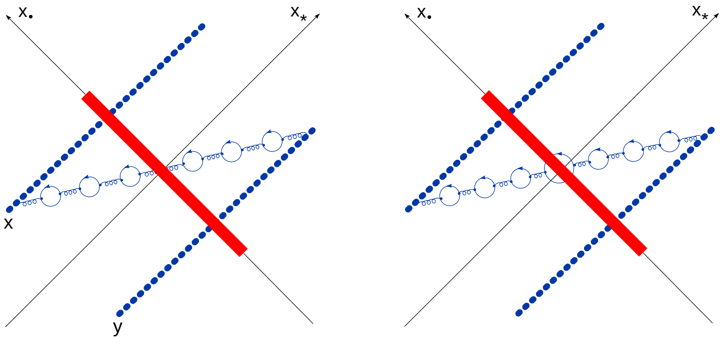

$$-\tfrac{2}{3}n_f \to b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\} \;=\; \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,\left[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{Tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}\right]$$
$$\times\left[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\left(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\right) + \frac{1}{z_{23}^2}\left(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\right)\right] + \dots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]$$

$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$

$$\frac{\alpha_s(z_{13})^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$

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Argument of coupling constant

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$$\times\Big[\frac{z_{12}^2}{z_{13}^2z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

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$$\frac{\alpha_s(z_{13})^2)}{2\pi^2z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23})^2)}{2\pi^2z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO kernel for the evolution of conformal composite dipoles in $\mathcal{N}=4$ SYM is Möbius invariant in the transverse plane.
- The NLO kernel agrees with NLO BFKL eigenvalues.
- The NLO kernel in QCD is a sum of the conformal part and running-coupling part.

Happy Birthday!