# Gluon and dipole cascades on the light-front

Anna Staśto
Penn State & RIKEN BNL

Work done in collaboration with Leszek Motyka, Phys. Rev. D 79, 085016 (2009); arXiv:0901.4949

*Alfred Mueller's birthday workshop, Columbia University, October 23, 2009*

# Outline

Motivation: better kinematics in gluon(dipole) cascades. Modified kernel for dipole evolution at small x.

Light-front wave-functions and gluon fragmentation with exact kinematics.

Relation with the maximally helicity violating (MHV) amplitudes.

# SOFT GLUONS IN THE INFINITE MOMENTUM WAVEFUNCTION AND THE BFKL POMERON[1]

A.H. Mueller
*Stanford Linear Accelerator Center*
*Stanford, California 94309*

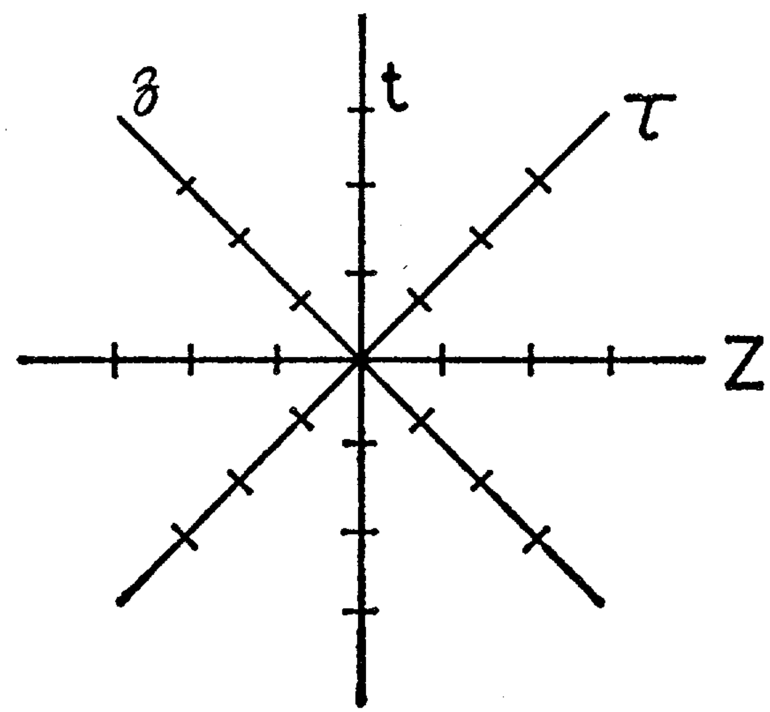
and

*Department of Physics, Columbia University[2]*
*New York, New York 10027*

We construct the infinite momentum wavefunction for arbitrary numbers of soft gluons in a heavy quark-antiquark, onium, state. The soft gluon part of the wavefunction is constructed exactly within the leading logarithmic and large $N_c$ limits. The BFKL pomeron emerges when gluon number densities are evaluated.

# Light-front formalism

*Dirac*　　possibly avoid the negative energies



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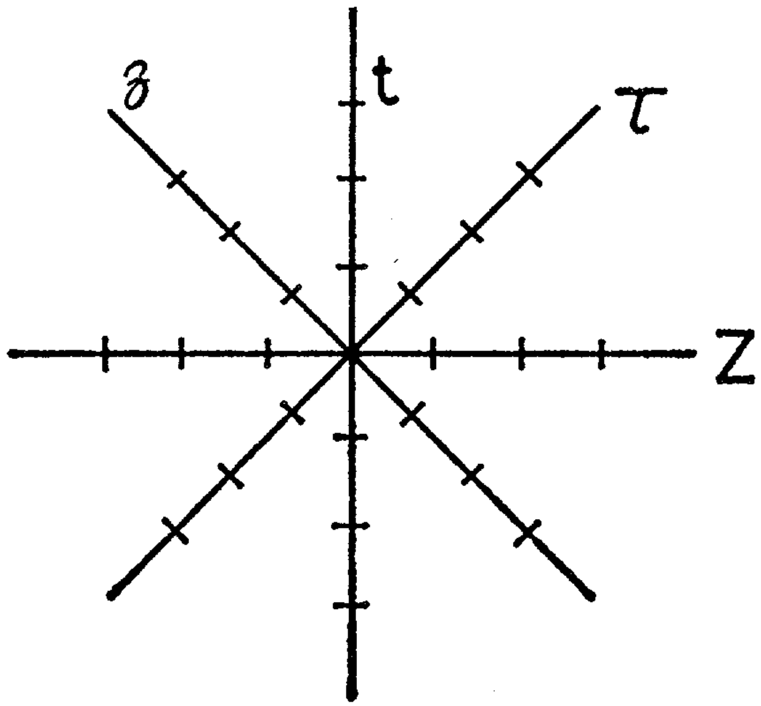


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Infinite momentum frame: a limit of a Lorentz frame moving in the -z direction with a (nearly) the speed of light.

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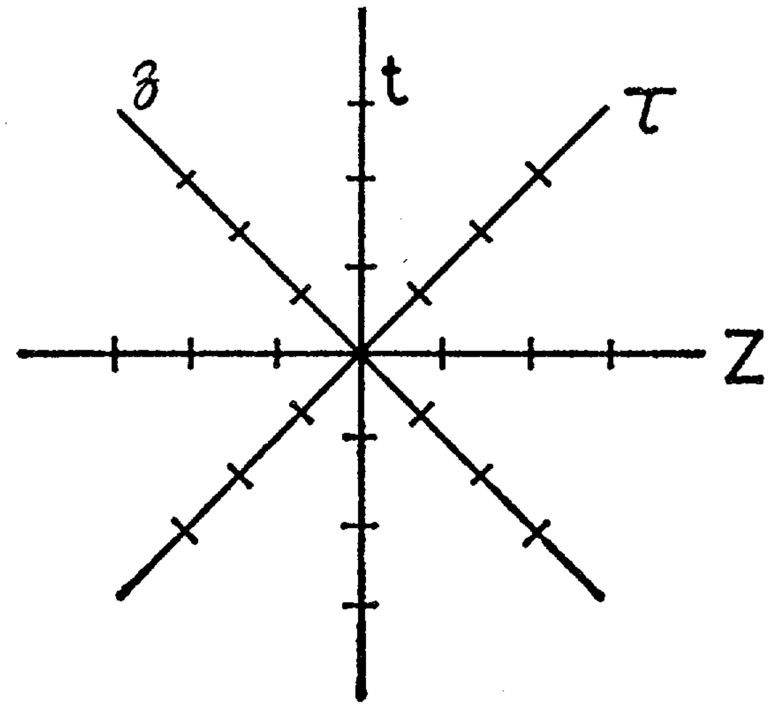


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Infinite momentum frame: a limit of a Lorentz frame moving in the -z direction with a (nearly) the speed of light.

*Susskind*

Isomorphism with the Galilean dynamics in 2 dimensions:

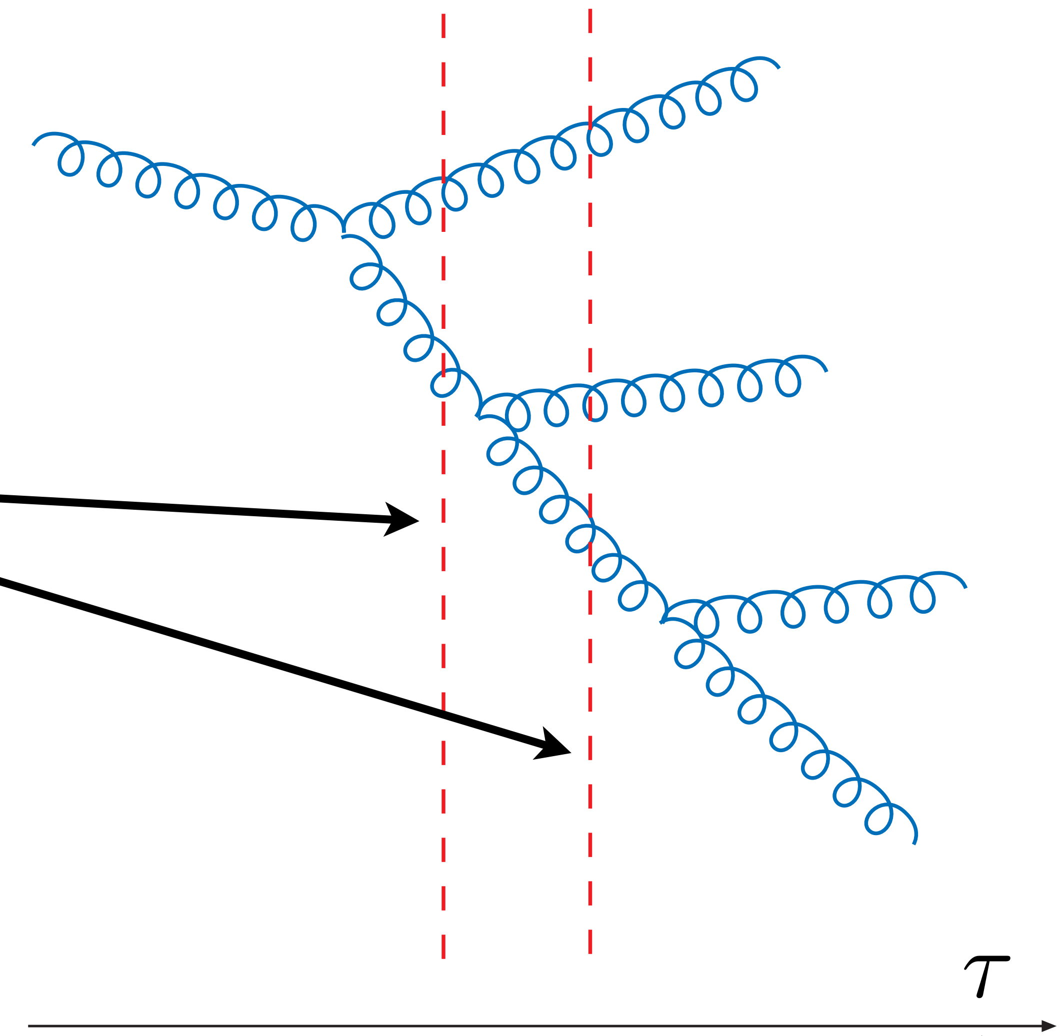
$P^-$ ⟶ Hamiltonian

$P^+$ ⟶ Mass

$P_T$ ⟶ 2-dim. momentum

# Non-covariant (light-front) time ordered diagram

Energy denominators




light-front time

Difference of light - cone energies:

$$D_n = P^- - \sum_i k_i^-$$

Initial state

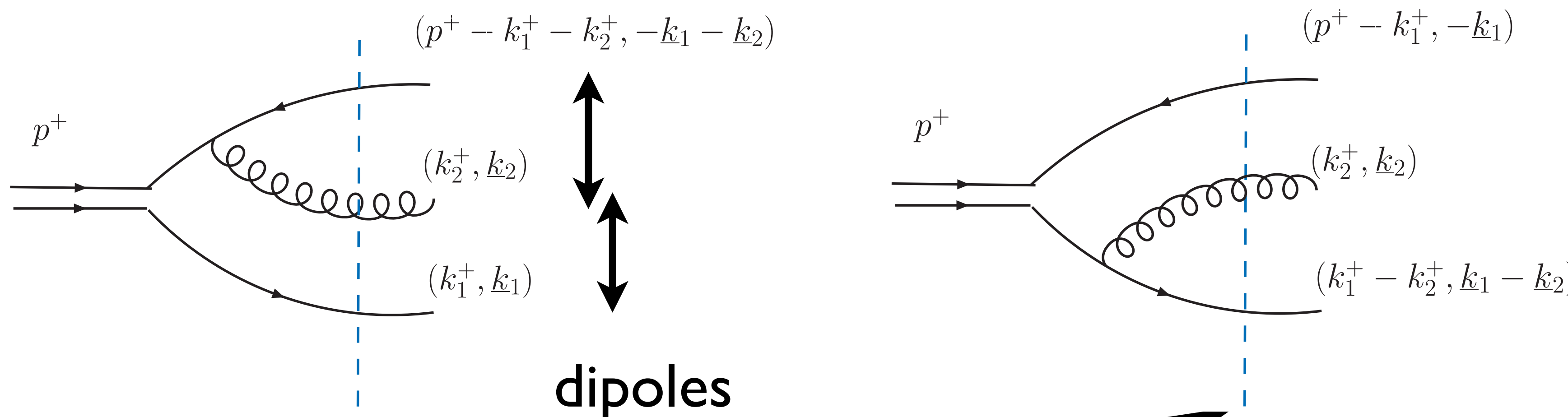
Intermediate states

Energy denominators entangle the momenta of all the particles in the cascades.
Need to consider all possible time ordered diagrams.

# Dipole evolution at high energy

*Mueller*

Quark-antiquark pair (dipole) emitting longitudinally soft gluon.




Energy denominators in light cone perturbation theory

$$\bar{D}_1 = \frac{1}{P^- - [(P - k_1 - k_2)^- + k_1^- + k_2^-]} \simeq \frac{1}{k_2^-}$$

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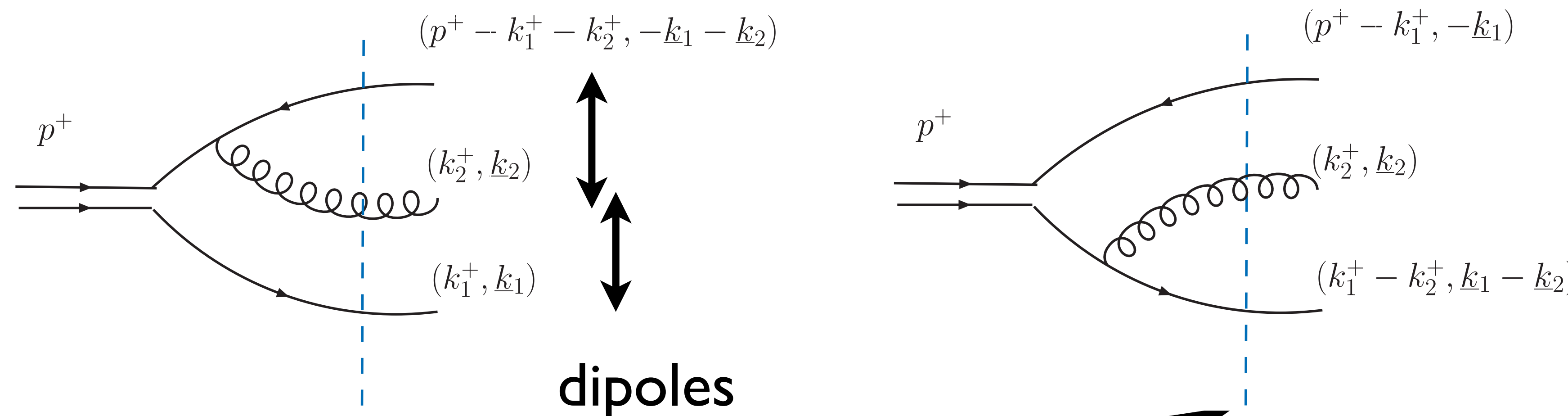


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High energy limit: strong ordering in longitudinal momenta

$$k_2^+ \ll k_1^+, P^+ \qquad k_2^- \equiv \frac{\underline{k}_2^2}{2k_2^+}$$

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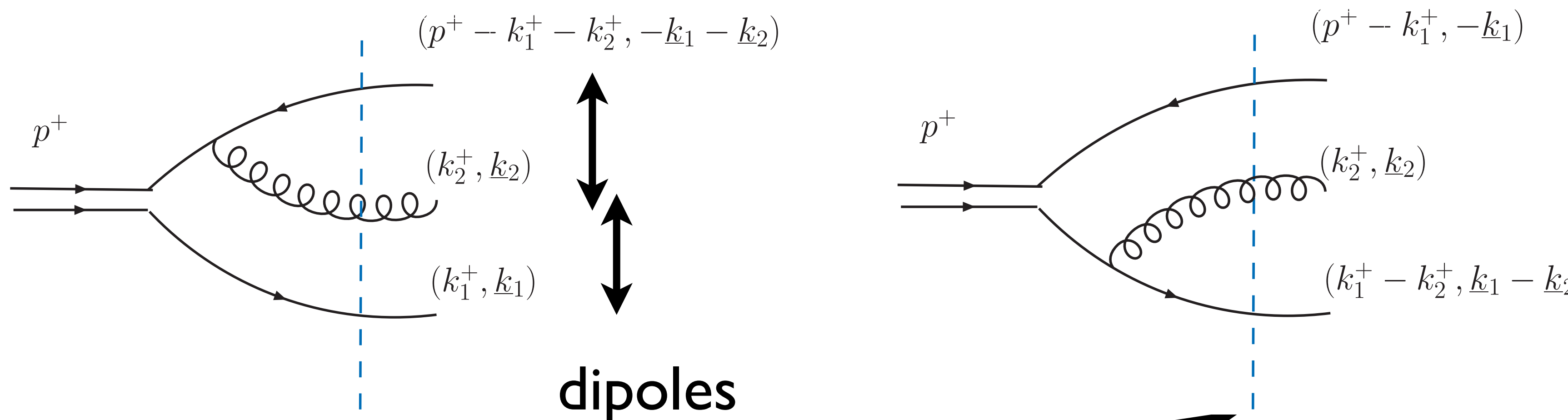


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Wave function with 1 gluon

Wave function without gluons

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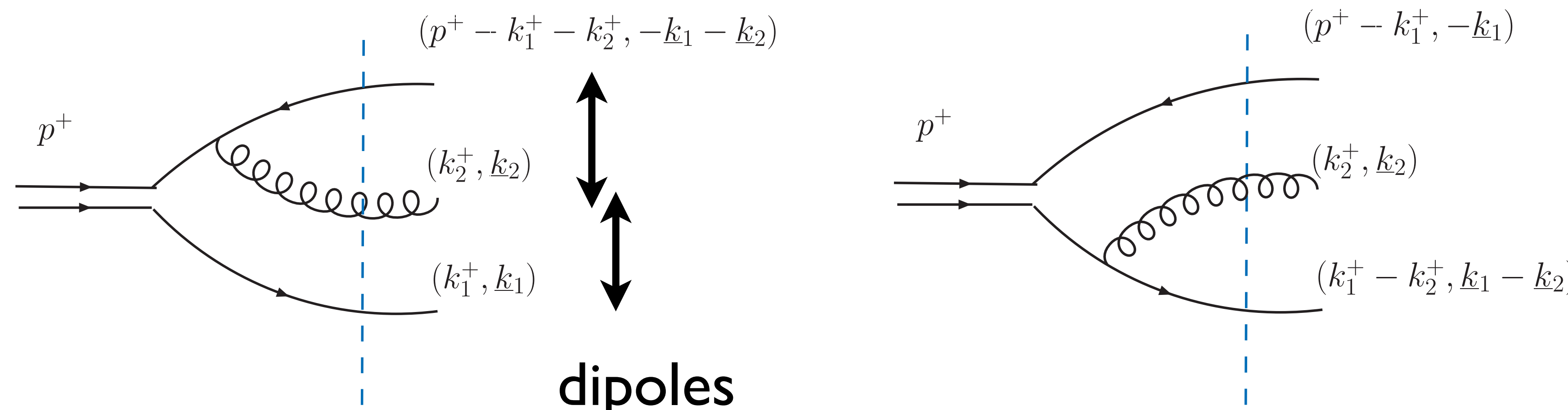


dipoles

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Wave function with 1 gluon

Wave function without gluons

$$z_1 = \frac{k_1^+}{P^+} \qquad z_2 = \frac{k_2^+}{P^+}$$

# In transverse coordinate space

$$\Phi^{(1)}(\underline{x}_{01}, \underline{x}_{02}; z_1, z_2) \;= -\frac{igt_a}{\pi}\left(\frac{\underline{x}_{20}}{x_{20}^2} - \frac{\underline{x}_{21}}{x_{21}^2}\right) \cdot \underline{\epsilon}_2 \, \Psi^{(0)}(\underline{x}_{01}; z_1)$$

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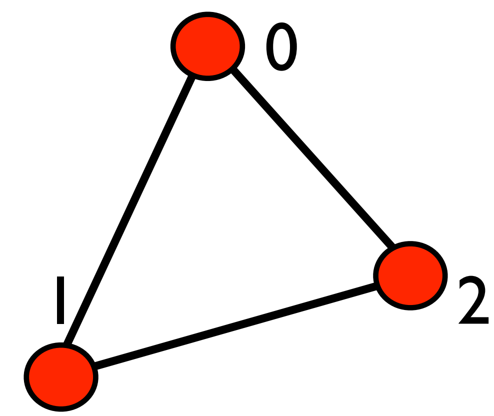
Soft gluons factorize in the transverse spae




Dipole kernel in the limit of high energy:

$$\frac{d^2\underline{x}_{02}\,\underline{x}_{01}^2}{\underline{x}_{02}^2\,\underline{x}_{12}^2}$$

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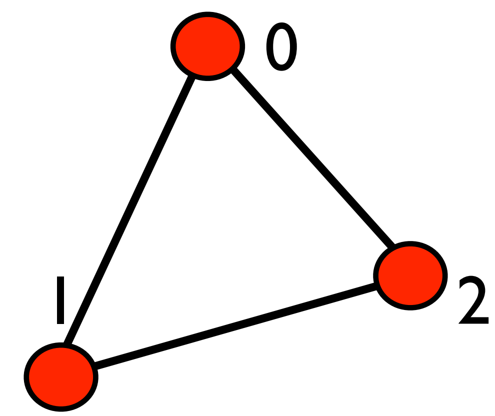


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$$\frac{\partial N_{01}}{\partial Y} = \bar{\alpha}_s\int\frac{d^2x_2}{2\pi}\frac{x_{01}^2}{x_{02}^2x_{12}^2}[N_{02}+N_{12}-N_{01}]$$

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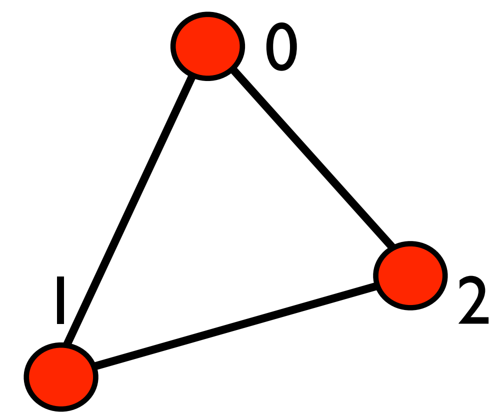


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Dipole version of the BFKL equation.

$Y$ rapidity

$N_{01}$ dipole scattering amplitude (related to the gluon density)

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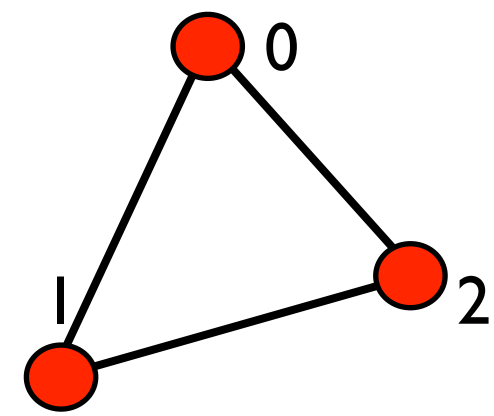


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$N_{01}$ dipole scattering amplitude (related to the gluon density)

No restrictions on the transverse coordinates (or momenta).

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$$\frac{\underline{k}_2^2}{k_2^+} > \frac{\underline{k}_1^2}{k_1^+}$$

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Ordering in the fluctuation time: Dokshitzer, Marchesini, Salam

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Longitudinal and transverse momenta tied together.

# Modified dipole kernel

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Approximate Fourier transform to coordinate space

$$\Phi^{(1)}(\underline{x}_{02}, \underline{x}_{12}; z) \sim g t^a \left( \bar{Q}_{01} K_1(\bar{Q}_{01} x_{02}) \frac{\underline{\epsilon}_2 \cdot \underline{x}_{02}}{x_{02}} - \bar{Q}_{01} K_1(\bar{Q}_{01} x_{12}) \frac{\underline{\epsilon}_2 \cdot \underline{x}_{12}}{x_{12}} \right) \Phi^{(0)}(\underline{x}_{01}; z)$$

Modified Bessel functions of the second kind.

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$z$ longitudinal momentum fraction

$\ln 1/z \sim y$ rapidity

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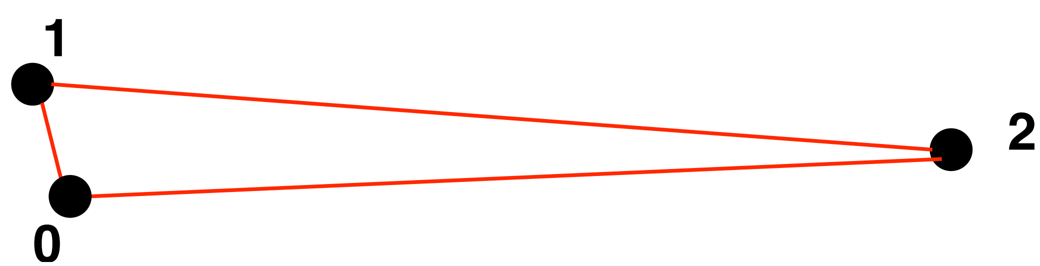
- Energy dependent cutoff in impact parameter: exponential tails, range depends on the energy.
- Violation of conformal invariance in 2-dimensions.
- Recovering original dipole kernel in the high energy limit.

# Impact parameter and NLL correction

Cutoff on configuration of large dipoles




$$x_{02} \gg x_{01} \; \& \; x_{12} \sim x_{02}$$

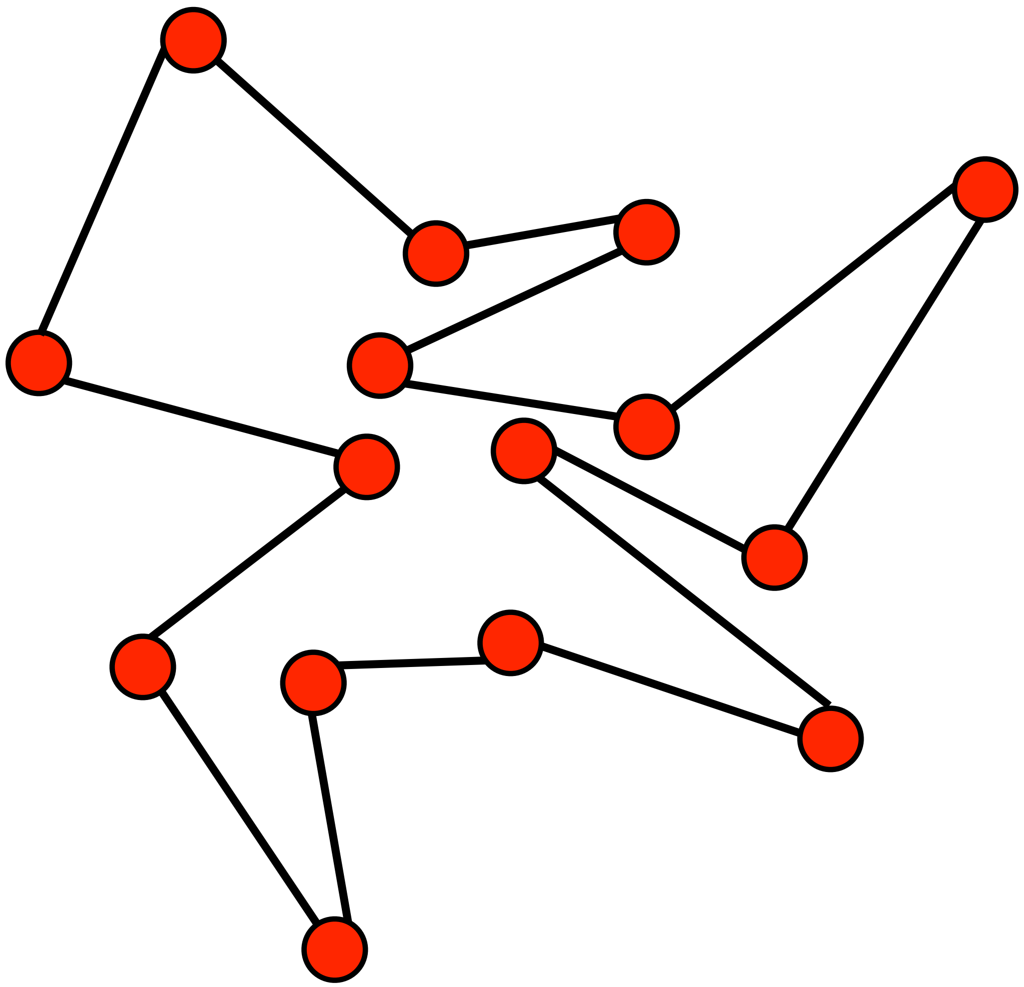
exponential behavior

$$\mathcal{K} \sim \exp\left(-\frac{x_{02}}{x_{01}}\sqrt{z}\right)$$

Recovering part of NLL contribution from explicit calculation by Balitsky and Chirilli (non-conformal part).

$$\mathcal{K}^{\rm NLO}_{\rm non-conf.} \otimes N_Y \quad \rightarrow \quad -\frac{\bar{\alpha}_s^2}{\pi} \int \frac{d^2\underline{x}_2 \, x_{01}^2}{x_{02}^4} \log^2(\frac{x_{02}}{x_{01}}) \, [\ldots]$$

# Random walk-diffusion in impact parameter space



$$\mathcal{K} \sim \frac{d^2\underline{x}_{02}\,\underline{x}_{01}^2}{\underline{x}_{02}^2\,\underline{x}_{12}^2}$$

$$\mathcal{K} \sim \exp\left(-\frac{x_{02}}{x_{01}}\sqrt{z}\right)$$

The growth of the interaction area is slowed down due to the different functional form of the branching kernel.

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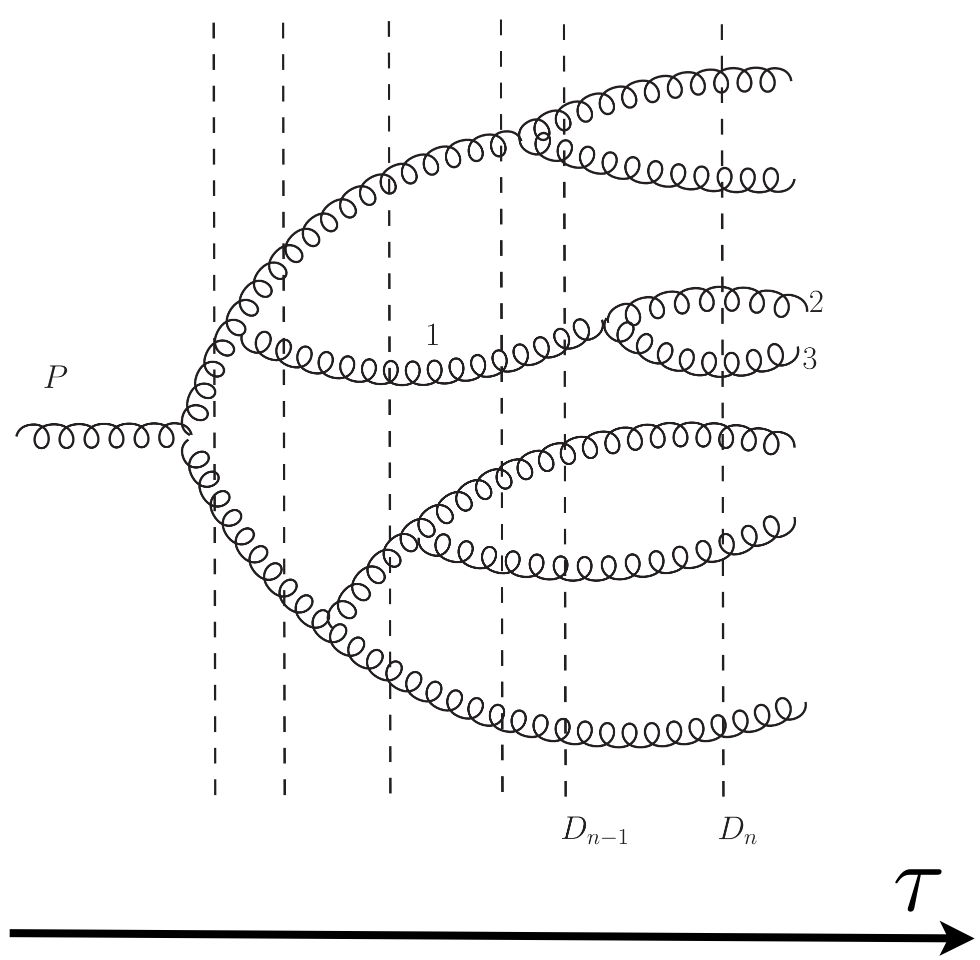

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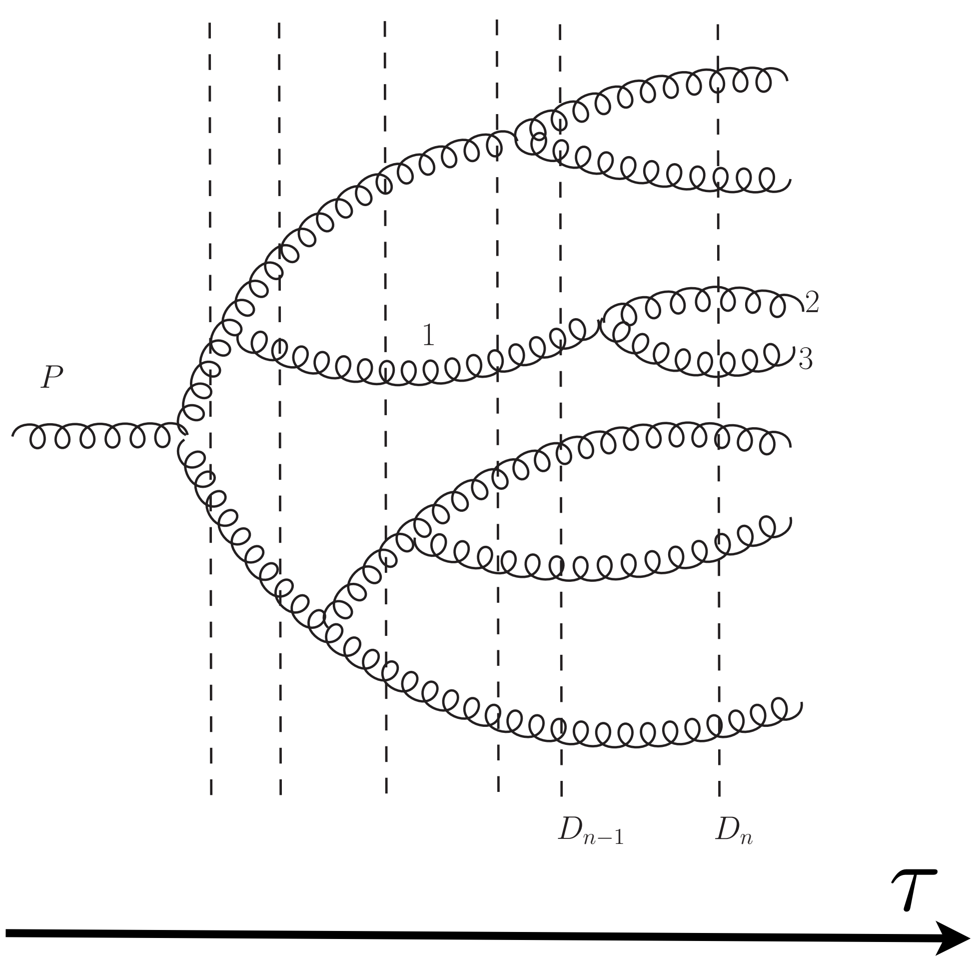


Special helicity configuration: helicity conserved through the whole cascade.

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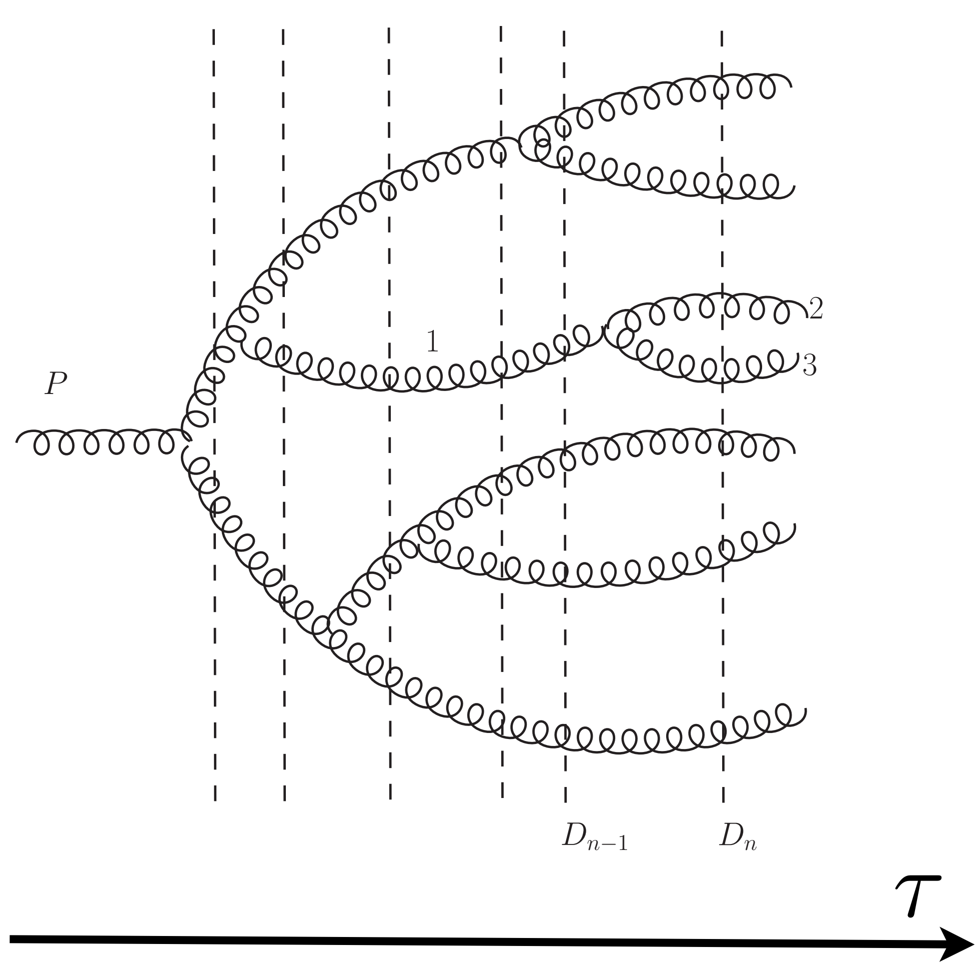


Special helicity configuration: helicity conserved through the whole cascade.

Unlike the small x limit now the energy denominators entangle all particle momenta.

# Recurrence relations between wave functions

$$\Psi_{n+1}(k_0, k_1, \ldots, k_n) = \frac{g}{\sqrt{\xi_{01}}} \frac{\underline{\epsilon}^{(-)} \underline{v}_{01}}{D_n + \xi_{01}\, \underline{v}_{01}^2}\, \Psi_n(k_{01}, k_2, \ldots, k_n)$$

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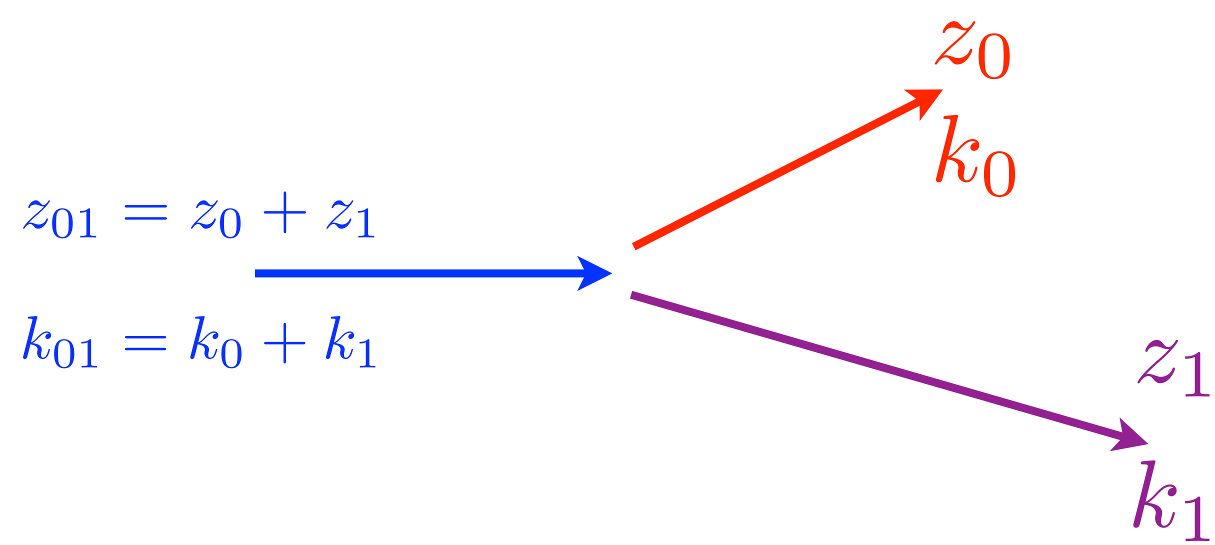

# Recurrence relations between wave functions

$$\Psi_{n+1}(k_0, k_1, \ldots, k_n) = \frac{g}{\sqrt{\xi_{01}}} \frac{\underline{\epsilon}^{(-)} \underline{v}_{01}}{D_n + \xi_{01}\, \underline{v}_{01}^2} \Psi_n(k_{01}, k_2, \ldots, k_n)$$

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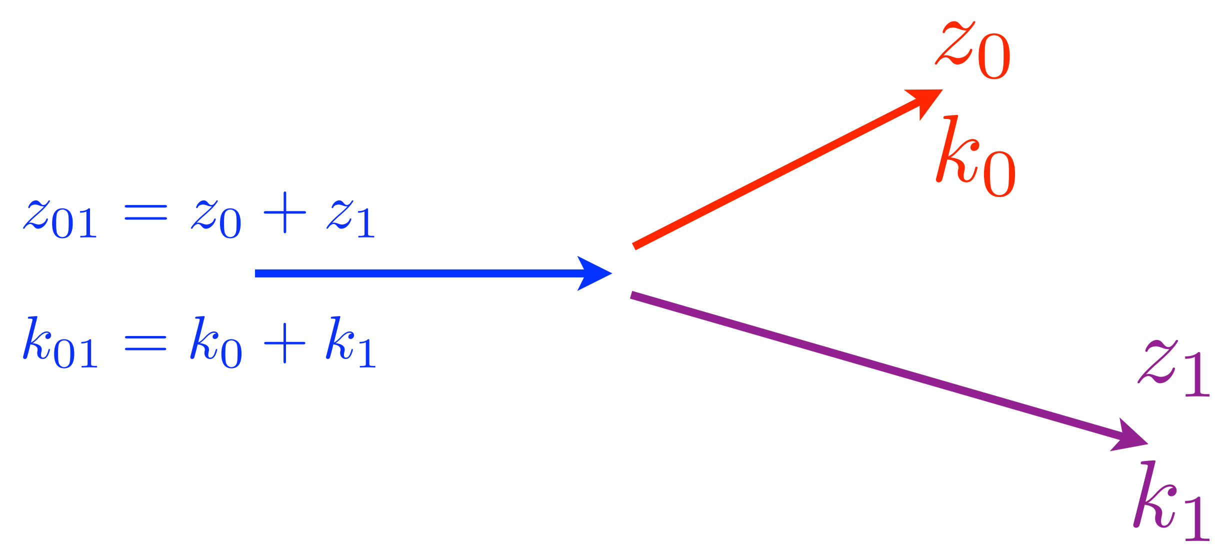


reduced lightfront 'mass'

$$\xi_{01} = \frac{z_0 z_1}{z_0 + z_1}$$

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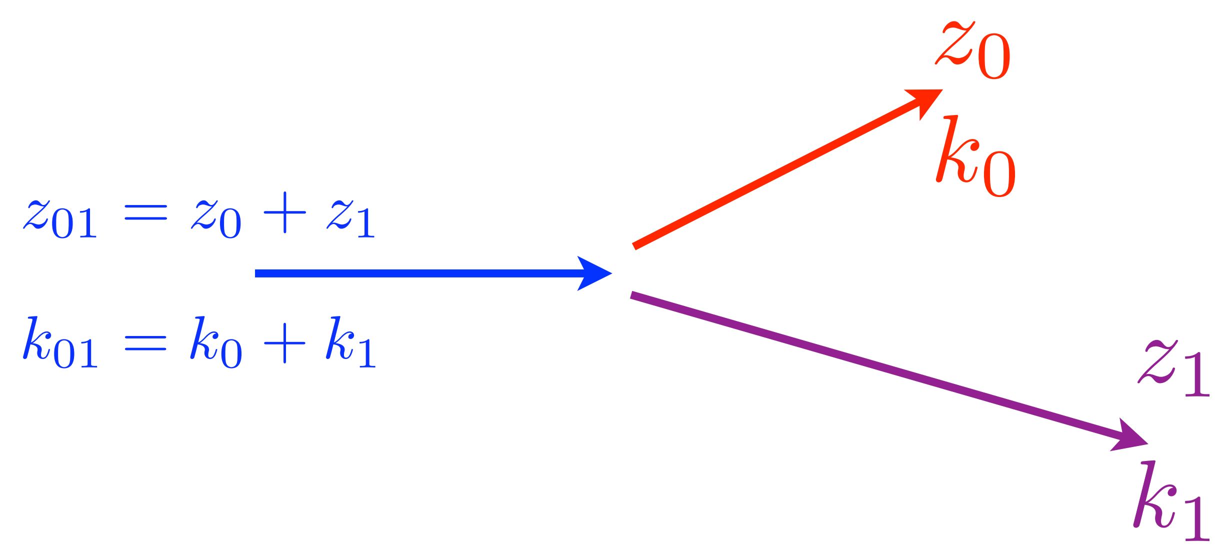


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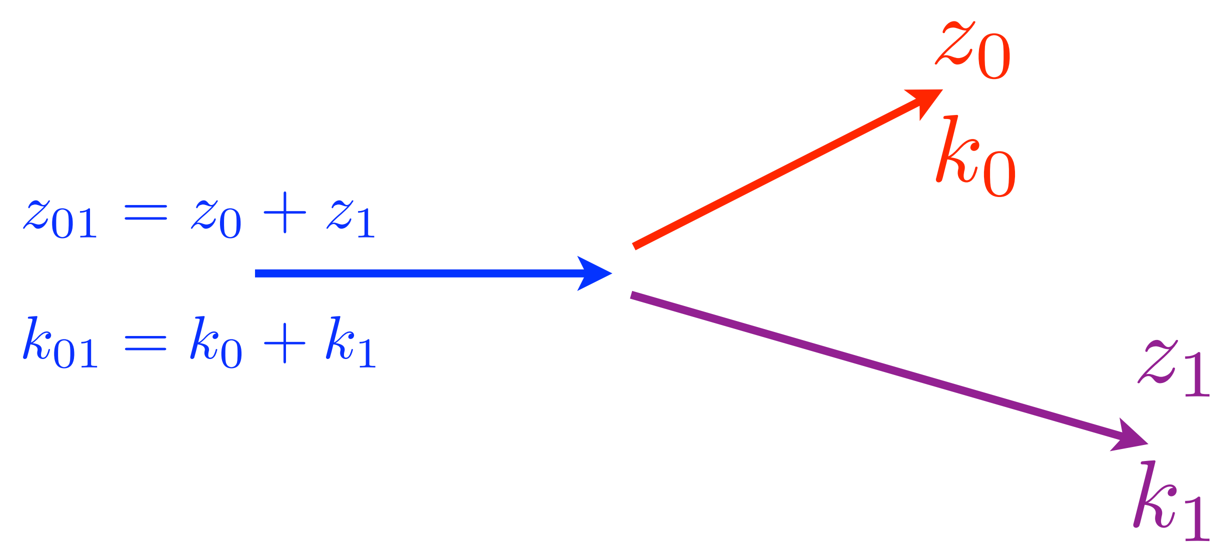


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Isomorphism with non-relativistic dynamics apparent in the case of the exact kinematics.

# Transverse coordinates

n-component wave function in transverse space

$$\Phi_n(1,\ldots,n) \equiv \Phi_n(z_1,\underline{r}_1;z_2,\underline{r}_2;\ldots,z_n,\underline{r}_n) = \int \frac{d^2\underline{k}_1}{(2\pi)^2}\frac{d^2\underline{k}_2}{(2\pi)^2}\ldots\frac{d^2\underline{k}_n}{(2\pi)^2}\exp(i\underline{k}_1\cdot\underline{r}_1+i\underline{k}_2\cdot\underline{r}_2+\ldots+i\underline{k}_n\cdot\underline{r}_n)\,\Psi_n(k_1,\ldots,k_n),$$

Recurrence relation in the transverse space with off-shell incoming particle $P^- = -\frac{Q^2}{2P^+}$

$$\Phi_n(1,\ldots,n) = i\,\frac{\underline{\epsilon}^{(-)}\cdot\underline{r}_{12}}{\sqrt{\xi_{12}}}\,z_1z_2\ldots z_n\int\frac{d^2\underline{r}'_1\ldots d^2\underline{r}'_{n-1}}{(2\pi)^n}\left(\frac{Q^2}{A}\right)^{\frac{n}{2}}K_n(\sqrt{Q^2A})\,\Phi_{n-1}(1',2',\ldots,(n-1)')\,,$$

$$A \equiv \xi_{12}\underline{r}_{12}^2 + z_{12}(\underline{r}'_1-\underline{R}_{12})^2 + z_3(\underline{r}'_2-\underline{r}_3)^2+\ldots+z_n(\underline{r}'_{n-1}-\underline{r}_n)^2$$

Recurrence relation is not easy to solve exactly in this case...

# Case of the on-shell incoming gluon. $Q^2 \to 0$

# Can resum the wave function completely.

$$- D_{n+1}\,\Psi_{n+1}(1,2,\ldots,n+1) \;=\; g\sum_{i=1}^{n}\frac{v^*_{(i,i+1)}}{\sqrt{\xi_{(i,i+1)}}}\Psi_n(1,2,\ldots,(i\,i+1),\ldots,n+1) \qquad n \to n+1$$

$$-D_n\Psi_n(1,2,\ldots,n) \;=\; g\sum_{k=1}^{n-1}\frac{v^*_{(k,k+1)}}{\sqrt{\xi_{(k,k+1)}}}\Psi_{n-1}(1,2,\ldots,(k\,k+1),\ldots,n) \qquad n-1 \to n$$

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...

**Tree-level gluon wave function with exact kinematics**

$$\Psi_n(1,2,\dots,n) \;=\; (-1)^{n-1}g^{n-1}\,\Delta^{(n)}\,\frac{1}{\sqrt{z_1 z_2\dots z_n}}\,\frac{1}{\xi_{(12\dots n-1)n}\,\xi_{(12\dots n-2)(n-1\,n)}\,\cdots\,\xi_{1(2\dots n)}}$$

$$\times\,\frac{1}{v_{(12\dots n-1)n}\,v_{(12\dots n-2)(n-1\,n)}\,\cdots\,v_{1(2\dots n)}}\,.$$

$$v_{(i_1 i_2\dots i_p)(j_1 j_2\dots j_q)} = \frac{k_{i_1}+k_{i_2}+\dots+k_{i_p}}{z_{i_1}+z_{i_2}+\dots+z_{i_p}} - \frac{k_{j_1}+k_{j_2}+\dots+k_{j_q}}{z_{j_1}+z_{j_2}+\dots+z_{j_q}},$$

$$\xi_{(i_1 i_2\dots i_p)(j_1 j_2\dots j_q)} = \frac{(z_{i_1}+z_{i_2}+\dots+z_{i_p})(z_{j_1}+z_{j_2}+\dots+z_{j_q})}{z_{i_1}+z_{i_2}+\dots+z_{i_p}+z_{j_1}+z_{j_2}+\dots+z_{j_q}}\,,$$

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...

**Tree-level gluon wave function with exact kinematics**

$$\Psi_n(1,2,\dots,n) \;=\; (-1)^{n-1} g^{n-1}\, \Delta^{(n)}\, \frac{1}{\sqrt{z_1 z_2 \dots z_n}}\, \frac{1}{\xi_{(12\dots n-1)n}\, \xi_{(12\dots n-2)(n-1\,n)} \;\cdots\; \xi_{1(2\dots n)}}$$

$$\times \frac{1}{v_{(12\dots n-1)n}\, v_{(12\dots n-2)(n-1\,n)} \;\cdots\; v_{1(2\dots n)}}\,.$$

$$v_{(i_1 i_2 \dots i_p)(j_1 j_2 \dots j_q)} = \frac{k_{i_1} + k_{i_2} + \dots + k_{i_p}}{z_{i_1} + z_{i_2} + \dots + z_{i_p}} - \frac{k_{j_1} + k_{j_2} + \dots + k_{j_q}}{z_{j_1} + z_{j_2} + \dots + z_{j_q}},$$

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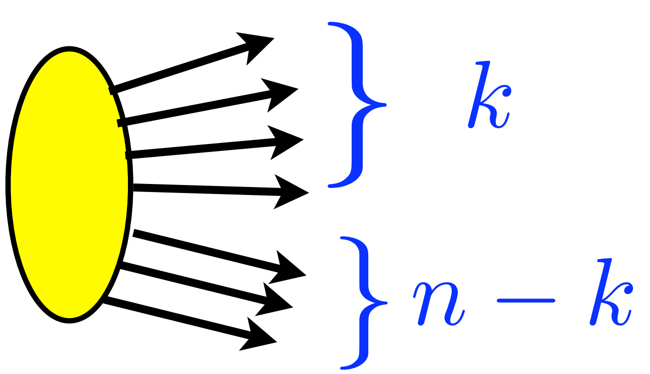

# Transverse coordinates again

$$\Phi_n(1,\ldots,n) \equiv \Phi_n(z_1,\underline{r}_1;z_2,\underline{r}_2;\ldots,z_n,\underline{r}_n)$$
$$= \int \frac{d^2\underline{k}_1}{(2\pi)^2}\frac{d^2\underline{k}_2}{(2\pi)^2}\cdots\frac{d^2\underline{k}_n}{(2\pi)^2}\exp(i\underline{k}_1\cdot\underline{r}_1+i\underline{k}_2\cdot\underline{r}_2+\ldots+i\underline{k}_n\cdot\underline{r}_n)\,\Psi_n(k_1,\ldots,k_n),$$

Closed expression is simple:

$$\Phi_n(z_1,\underline{r}_1;\ldots;z_n,\underline{r}_n) = (-1)^{n-1}g^{n-1}\delta\left(1-\sum_{i=1}^n z_i\right)\frac{1}{\sqrt{z_1z_2\ldots z_n}}\frac{\underline{\epsilon}^{(-)}\underline{r}_{12}\,\underline{\epsilon}^{(-)}\underline{r}_{23}\;\ldots\;\underline{\epsilon}^{(-)}\underline{r}_{n-1\,n}}{r_{12}^2\,r_{23}^2\;\ldots\;r_{n-1\,n}^2}$$

The wave function has similar form as in the small x limit even though we have exact kinematics.

(The evolution will be different...work in progress)

# Relation to Parke-Taylor amplitudes

$$\mathcal{M}_n = \sum_{\{1,\ldots,n\}} \mathrm{tr}(t^{a_1} t^{a_2} \ldots t^{a_n})\, m(p_1, \epsilon_1; p_2, \epsilon_2; \ldots; p_n, \epsilon_n) \ ,$$

Color part

Kinematical part

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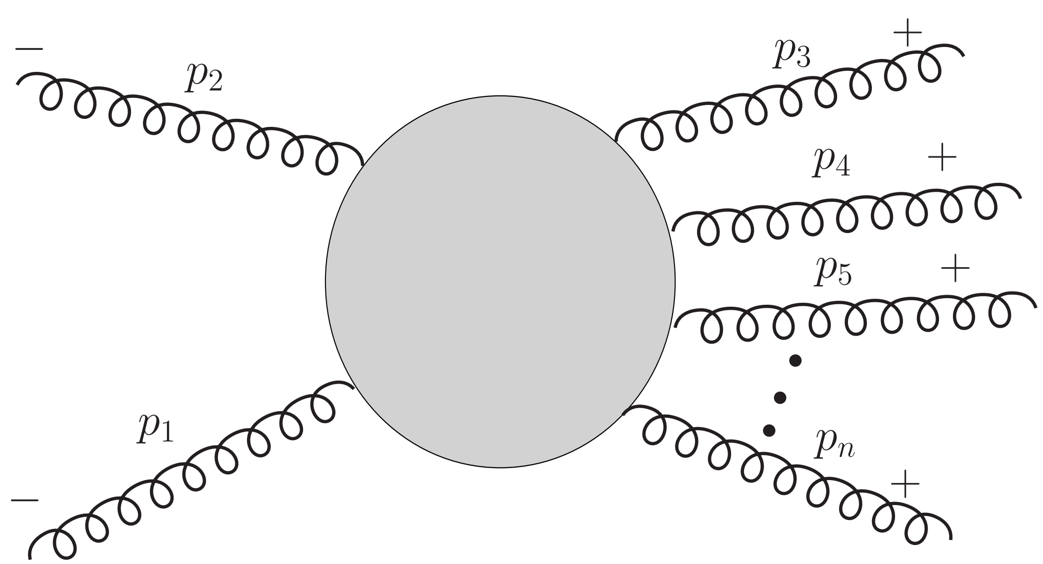
Maximally Helicity Violating amplitude for gluons: 2 to n

Here: all gluons are outgoing



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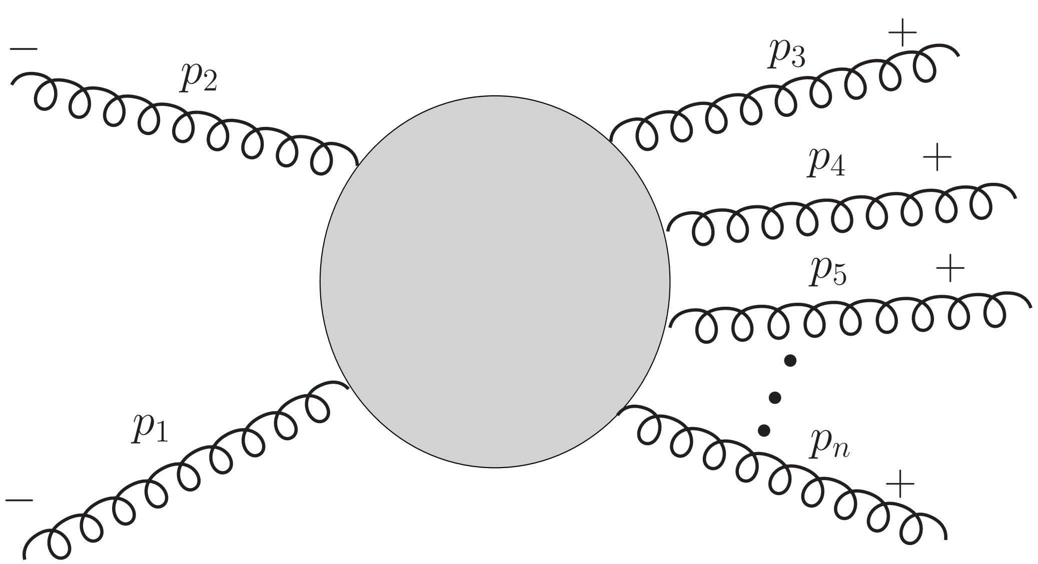



spinor products

Tree level, Parke-Taylor formula

$$m(1^-, 2^-, 3^+, \ldots, n^+) \;=\; i g^{n-2} \, \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n{-}2\; n{-}1 \rangle \langle n{-}1\; n \rangle \langle n1 \rangle} \; ,$$

# Light-front to MHV dictionary...

$$|i\pm\rangle \;=\; \psi_\pm(k_i) \;=\; \frac{1}{2}(1\pm\gamma_5)\psi(k_i)\;, \quad \langle\pm i| \;=\; \overline{\psi_\pm(k_i)}\,,$$

$$\langle i|j\rangle = \langle i-|j+\rangle\,, \quad [ij] = \langle i+|j-\rangle$$

spinor products

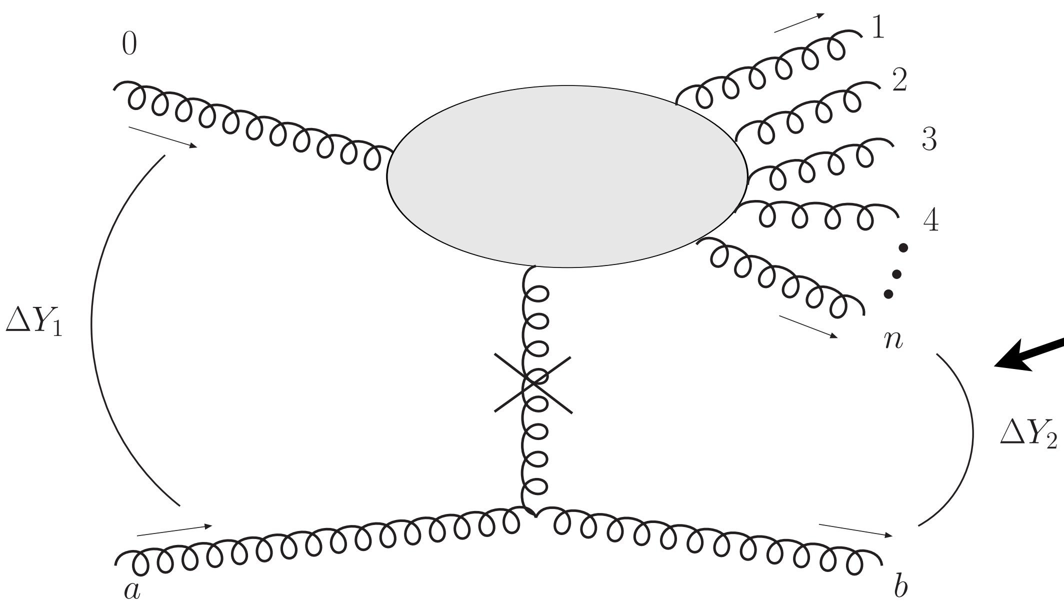
$$\langle ij\rangle = \sqrt{z_i z_j}\;\underline{\epsilon}^{(+)}\cdot\left(\frac{\underline{k}_i}{z_i}-\frac{\underline{k}_j}{z_j}\right)$$

$$[ij] = \sqrt{z_i z_j}\;\underline{\epsilon}^{(-)}\cdot\left(\frac{\underline{k}_i}{z_i}-\frac{\underline{k}_j}{z_j}\right)$$

$$\langle\, ij\,\rangle = \sqrt{z_i z_j}\;\underline{\epsilon}^{(+)}\cdot\underline{v}_{ij},$$

$$[\,ij\,] = \sqrt{z_i z_j}\;\underline{\epsilon}^{(-)}\cdot\underline{v}_{ij}\,,$$

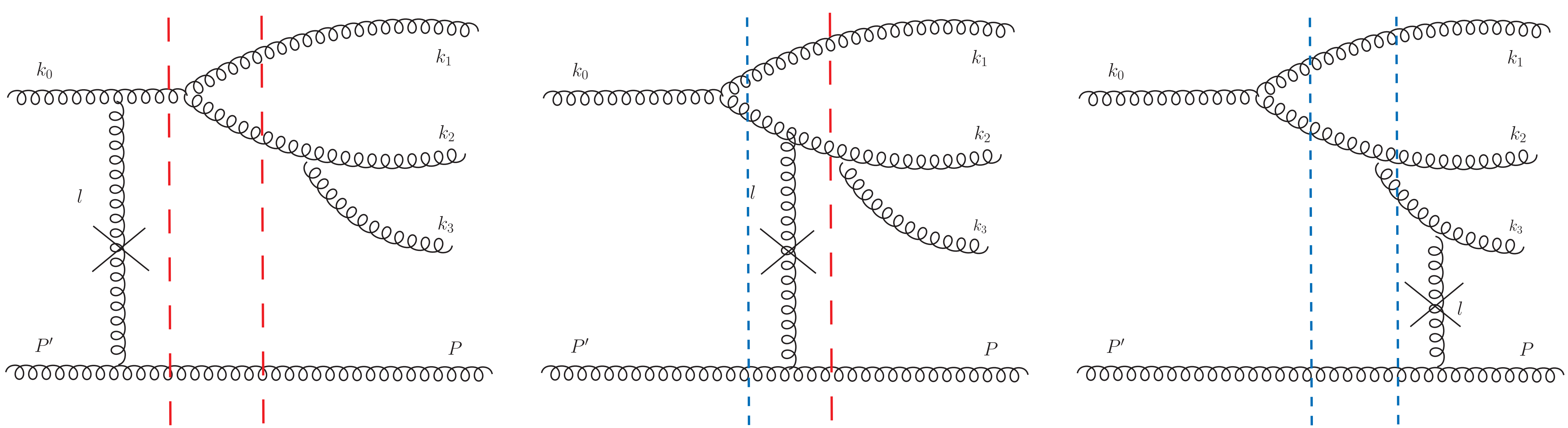
# Scattering from light -cone wave functions



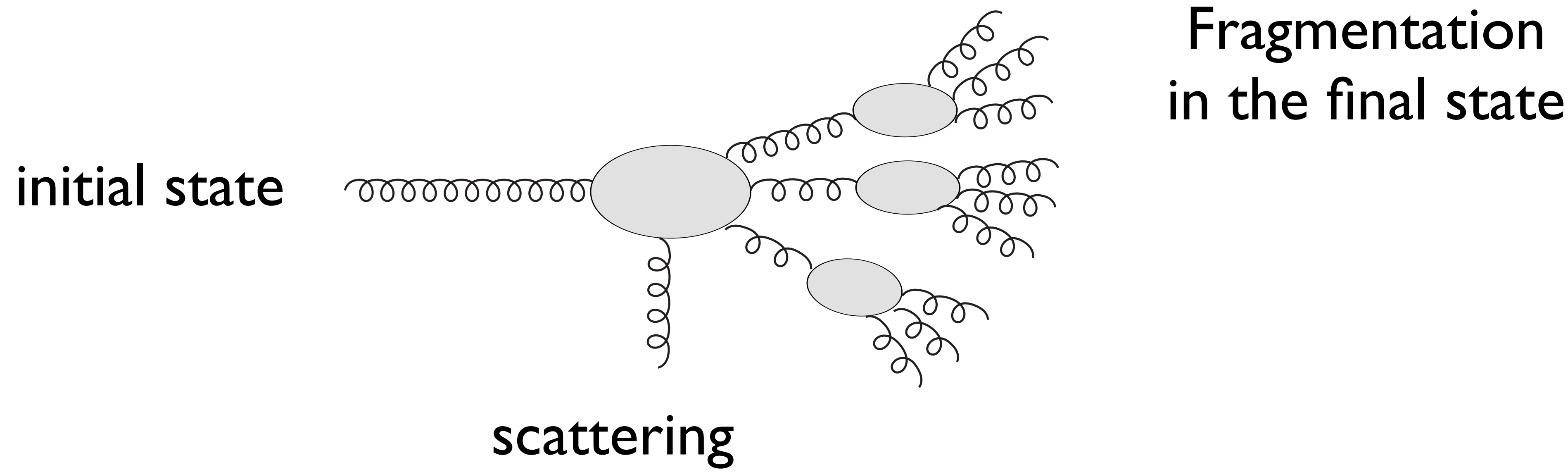

Rapidity gap

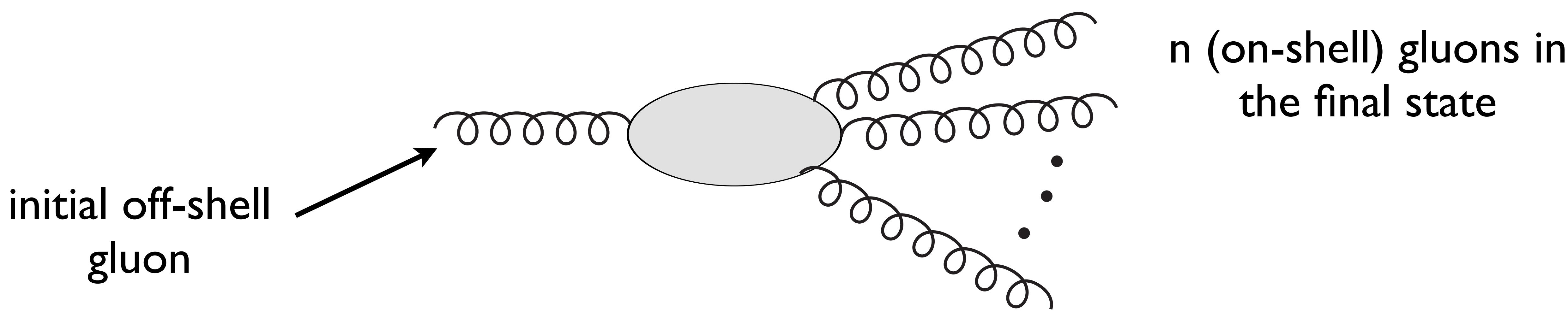
Helicity conserving.
High energy approximation: instantaneous gluon in the light-cone gauge

Sum over initial and final state emissions



# Final state emissions: gluon fragmentation




Amplitude for fragmentation

$$T[(12\ldots n) \to 1, 2, \ldots, n] \;=\; g^{n-1} \left( \frac{z_{(12\ldots n)}}{z_1 z_2 \ldots z_n} \right)^{3/2} \frac{1}{v_{12} v_{23} \ldots v_{n-1\,n}}$$

# Duality: wave function vs fragmentation

| Wave function initial state | ⟷ | Fragmentation final state |
|---|---|---|

$$\Psi_n \sim \frac{1}{v_{(12\ldots n-1)n}\, v_{(12\ldots n-2)(n-1\,n)} \;\cdots\; v_{1(2\ldots n)}}$$

$$T_n \sim \frac{1}{v_{12}v_{23}\ldots v_{n-1\,n}}$$

Nearly identical expressions (the same topology of graphs): different combinations of momenta

$$v_{(i_1 i_2\ldots i_p)(j_1 j_2 \ldots j_q)} = \frac{k_{i_1}+k_{i_2}+\ldots+k_{i_p}}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}} - \frac{k_{j_1}+k_{j_2}+\ldots+k_{j_q}}{z_{j_1}+z_{j_2}+\ldots+z_{j_q}},$$

# Relation with MHV

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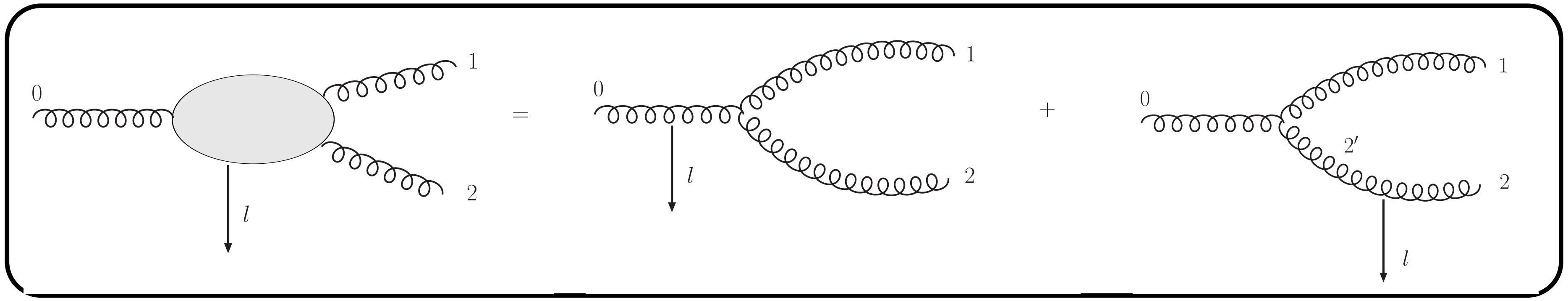
Two gluons




$$\tilde{\Psi}_2(1,2) \;=\; T[(12) \to 1,2] \,+\, \Psi_2(1,2')\,,$$

# Relation with MHV

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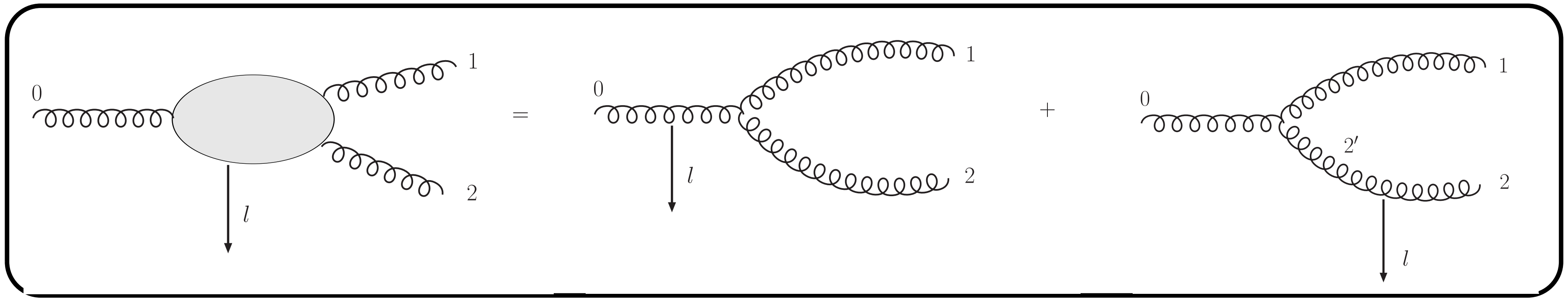


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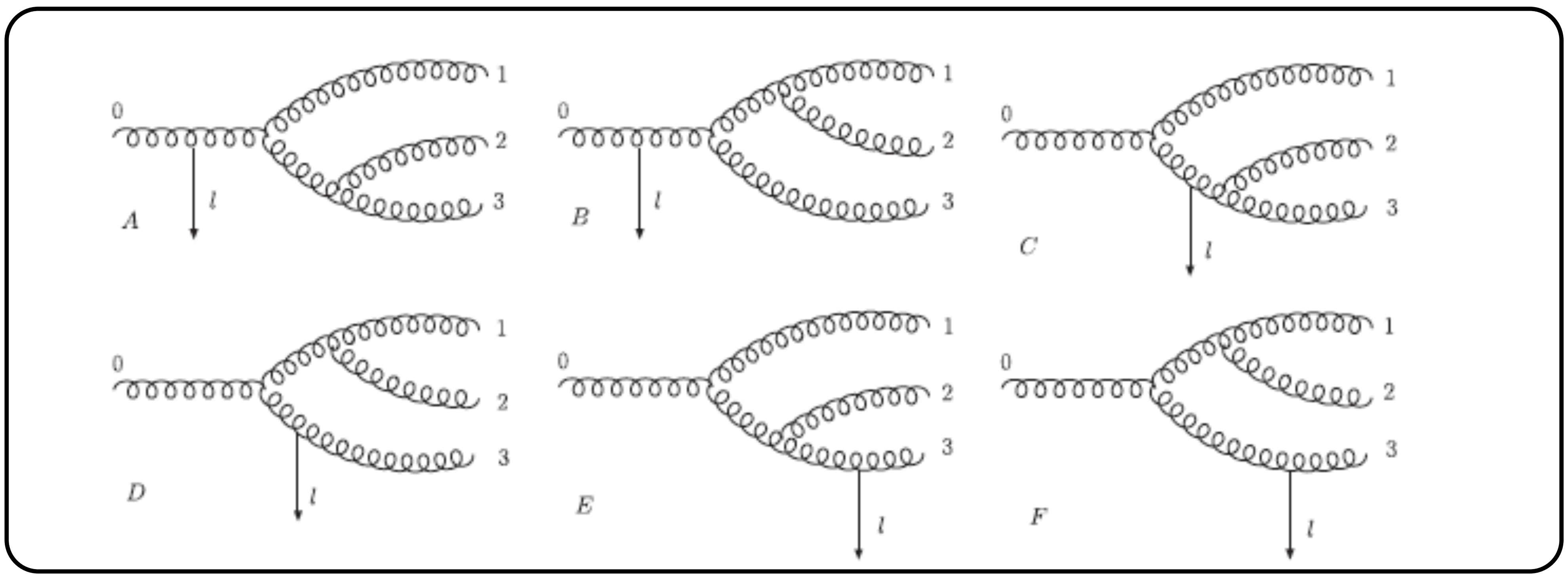
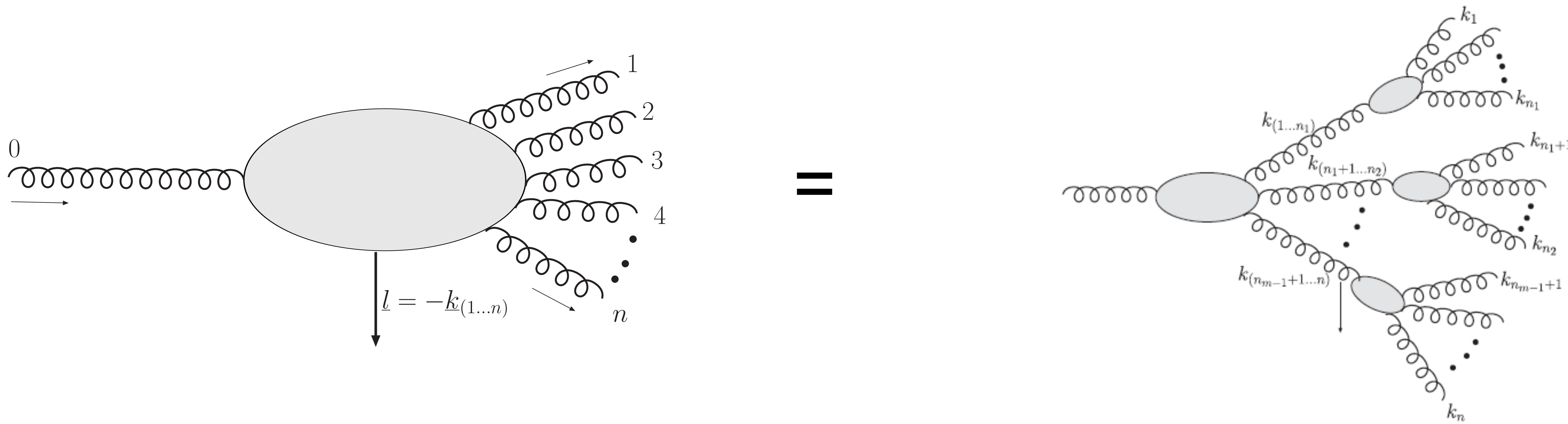
Three gluons



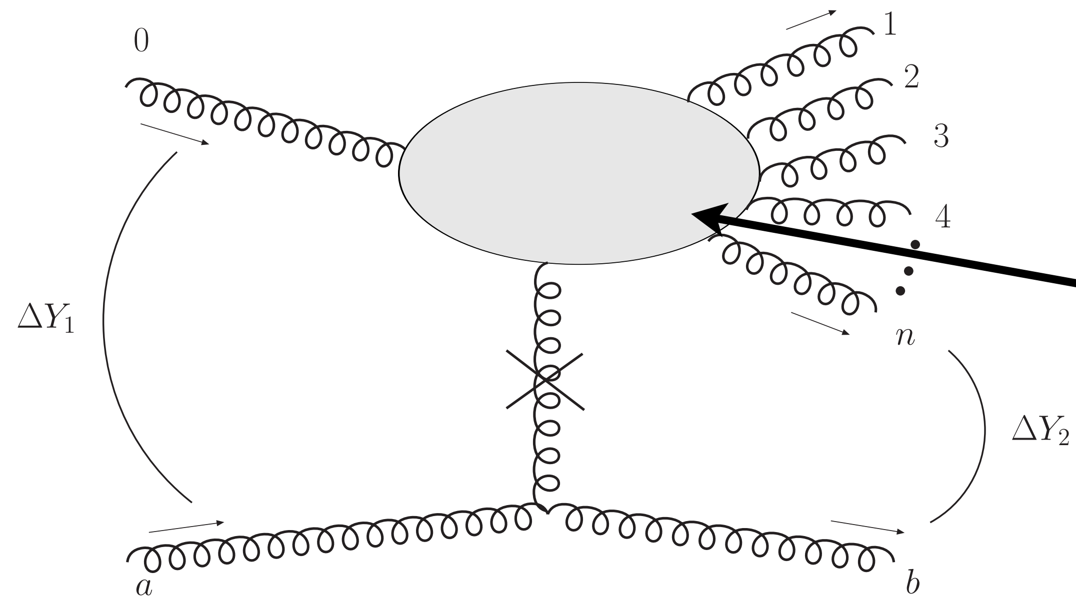

$$\begin{aligned}\tilde{\Psi}_3(1,2,3) = {} & \Psi_1(123')T[(123) \to 1,2,3] \\ & + \Psi_2(1,(23)')T[(23) \to 2,3] \\ & + \Psi_2(12,3')T[(12) \to 1,2] + \Psi_3(1,2,3').\end{aligned}$$

# Master formula for arbitrary number of gluons




Sum over all possible attachments of the exchanged gluon.
Sum over all possible combinations of wave-functions and fragmentation.

$$
\begin{aligned}
\tilde{\Psi}_n(1,2,\dots,n) \;=\; & \sum_{m=1}^{n} \sum_{(1\le n_1<n_2<\dots<n_{m-1}\le n)} \Psi_m((1\dots n_1)(n_1+1\dots n_2)\dots(n_{m-1}+1\dots n)) \\
\times\; & T[(1\dots n_1)\to 1,\dots,n_1]\; T[(n_1+1\dots n_2)\to n_1+1,\dots,n_2]\;\dots\; T[(n_{m-1}+1\dots n)\to n_{m-1}+1,\dots,n]\,.
\end{aligned}
$$




Upper part: 1 to n with momentum transfer. Obtained by summing all possible attachments.

$$M(0; a \to 1, \ldots, n; b) \simeq \frac{s}{t} \times \tilde{\Psi}_n$$

2 to 2 amplitude

Spinor products:

$$\langle i i+1 \rangle = \sqrt{z_i z_{i+1}} v_{ii+1}$$

Recover MHV amplitude in the light cone formalism

$$M(0; a \to 1, \ldots, n; b) \;\simeq\; g^{n+1} \frac{\langle a0 \rangle^4}{\langle a0 \rangle \langle 01 \rangle \langle 12 \rangle \langle n-1\, n \rangle \langle nb \rangle \langle ba \rangle} \,,$$

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- Consistency check with (new derivation of) the MHV amplitudes.