Factorization – and some comments

Alfest, Columbia Univ., Oct. 25, 2009

George Sterman, Stony Brook

- A bit of physics in tribute on a birthday.

I. " From the Mueller files" and factorization

II. Factorization: the classical story

III. Some recent thoughts on factorization in pQCD

IV. Glancing back and looking forward

- **From the Mueller files: groundbreaking work that evolved into the idea of factorization at the cusp of the standard model.**
- **(In the spirit of the season ...) loading the bases for the home run of asymptotic freedom:**

PHYSICAL REVIEW D VOLUME 6, NUMBER 12 15 DECEMBER 1972

Light-Cone Behavior of Perturbation Theory

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A technique introduced by Symanzik is used to derive a series of equations obeyed order by order in perturbation theory by the structure functions W_1 and νW_2 entering the cross section for inelastic electron scattering. These equations relate the q^2, ν, and coupling-constant dependence of W_1 and νW_2 in a manner reminiscent of the renormalization-group results of Gell-Mann and Low. The equations are used to compute the leading logarithmic contribution to νW_2 in a theory of fermions coupled to pseudoscalar particles and a theory of fermions coupled to vector particles.

I. INTRODUCTION

The simple scaling behavior[1] of the structure functions W_1 and νW_2 (Ref. 2) observed[3] for q^2 and $m\nu \gtrsim 2$ BeV^2 has caused considerable interest in the large q^2 and ν dependence of the matrix element

$$\frac{1}{8\pi m}\sum_{s=\pm 1/2}\int e^{-iq\cdot x}d^4x\,\langle p,s|J_\mu(x)J_\nu(0)|p,s\rangle=\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1(q^2,\omega)+\frac{1}{m^2}\left(p^\mu-q^\mu\frac{p\cdot q}{q^2}\right)\left(p^\nu-q^\nu\frac{p\cdot q}{q^2}\right)W_2(q^2,\omega), \tag{1}$$

where $|p,s\rangle$ is a single nucleon state with four-momentum p and z component of spin s, $J_\mu(x)$ is the usual electromagnetic current.[4] In this paper we investigate the behavior of W_1 and νW_2 for large q^2 and fixed $\omega=2m\nu/q^2$ as computed to arbitrary order in the perturbation expansion of a renormalizable field theory.

As is well known,[5] the large q^2 and ν behavior of the matrix element (1) can be determined from the singularity of the product $J_\mu(x)J_\nu(0)$ on the light cone, $x^2=0$. We begin with Wilson's operator expansion[6,7] for the short-distance limit of the product $J_\mu(\frac{1}{2}(x+y))J_\nu(\frac{1}{2}(-x+y))$:

$$\begin{aligned}J_\mu\!\left(\frac{x+y}{2}\right)J_\nu\!\left(\frac{-x+y}{2}\right)&=\left(\delta^{\mu\nu}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}-\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\frac{1}{x^2+i\epsilon x_0}\left\{\sum_{n=0}^{N}\sum_{i=0}^{u_n}F_n^{(i)}(x^2+i\epsilon x_0)\,O^{(i)}_{\mu_1\cdots\mu_n}(y)x_{\mu_1}\cdots x_{\mu_n}+R_N'^{(i)}(x,y)\right\}\\&\quad+\left(\delta^{\mu\nu}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}+\delta_{\alpha\mu}\delta_{\beta\nu}\frac{\partial}{\partial x_\rho}\frac{\partial}{\partial x_\rho}-\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\mu}\delta_{\beta\nu}-\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\nu}\delta_{\beta\mu}\right)\\&\quad\times\left\{\sum_{n=0}^{N}\sum_{i=0}^{u_{n+2}}E_n^{(i)}(x^2+i\epsilon x_0)\,O^{(i)}_{\alpha,\beta,\mu_1,\ldots,\mu_n}(y)x_{\mu_1}\cdots x_{\mu_n}+R_N^{(i)}(x,y)\right\},\end{aligned} \tag{2}$$

The missing fixed point . . .

. . .

$\tilde{E}_n^{(i)}(q^2)$ results if we assume that g_∞ is a simple root of $\beta(g)$ and that $A_n^{(i)}(g)$ and $B_n^{(i)}(g)$ are regular at g_∞. As is shown in Appendix D, these assumptions when combined with Eqs. (79) and (80) imply a simple power behavior for $\tilde{E}_n^{(i)}(q^2)$.

• Opening the door to the final state . . .

PHYSICAL REVIEW D VOLUME 9, NUMBER 4 15 FEBRUARY 1974

Inclusive annihilation processes in ϕ^4 field theory

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The process $\phi(q) \to \phi(p)$ + anything, the process in ϕ^4 theory analogous to $e^+ + e^- \to$ hadron + anything, is examined in ϕ^4 field theory for large values of q^2. Some heuristic arguments as to the strength of mass singularities in a particular two-particle irreducible amplitude make it possible to argue that a light-cone-like expansion exists when $q^2 \to \infty$. This light-cone expansion has virtually all of the properties of the usual light-cone expansion except that it is not an expansion in terms of invariant amplitudes associated with local operators. In case ϕ^4 theory has an eigenvalue, $\beta(g_\infty) = 0$, the moments of the annihilation cross section will have a power behavior in q^2, a power unrelated to the powers of q^2 appearing in any deeply inelastic scattering process. Also, at an eigenvalue the average multiplicity of particles produced, a quantity governed by the Callan-Symanzik equation in this theory, grows like a fractional power of q^2.

- **Including the factorized "distribution function" in a picture**

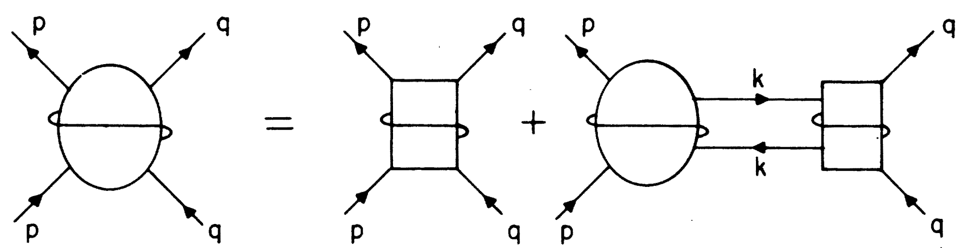

FIG. 1. An illustration of Eq. (5).

ogous to amplitudes which occur in studies of deeply inelastic electron scattering. Further, define the completely off-shell amplitude

$$T(p^2, p\cdot q, q^2) = i\int d^4x\, d^4y\, d^4z\, e^{iq\cdot x + ip\cdot(y-z)} \times \langle \overline{T}(\phi(x)\phi(y))T(\phi(0)\phi(z))\rangle_0 \times [\Delta_F'(p^2)\Delta_F'(q^2)]^{-2}\,, \qquad (2)$$

where Δ_F' is the full, renormalized propagator for the ϕ field, and $\overline{T}$ denotes the anti-time-ordered product. Now, when q^2 and p^2 are below their thresholds

$$\mathrm{disc}_{\ldots}\, \tilde{T}(p^2, p\cdot q, q^2) = 2i\,\mathrm{Im}\, \tilde{T}(p^2, p\cdot q, q^2)$$

ments of T (Refs. 4 and 26) utilized the light-cone expansion and thus cannot be easily generalized.

A. Integral equation and diagonalization

An integral equation for T can be given in terms of a two-particle irreducible kernel, the potential V:

$$T(p^2, p\cdot q, q^2) = V(p^2, p\cdot q, q^2) + \int d^4k\, T(p^2, p\cdot k, k^2)|\Delta_F'(k^2)|^2 \times V(k^2, k\cdot q, q^2)\,. \qquad (5)$$

- **Produced hadron q only traces its lineage back to a single "ancestor," k. All the rest of history is forgotten (the blob on the left). This is the essence of factorization.**

• **And a little later its evolution . . .**

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Cut vertices and their renormalization: A generalization of the Wilson expansion

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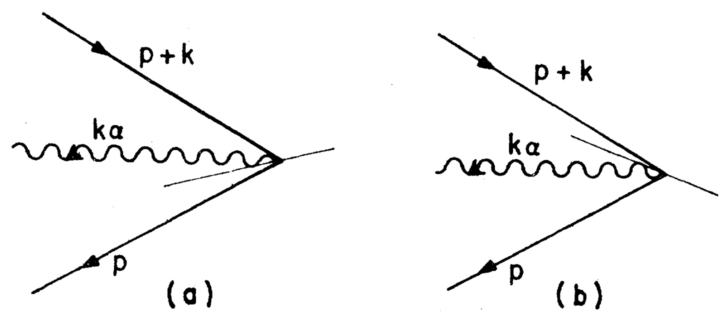
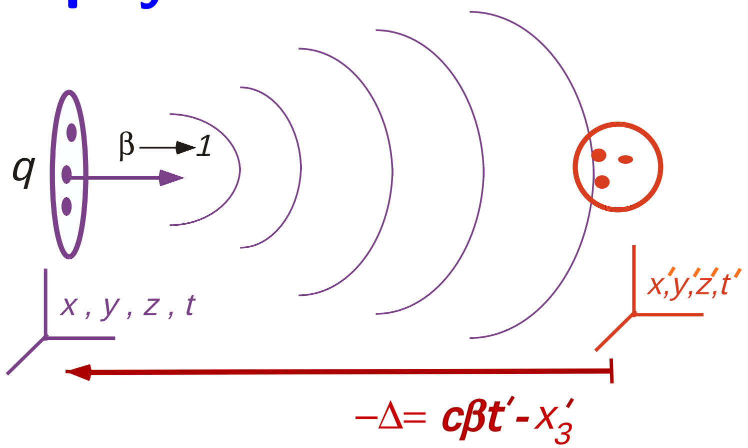
Cut vertices, a generalization of matrix elements of composite operators, are introduced. Their renormalization is discussed. The Bogolubov-Parasiuk-Hepp-Zimmermann method of renormalization of cut vertices allows one to obtain a generalization of the Wilson expansion where cut vertices multiplied by singular functions appear rather than local operators times singular functions. A Callan-Symanzik equation for the moments of the structure function in $e^+ + e^- \rightarrow$ hadron $(p)+$ anything is derived. This equation is valid to all orders of perturbation theory in both gauge and nongauge theories. Examples of renormalization through the two-loop level are given.



• **Here Al cites advances in understanding factorization in gauge theories. How is factorization consistent with long-range forces?**

II. Factorization as a classical story

- **Its physical basis in hadronic collisions**



$$\Delta \equiv x'_3 - \beta c t'$$

- **Why a classical picture isn't far-fetched . . .**

The correspondence principle is the key to to IR divergences.

An accelerated charge must produce classical radiation,

and an infinite numbers of soft gluons are required to make a classical field.

- **Transformation of a <u>scalar</u> field:**

$$\phi(x) = \frac{q}{(x_T^2 + x_3^2)^{1/2}} = \phi'(x') = \frac{q}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$$

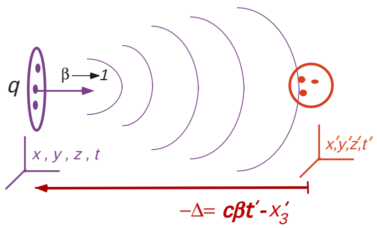
From the Lorentz transformation:

$$x_3 = \gamma(\beta c t' - x_3') \equiv -\gamma \Delta .$$

Closest approach is at $\Delta = 0$, i.e. $t' = \frac{1}{\beta c} x_3'$.

The scalar field transforms "like a ruler": At any fixed $\Delta \neq 0$, the field decreases like $1/\gamma = \sqrt{1 - \beta^2}$.

Why? Because when the source sees a distance x_3, the observer sees a much larger distance.



field	x frame	x' frame
scalar	$\frac{q}{\lvert\vec{x}\rvert}$	$\frac{q}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
gauge (0)	$A^0(x) = \frac{q}{\lvert\vec{x}\rvert}$	$A'^0(x') = \frac{-q\gamma}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
field strength	$E_3(x) = \frac{q}{\lvert\vec{x}\rvert^2}$	$E_3'(x') = \frac{-q\gamma\Delta}{(x_T^2+\gamma^2\Delta^2)^{3/2}}$
Gauge fields :	$E_3 \sim \gamma^0,$	$E_3 \sim \gamma^{-2}$

- **The "gluon" $\vec{A}$ is enhanced, yet is a total derivative:**

$$A^\mu = q\frac{\partial}{\partial x'_\mu} \ln\left(\Delta(t', x'_3)\right) + \mathcal{O}(1-\beta) \sim A^-$$

- **The "large" part of A^μ can be removed by a gauge transformation!**

- The "force" $\vec{\mathrm{E}}$ field of the incident particle does not overlap the "target" until the moment of the scattering.

- "Advanced" effects are corrections to the total derivative:

$$1 - \beta \sim \frac{1}{2}\left[\sqrt{1-\beta^2}\right]^2 \sim \frac{m^2}{2E^2}$$

- Power-suppressed! These are corrections to factorization.

- At the same time, a gauge transformation also induces a phase on charged fields:

$$q(x) \Rightarrow q(x)\, e^{i\,\ln(\Delta)}$$

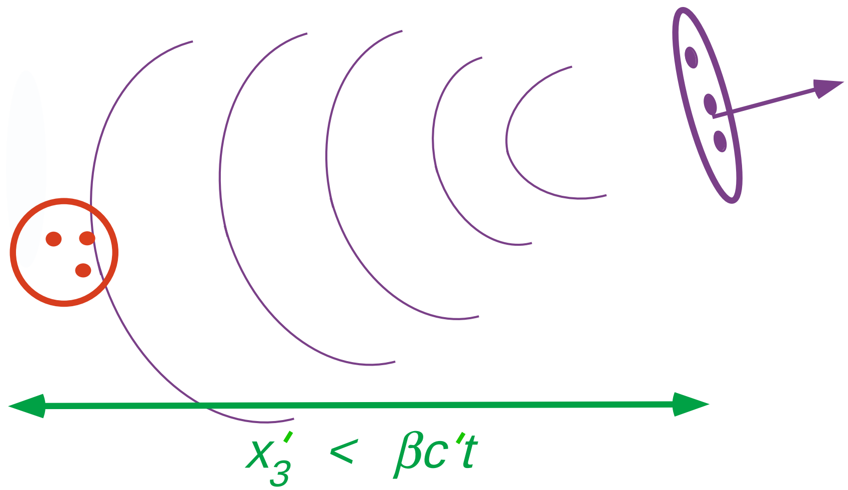
Cancelled if the fields are well-localized $\Leftrightarrow$ σ inclusive

- **Initial-state interactions decouple from hard scattering**

- **Summarized by multiplicative factors: the parton distributions**

- **But what about cross sections where we observe specific particles in the final state? Single hadrons, dihadron correlations, etc? Why does an outgoing hadron only know about a cut vertex?**

- **Much of the same reasoning holds:**



- **Subtle but important difference: Δ changes sign in the final state.**
- **Then the gauge function in $\ln(\Delta)$ gets an imaginary part.**
- $q(x) \Rightarrow q(x)\, e^{i \ln(\Delta)}$ **no longer a pure phase.**
- **Mismatch between initial- and final-state interactions.**
- **Indicates physical effects in the final state.**

- *Still* **cancels at high p_T for single hadrons, but not in general for distributions of momentum pairs.**

  But for single-particle inclusive . . .

  Interactions after the scattering are too late to affect large momentum transfer, creation of heavy particle, etc.

III. Factorization: some recent thoughts

- **The ongoing saga in brief:**

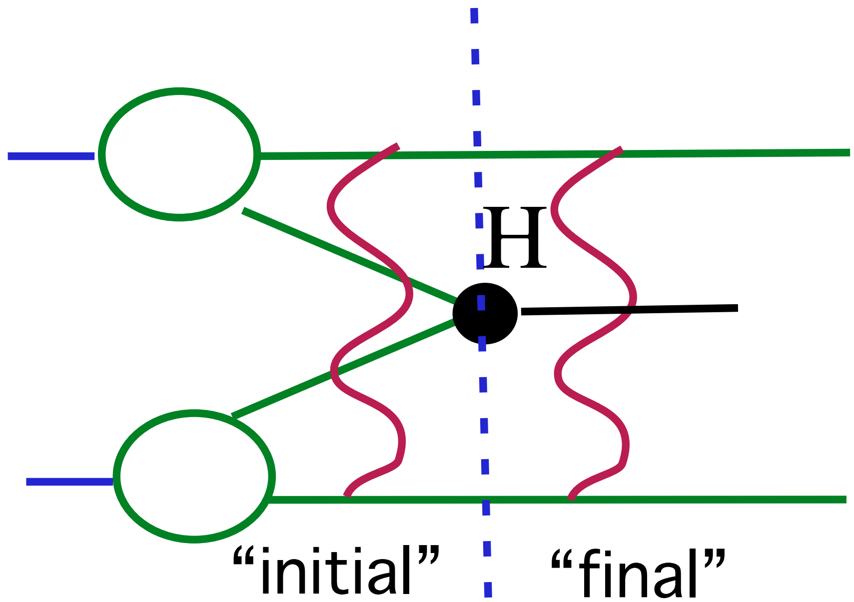
The late seventies: classifying soft and collinear singularities

The mid eighties: cancelling soft gluons

- **Light-cone ordered forms for amplitudes (viz. Koplik and Mueller) H is the hard-scattering – defines initial- and final states.**
- **Separates amplitudes M at fixed transverse momenta into initial and final states ($s_j \equiv \sum_{i \in j} k_{it}^2/2k_i^+$, $\mathcal{E}$ a source of soft gluons):**

$$M(p_t) = \int_q \mathcal{E}_M^{(f')}(\{q_a, q_b\}) \prod_{i<H} \frac{1}{(- \sum_{a \in i} q_a^+ - s_i + i\epsilon)}$$
$$\times \prod_{j>H} \frac{1}{(\sum_{b \in j} q_b^+ + k_{N_T}^+ - s_j + i\epsilon)} .$$

- The big issue: single diagrams in covariant perturbation theory that include linear superpositions of initial- and final-state gluon exchanges ("Glauber" or "Coulomb" gluons).



There is no longer a single "phase" to eliminate both initial- and final-state non-factoring guons.

The essential point: final state singularities cancel in the sum over final states. So it's really an initial-state phase.

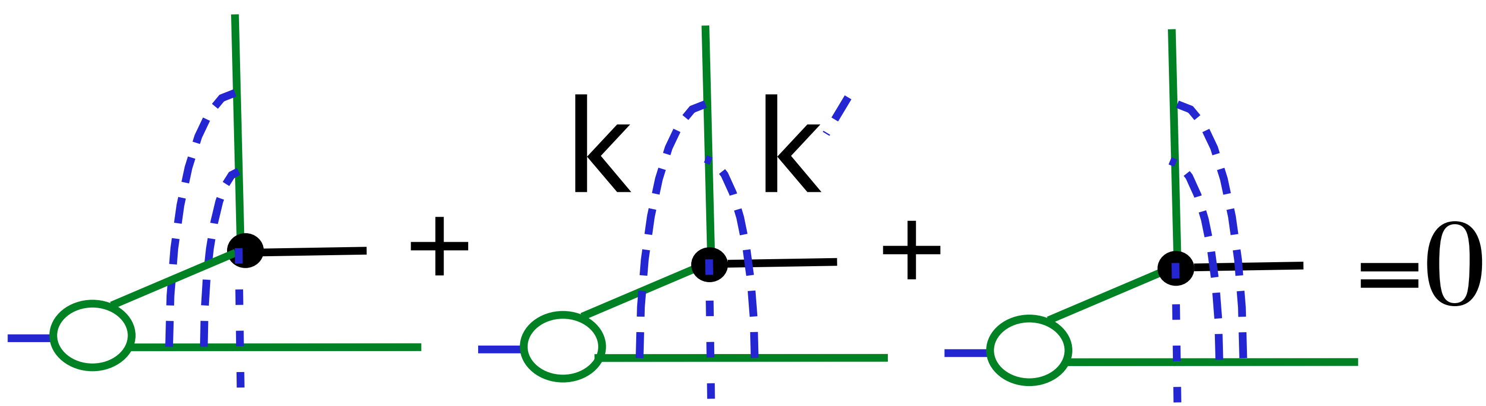
- **And the past few years: revisiting (sometimes rediscovering) the old arguments**
  - **A renewed interest in the context of "soft-colinear effective theories"** [Bauer, Fleming, C. Lee, Rothstein, Stewart et al.]
  - **and extensions to factorization at fixed transverse momentum**
  - **Back to those Glauber/Coulomb gluons.** [S. Mert Aybat and GS] **OK, they cancel, but what do they do before they cancel?**

- **"Causal identity": Makes LC momentum integrals converge**

(no "pseudo-collinear" subtractions necessary)

$$\sum_{a=0}^{n} \prod_{k=1}^{a} \frac{1}{\left(-\sum_{i=1}^{k} k_i^{+}\right)} \prod_{l=a+1}^{n} \frac{1}{\left(\sum_{j=l}^{n} k_j^{+}\right)} = 0\,,$$

LCOPT only for the Green lines; then at fixed k and k':



- **What's left over in the amplitude: phases for each spectator.**

$$\mathcal{M}(p_a,\{p_t\}) = \langle 0|\Phi^{(f')}(0,-\infty)\, C_{f'f}(p_a)\, \Phi^{(f)}(0,-\infty)$$
$$\times \prod_t \int d^2\mathrm{x}_t\, e^{i\mathrm{p}_{t,\perp}\cdot \mathrm{x}_t}\, W_-^{(t)}(\mathrm{x}_t)|0\rangle\, \Psi_f(\{p^-{}_t,\mathrm{x}_t\})\,,$$

- **with $\Psi_f(\{p^-{}_t,\mathrm{x}_t\})$ a light-cone wave function,**
- **in ($\perp$) convolution with Wilson lines,**

$$W_-^{(t)}(\mathrm{x}_t) = \Phi^{(t)}(\infty,\mathrm{x}_t)\Phi^{(t)}(\mathrm{x}_t,-\infty)\,.$$

- **Perhaps a link to dipole-based pictures of hadron-hadron scattering.**

IV. Glancing back (with admiration) and looking forward

- What a turn-out!

- In recognition of a founding role in modern strong interaction physics, seeing us through from Regge to gauge theory and, who knows, to string pictures that combine them,

- and a key role in the evolving reengagment particle and nuclear physics,

- for giving us so many ideas to build on, and for finding depths in our ideas we didn't know were there,

- **for terrific ideas and terrific students that and who just keep coming,**

- **all for as long as I can remember, and then some,**

- **and in tribute to the physics, to the encouragement and the generosity,**

- **and for just generally showing us how to do science with style . . .**

- **Here's a toast to you, Al . . .**

- **Happy birthday! . . . and many, many more . . .**

- **happy occasions for you and Julia . . .**