ADVANCED MATH TUTORIAL

S'Cool LAB Summer Camp 2018

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1 Complex numbers

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i is a solution of the equation $x^2 = -1$. This can be written as follows:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$$

The complex number system can be defined as the algebraic extension of the ordinary real numbers by an imaginary number i, this means that complex numbers can be added, subtracted, and multiplied, as polynomials in the variable i. Furthermore, complex numbers can also be divided by nonzero complex numbers. Overall, the complex number system is a field.

Definition 1.1. A **field** is a set F together with two operations called addition $(+)$ and multiplication $(\cdot)$, required to satisfy the following properties:

1. Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in F$
2. Commutativity of addition and multiplication: $a + b = b + a$ and $a \cdot b = b \cdot a \quad \forall a, b \in F$
3. Additive and multiplicative identity.
   There will exist two different elements in F such as:
   - The additive identity, denoted by 0, will satisfy $a + 0 = a \quad \forall a \in F$
   - The multiplicative identity, denoted by 1, will satisfy $a \cdot 1 = a \quad \forall a \in F$
4. Additive and multiplicative inverses:
   - For every $a \neq 0 \in F$ there exists an element in F, denoted by $-a$, such that $a + (-a) = 0$.
   - For every $a \neq 0 \in F$ there exists an element in F, denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$
5. Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \forall a, b, c \in F$

   Examples of fields are: $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ but not $\mathbb{Z}$.

Theorem 1.1. ***Fundamental Theorem of Algebra***
Every nonconstant polynomial equation having complex coefficients has at least one complex root.

A field with the property that every nonconstant polynomial with coefficients in has a root in is called **algebraically closed**, so the fundamental theorem of algebra states that the complex field is algebraically closed.

The theorem is equivalent to the following statement:
A polynomial $P(x)$ of degree n has n values $x_1, ..., x_n$ for which $P(x_i) = 0, \quad \forall i \in 1, ..., n$. Those values are called **polynomial roots**.

1.1 Representation

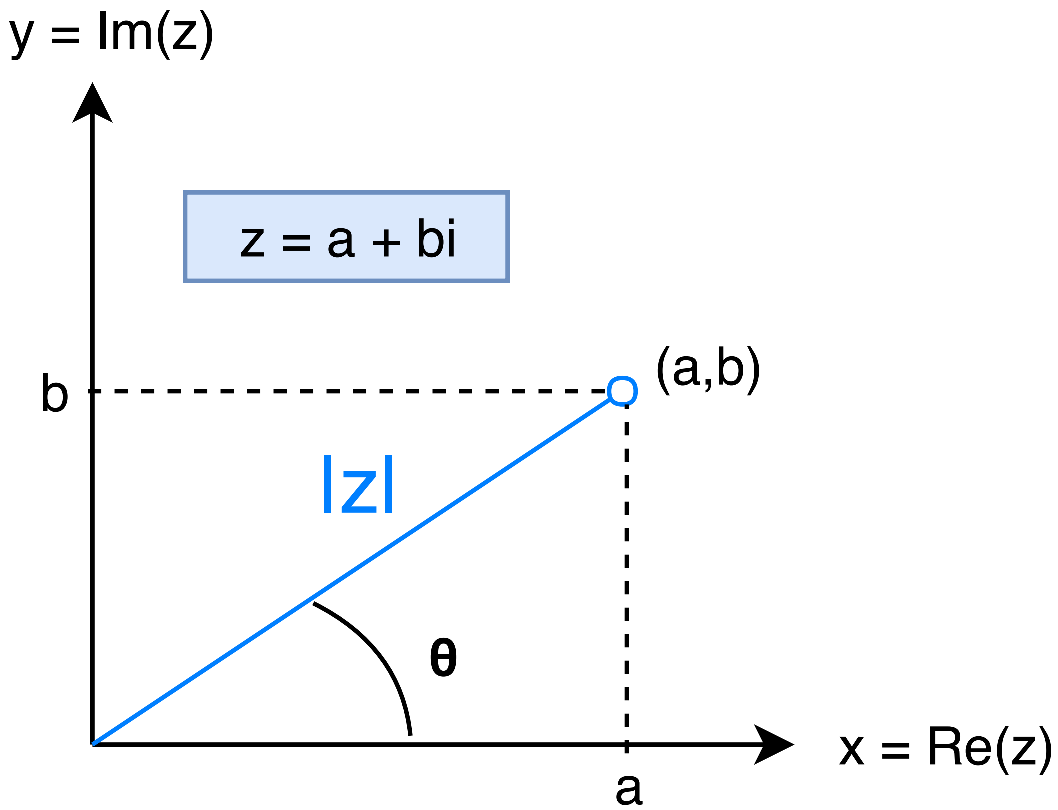
Complex numbers can be expressed using different notations. Some of them are:

- Cartesian form
- Polar form
- Exponential form

1.1.1 Cartesian form

Complex numbers can naturally be represented geometrically by points in the plane: The complex number $z = a + bi$ is represented by the point (a, b) in cartesian coordinates as seen in the image below.

Nevertheless, it is also possible to take into account the vector that can be built from the origin to the point $(a, b), z$. With the representation of the vector, it is easy to deduce that it is possible to represent this numbers by using polar coordinates, defining the angle of rotation θ and the length of the vector $|z|$.


y = Im(z)
z = a + bi
(a,b)
b
|z|
θ
x = Re(z)
a

1.1.2 Polar form

Using our trigonometrical knowledge it is possible to see that we have everything we need in the image above to represent a complex number in polar coordinates. We have the modulus or absolute value:

$$|z| := r = \sqrt{a^2 + b^2}$$

We call θ the polar angle or the argument of $a + bi$. The modulus, or absolute value is determined by $a + bi$ but the polar angle is not since it can be increased by any integer multiple of 2π. To make θ unique, one can specify:

$$0 \leq \theta < 2\pi$$

In this case:

$$\theta = arg(z) = \arctan^{-1}\left(\frac{b}{a}\right)$$

Changing between cartesian and polar representation of complex numbers is essentially the same as changing between cartesian and polar coordinates.

Using the following equations that can be deduced from the expressions above:

$$\cos\theta = \frac{a}{r} \to a = r\cos\theta$$

$$\sin\theta = \frac{b}{r} \to b = r\sin\theta$$

We can now build the polar representation of a complex number:

$$\boxed{z = a + bi = r\cos\theta + r\sin\theta \cdot i = r(\cos\theta + i\sin\theta)}$$

1.2 Euler's formula

If we take a step further, we can still modify this polar representation into a more interesting equation, using the following statement:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \textbf{Euler's formula} \tag{1}$$

The Euler equation should be regarded as the *definition* of the exponential of an imaginary power. To better see how these two entities (the sum of sin and cosine and the exponential) are related, we have to check the infinite Taylor series of the exponential function:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + ...$$

Substituting t by $i\theta$ in the series:

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + ...$$

And remembering that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$ and so on, we get:

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\right) \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

in view of the infinite series representations for $\cos(\theta)$ and $\sin(\theta)$. This is not a proof since we only know that the series expansion for e^t is valid when t is a real number, but it shows that Euler's formula is formally compatible with the series expansions.

Now we can define the exponential representation:

$$z = a + bi = re^{i\theta}$$

2 Differential Equations

2.1 Preliminaires

Definition 2.1. We call a **differential equation** to any equation containing derivatives, either ordinary (one variable) or partial (several variables).

In applications, the functions usually represent physical quantities, the derivatives represent their rates of change and the equation defines a relationship between the two.

Notation. The time derivative notation in physics is usually $\dot{x}$ in addition to the Leibniz's notation: $\frac{dx}{dt}$

Definition 2.2. The **order** of a differential equation is the order of the highest derivative in the equation

Differential equations can be divided into several types. Apart from describing the properties of the equation itself, these classes of differential equations can help inform the choice of approach to a solution. Between all the possible types of differential equations we would like to highlight the following ones to get a general vision of the main types:

1. **Ordinary differential equations vs Partial differential equations**

   Definition 2.3. If a differential equation contains derivatives respect to a unique independent variable, then it is called **ordinary differential equation**

   Example 2.1. These are examples of ordinary differential equations with only derivatives respect to the "x" independent variable.

$$\frac{dy}{dx} - 4y = 2 \tag{2}$$

$$(x + 2y)dx = 3ydy = 0 \tag{3}$$

   Definition 2.4. If a differential equation contains derivatives respect to two or more independent variables then it is called **partial differential equation**

   Example 2.2. These are examples of partial differential equations with derivatives respect to different independent variables.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \tag{4}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial u}{\partial t} \tag{5}$$

2. **Linear differential equations vs Non-linear**

   Definition 2.5. A differential equation is **linear** if it can be written in the following way:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

   This means, if the differential equation is defined by a linear polynomial in the unknown function and its derivatives. Any other equation not satisfying this definition is considered to be **non-linear**

   Example 2.3. Linear:

$$y'' - 2y = ln(x)$$

   Non-linear:

$$3 + yy' = x - y$$

3. **Homogeneous vs Inhomogeneous**

   Definition 2.6. A differential equation is **homogeneous** if in the previous definition, the term $g(x) = 0$. Otherwise, the equation is considered non-homogeneous.

2.2 Solution of a differential equation

To solve a differential equation we should first define what would be a solution and for this purpose we will need some additional definitions.

Definition 2.7. A function f is **continuous** in $x = x_0$ if the three following statements are true:

i) $f(x_0)$ is defined.

ii) $\lim_{x \to x_0} f(x)$ exists.

iii) $f(x_0) = \lim_{x \to x_0} f(x)$

A function f is said to be continuous on an interval I if it is continuous at each point x in I. Some other properties are related to continuity:

- The sum, difference and product of continuous functions is continuous.
- The quotient of continuous functions is continuous at all points where the denominator is not zero.
- The composition of continuous functions is continuous at all points where the composition is properly defined.

Example 2.4. Any polinomial is continuous for all values of x. For instance: $f(x) = x^4 + 7x^2 - 3$. The functions $f(x) = e^x$, $f(x) = \sin(x)$ and $f(x) = \cos(x)$ are continuous in $\mathbb{R}$.

Example 2.5. The function $f(x) = \dfrac{1}{x-1}$ is not continuous in $x = 1$.
The function

$$f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2 - (x-1)^2 & \text{for } x > 1 \end{cases}$$

is not continuous in $x = 1$.

If you are interested, the discontinuities are classificated into several types, but we are not going to review them here.

Definition 2.8. A function of one real variable is **differentiable** if its derivative exists at each point in its domain. As a result, the graph of a differentiable function must have a (non-vertical) tangent line at each point in its domain, be relatively smooth, and cannot contain any breaks, bends, or cusps.

Definition 2.9. In calculus, an antiderivative, primitive function or **primitive integral** of a function f is a differentiable function F whose derivative is equal to the original function f.

Definition 2.10. A **solution of a differential equation** of order n consists of a function defined and n-times differentiable on a domain D having the property that the functional equation obtained by substituting the function and its n derivatives into the differential equation holds for every point in D. This means that, to find the solution of a differential equation, we need to find the primitive of the function by using integrals.

A Very Easy Example. To start easily, let's study the following differential equation:

$$y' = 6x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 6x \implies \mathrm{d}y = 6x\,\mathrm{d}x$$

The order of this equation is one as we only find the first derivative of y in the equation. Then we should find a function defined and at least one time differentiable in $\mathbb{R}$. We apply integrals to both sides of the equality to find this function:

$$\int \mathrm{d}y = \int 6x\,\mathrm{d}x \implies y = \int 6x\,\mathrm{d}x = 6\int x\,\mathrm{d}x$$
$$\implies \boxed{y = 3x^2 + C}$$

As your first exercise, you can verify this is actually a solution of the differential equation presented.

2.3 Explicitly Solvable First Order Differential Equations

In this chapter we will introduce the first order differential equations and several type of it that can be solved explicitly using integration methods covered in standard Calculus course.

2.3.1 Separable equations

If we have the differential equation:

$$g(x) = h(y)y'$$

Formally, we could write:

$$g(x)\,\mathrm{d}x = h(y)\,\mathrm{d}y$$

If we suppose G a primitive of g and H a primitive of h, integrating in the equation above we would have that $\boxed{G(x) = H(y) + C}$ is the general solution of the equation.

Example 2.6.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x)y = 0$$

Rearranging the terms we can see it is a separable equation:

$$\frac{\mathrm{d}y}{y} = -\sin(x)\,\mathrm{d}x$$

So let's apply what we learned!

$$\int \frac{1}{y}\,\mathrm{d}y = -\int \sin x\,\mathrm{d}x \implies \log|y| = \cos x + C$$

$$\implies y = e^{\cos(x)+C} = e^{\cos(x)}e^{C} \implies \boxed{y = Ke^{\cos(x)}\ \forall K \in \mathbb{R}}$$

It is possible that your problem has some initial conditions that helps us to have a particular solution of the equation (These problems are called Initial Value Problem). For example:

Example 2.7. Find the unique solution of the previous differential equation taking into account $y\left(\frac{\pi}{2}\right) = 2$.

$$y = Ke^{\cos(x)} \implies y(0) = Ke^{\cos\left(\frac{\pi}{2}\right)} = 2 \implies K = 2 \implies \boxed{y = 2e^{\cos(x)}}$$

2.3.2 Convertible to separable

There are some equations that seems not separable at first but we can easily convert them to use the same method before. These equations have the following structure:

$$y' = f(ax + by)$$

If $a = 0$ or $b = 0$, then the equation is already separable. Otherwise, we apply the change of variables $z = ax+by$.

Example 2.8. Solve $y' - e^x e^y = -1$

$$y' - e^x e^y = -1 \implies y' + 1 = e^{x+y}$$

Applying as suggested the change of variables $z = x + y \implies y = z - x$ and then $y' = z' - 1$. Substituting in the equation, we transform it and now have a differential equation in z.

$$z' = e^z$$

And now we can solve it using the separable equations method. Solve it as an exercise and check that the general solution should be $\boxed{y = -log(C - x) - x}$

2.3.3 Exact equations

Let $D \subset \mathbb{R}^2$ be a domain and let us consider the following equation:

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$$

where $P, Q : D \to \mathbb{R}$ are given continuous functions and $Q(x, y) \neq 0$. This equation is said to be an **exact differential equation** if there exists a function $F : D \to \mathbb{R}, F \in C^1(D)$ such that:

$$dF(x, y) = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$$

Notice that F is called a *first integral* of the differential equation and we have:

$$\begin{cases} \dfrac{\partial F}{\partial x}(x, y) = P(x, y) \\ \\ \dfrac{\partial F}{\partial y}(x, y) = Q(x, y) \end{cases} \tag{6}$$

In this case, the equation takes the form: $dF(x, y) = 0$. If the function $y(x)$ is a solution then $dF(x, y(x)) = 0$ and consequently $F(x, y(x)) = C,\ C \in \mathbb{R}$.

Because of the nature of the equation, there is a test we can do to any equation to check if it is exact or not. Although here the details are not going to be explained, the following equality should be true: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Example 2.9. Solve $3y + e^x + (3x + \cos y)y' = 0$

Let's find which part would be $P(x, y)$ and which would be $Q(x, y)$. The equation above could be written as:

$$\frac{dy}{dx} = -\frac{3y + e^x}{3x + \cos(y)} \implies P(x, y) = 3y + e^x, Q(x, y) = 3x + \cos(y)$$

Applying our test to check if the equation is exact we see that indeed $P_y = Q_x = 3$, then it is an exact equation. Let's start to compute the solution:

$$\frac{\partial F}{\partial x}(x, y) = 3y + e^x \implies F = \int 3y + e^x dx \implies F = 3xy + e^x + \varphi(y)$$

Where $\varphi(y)$ is a constant in the previous integral. At the same time, if we derive our "proposed" F respect to y, it should be equal to $Q(x, y)$, thus:

$$\frac{\partial F}{\partial y}(x, y) = 3x + \varphi'(y) = 3x + \cos y = Q(x, y) \implies \varphi'(y) = \cos y \implies \varphi(y) = \sin y$$

Then, we have our function F, and then our general solution:

$$F(x, y) = \boxed{3xy + e^x + \sin y = C}$$

2.3.4 Reductible to exact: Integrating factors

If $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$ is not exact, we can still find a function $\mu(x, y)$ such as:

$$\mu(x, y)P(x, y)\,\mathrm{d}x + \mu(x, y)Q(x, y)\,\mathrm{d}y = 0$$

is exact.

Let's imagine that $\mu(x, y)$ only depends on variable x, this is $\mu(x, y) = \mu(x)$. To be exact, the equation should verify:

$$\frac{\partial}{\partial y}(\mu(x)P(x, y)) = \frac{\partial}{\partial x}(\mu(x)Q(x, y)) \implies \mu(x)P_y(x, y) = \mu'(x)Q(x, y) + \mu(x)Q_x(x, y)$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{P_y(x,y) - Q_x(x,y)}{Q(x,y)} = h(x)$$

Then, the integrating factor would be:

$$\mu(x) = e^{\int h(x)dx}$$

The same argumentation could be followed in case of an integrating factor $\mu(y)$ or $\mu(xy), \mu(x+y)$, etc. Precisely, in the case the integrating factor only depends on y, it will be:

$$\mu(y) = e^{\int h(y)dy}, \quad h(y) = \frac{Q_x - P_y}{P}$$

Example 2.10. Solve $(2x^2 + y)\,\mathrm{d}x + (x^2y - x)\,\mathrm{d}y = 0$

As it can be seen:

$$P_y = 1 \neq 2xy - 1 = Q_x$$

Let's try to find an integrating factor $\mu(x)$:

$$\mu(x) = e^{\int h(x)}, \quad h(x) = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2}{x}$$

$$\mu(x) = \frac{1}{x^2}$$

Then, as an exercise for the lector, it is possible to verify the fact that multiplying by the integrating factor the new equation is exact.

2.3.5 First order linear equations

The equation:

$$y' + a(x)y = b(x)$$

can be solved using three different methods:

(i) Find an integrating factor of the form $\mu(x)$. By writing the equation above in the following manner:

$$(a(x)y - b(x))\,\mathrm{d}x + \mathrm{d}y = 0$$

then we can denote $P(x,y) = a(x)y - b(x)$ and $Q(x,y) = 1$, with $a(x) = \dfrac{P_y - Q_x}{Q}$ and integrating factor $\mu(x) = \exp\left(\int a(x)\,\mathrm{d}x\right)$.

(ii) The second method to resolve this equation would be solving first the lineal homogeneous equation associated ($b(x) = 0$). This is a separable equation as we can write in an equivalent way:

$$y' + a(x)y = 0 \implies \frac{\mathrm{d}y}{y} = -a(x)\,\mathrm{d}x \implies y = C\exp\left(-\int (a(x)\,\mathrm{d}x\right)$$

Then we can apply the variation of constants method to find the solution to the non-homogeneous equation.

The variation of constants method: Substitute C by $C(x)$ in the general solution and then apply this solution into the differential equation. This leads us to have:

$$C(x) = \int b(x)\exp\left(\int (a(x)\,\mathrm{d}x\right) + C$$

(iii) The third procedure would be to use a particular solution of the differential equation $y_p(x)$ we have already found. Then, the general solution will be:

$$y = y_p + C\exp\left(-\int a(x)\,\mathrm{d}x\right)$$

The proof of this result will be left as an exercise for the reader.

(iv) The last possible method to solve this kind of equations would be to perform a decomposition $y(x) = u(x)v(x)$. Then when derivating: $y' = u'v + uv'$, and if we substitute that into the equation:

$$u'v + uv' + a(x)uv = b(x) \implies u'v + (v' + a(x)v)u = b(x)$$

Then we are going to choose $v(x)$ such as the coefficient of u is zero, this means that we are going to resolve $v' + a(x)v = 0$, a separable equation with solution:

$$v(x) = exp\left(-\int a(x)dx\right)$$

We already have $v(x)$, now we have to find $u(x)$ using $u'v = b(x)$ and this leads us to:

$$u(x) = \int b(x)\exp\left(\int a(x)\,\mathrm{d}x\right)\mathrm{d}x + C$$

All these methods are going to give us the same solution for the same equation, so you can choose the one you like the most. The best thing is that you don't have to memorize the formula, it is enough to remember which approach to use.

Exercise 2.1. Solve $2xy' - 3y = 4x^2$ using the four methods above and verify you get the same solution in all of them. (Solution:$y = Cx^{3/2} + 4x^2$)