Continuity (exercises with detailed solutions)

1. Verify that $f(x) = \sqrt{x}$ is continuous at x_0 for every $x_0 \geq 0$.
2. Verify that $f(x) = \dfrac{1}{x} - \dfrac{1}{x_0}$ is continuous at x_0 for every $x_0 \neq 0$.
3. Draw the graph and study the discontinuity points of $f(x) = [\sin x]$.
4. Draw the graph and study the discontinuity points of $f(x) = \sin x - [\sin x]$.
5. Draw the graph and study the discontinuity points of $f(x) = \dfrac{2x^2 - 5x - 3}{x^2 - 4x + 3}$.
6. Draw the graph and study the discontinuity points of $f(x) = \dfrac{x+3}{3x^2 + x^3}$.
7. Find $k \in \mathbb{R}$ such that the function
$$f(x) = \begin{cases} 2x^2 + 4x, & \text{if } x \geq 1 \\ -x + k, & \text{if } x < 1 \end{cases}$$
is continuous on $\mathbb{R}$.
8. Find $a, b \in \mathbb{R}$ such that the function
$$f(x) = \begin{cases} \log(1+x), & \text{if } -1 < x \leq 0 \\ a \sin x + b \cos x & \text{if } 0 < x < \frac{\pi}{2} \\ x & \text{if } x \geq \frac{\pi}{2} \end{cases}$$
is continuous on its domain.
9. Determine the domain and study the continuity of the function $f(x) = \dfrac{\log(1+x^2)}{\sqrt{3 - \sin x}}$.
10. Draw the graph and study the continuity of the function
$$f(x) = \begin{cases} x \left[\dfrac{1}{x}\right], & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$
11. Draw the graph and study the continuity of the function
$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Solutions

1. In order to verify that $f(x) = \sqrt{x}$ is continuous at x_0, with $x_0 \geq 0$, we try to find an upper bound for $f(x)$ dependent on the difference $x - x_0$. We obtain
$$\sqrt{x} - \sqrt{x_0} = \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} - \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} = \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}.$$
Since $\sqrt{x} \geq 0$ for every $x \geq 0$ we have
$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}}.$$
We now fix $\varepsilon > 0$, and we want to determine $\delta > 0$, such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. From the previous inequality we have that we must find $\delta > 0$, such that if $|x - x_0| < \delta$ then
$$\frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon.$$
The last inequality is equivalent to $|x - x_0| < \sqrt{x_0}\varepsilon$, hence we choose $\delta \leq \sqrt{x_0}\varepsilon$.

2. As in exercise 1 we have to find an upper bound for $f(x) - f(x_0)$, dependent on the difference $x - x_0$ or with a function of $x - x_0$. We have
$$\frac{1}{x} - \frac{1}{x_0} = \frac{x_0 - x}{x x_0}.$$
If $x_0 > 0$ (when $x_0 < 0$ we proceed in the same way), then for every $x \in I =]x_0/2, 3/2x_0[$ we have
$$x \cdot x_0 > \frac{x_0}{2} \cdot x_0 = \frac{x_0^2}{2.} \quad \Rightarrow \quad \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{x x_0} < 2\frac{|x_0 - x|}{x_0^2}.$$
Hence, fixed $\varepsilon > 0$, if we find $\delta > 0$ such that $|x - x_0| < \delta$ implies $2\frac{|x_0 - x|}{x_0^2} < \varepsilon$, we have finished. This condition is equivalent to $|x - x_0| < \varepsilon\frac{x_0^2}{2}$, and the last inequality is satisfied for every $x \in I$ if we take $\delta \leq \min\{\varepsilon\frac{x_0^2}{2}, \frac{x_0}{2}\}$.

3. Since $\sin x$ is 2π-periodic, f is also 2π-periodic. We then study f only on the interval $[-\pi, \pi]$. Since $[n] = n$ for every $n \in \mathbb{Z}$, then $f(x) = \sin x$ when $x = -\pi, -\pi/2, 0, \pi/2, \pi$. Furthermore $[y] = 0$ for every $y \in [0, 1[$, hence $f(x) = 0$ for every x such that $\sin x \in [0, 1[$, that is for every $x \in [0, \pi] \setminus \{\pi/2\}$.

   Similarly, since $[y] = -1$ for every $y \in [-1, 0[$, we have $f(x) = -1$ for every x such that $\sin x \in [-1, 0[$, that is for every $x \in]-\pi, 0[$.

   We can then draw the graph of f. At $\pm\pi$ and 0 has a discontinuity f of the first kind, indeed
$$\lim_{x\to\pm\pi^-} f(x) = 0, \qquad \lim_{x\to\pm\pi^+} f(x) = -1, \qquad \lim_{x\to 0^-} f(x) = -1, \qquad \lim_{x\to 0^+} f(x) = 0,$$
At $x_0 = \pi/2$ f we have
$$\lim_{x\to\frac{\pi}{2}} f(x) = 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = 1$$
hence we can extend f at $\pi/2$ to a continuous function.

4. f is 2π-periodic and we study it on $[-\pi, \pi]$. To draw its graph we observe that $f(n) = 0$, $\forall n \in \mathbb{Z}$, hence $f(x) = 0$ for every x such that $\sin x \in \mathbb{Z}$, that is when $x = -\pi, -\pi/2, 0, \pi/2, \pi$. Furthermore, since if $y \in]0, 1[$ we have $y - [y] = y$, then for every $x \in]0, \pi[\setminus\{\pi/2\}$, we have $f(x) = \sin x$. Since if $y \in]-1, 0[$ we have $y - [y] = y + 1$, for every $x \in]\pi, 0[\setminus\{-\pi/2\}$, we have $f(x) = \sin x + 1$. Hence in $x = \pi, 0, \pi/2, \pi$ f has a discontinuity of the first kind, indeed
$$\lim_{x\to\pm\pi^-} f(x) = 0, \qquad \lim_{x\to\pm\pi^+} f(x) = 1, \qquad \lim_{x\to 0^-} f(x) = 1, \qquad \lim_{x\to 0^+} f(x) = 0.$$

At $x = \frac{\pi}{2}$, f can be extended to a continuous function since

$$\lim_{x\to\frac{\pi}{2}} f(x) = 1, \qquad \text{and} \qquad f\left(\frac{\pi}{2}\right) = 0.$$

5. $\text{dom}(f) = \mathbb{R}\setminus\{1,3\}$. Since the numerator vanishes when $x = 3$ we can simplify the fraction to obtain, for every $x \in \mathbb{R}\setminus\{1,3\}$

$$f(x) = \frac{(x-3)(2x+1)}{(x-3)(x-1)} = \frac{2x+1}{x-1} = 2 + \frac{3}{x-1}.$$

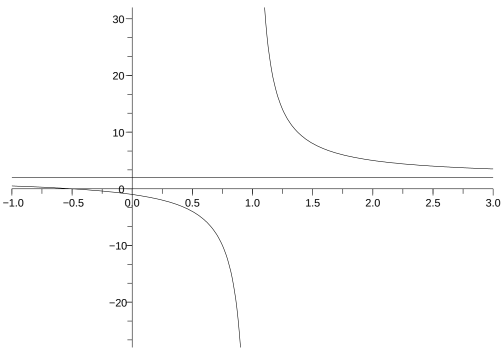
The graph of f can be obtained from the graph of $g(x) = 1/x$ with some translations and rescaling. At $x = 3$ we can extend f to a continuous function, indeed $3 \notin \text{dom}(f)$ but

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\left(2 + \frac{3}{x-1}\right) = \frac{7}{2}.$$

When $x = 1$, we have

$$\lim_{x\to 1^-} f(x) = -\infty, \qquad \lim_{x\to 1^+} f(x) = +\infty.$$

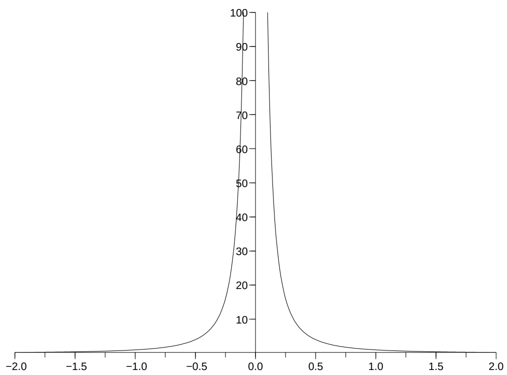
Thus $x = 1$ is a discontinuity point of the second kind.



6. $\text{dom}(f) = \mathbb{R}\setminus\{-3,0\}$, and for every $x \in \text{dom}(f)$ we have $f(x) = 1/x^2$. Hence we have

$$\lim_{x\to -3} f(x) = \frac{1}{9}, \quad \lim_{x\to 0} f(x) = +\infty.$$

we can extend f in $x = -3$ to a continuous function; $x = 0$ is a discontinuity point of the second kind.



7. f is continuous for every $x \neq 1$, since it is a composition of continuous functions. Hence we just study the continuity of f in $x = 1$. f is continuous in $x = 1$ if both limits

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(-x+k) = k-1, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x^2+4x) = 6$$

are equal to $f(1) = 6$. We then impose $k - 1 = 6$ that is $k = 7$.

8. $\text{dom}(f) =]-1,+\infty[$. Furthermore on $]-1,0[$, $]0,\frac{\pi}{2}[$, $]\frac{\pi}{2},+\infty[$ the function $f(x)$ is continuous because it is a composition of continuous functions. We then study the continuity of f at $x = 0$ and $x = \frac{\pi}{2}$.
We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\log(1+x) = 0, \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(a\sin x + b\cos x) = b.$$

Hence f is continuous at 0 if and only if $b = 0$. Furthermore

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-}(a\sin x + b\cos x) = a, \qquad \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} x = \frac{\pi}{2},$$

hence f is continuous in $x = \pi/2$ if and only if $a = \pi/2$.

9. $\mathrm{dom}(f) = \mathbb{R}$, indeed for every $x \in \mathbb{R}$ we have $1 + x^2 \geq 1 > 0$ and $3 - \sin x \geq 2 > 0$. For every $x \in \mathbb{R}$ f is continuous since it is a composition of continuous functions.

10. f is not continuous when $x = 1/n$, for every $n \in \mathbb{Z} \setminus \{0\}$. These points are discontinuities of the first kind. When $x \neq 1/n$, f is continuous.

11. f is continuous when $x \neq 0$; at 0 we have

$$\lim_{x \to 0} f(x) = 0, \qquad f(0) = 1.$$

hence we can extend f to a continuous function on the whole $\mathbb{R}$.