Two new approaches of estimating the polarization in high energy electron storage rings

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Issue: Is polarization possible in FCC-ee and CEPC?

Topic of talk: Three mathematical models and two new approaches to estimate the polarization in high energy electron storage rings

Polarization is viewed as a balance of three factors:

1. Sokolov-Ternov effect + Baier-Katkov correction
2. Spin diffusion
3. Kinetic polarization effect

These three factors are modeled mathematically in three ways:

1. Derbenev-Kondratenko formulas
2. Bloch equation for polarization density
3. Stochastic differential equations (SDEs) for orbit and spin

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We believe that the Derbenev-Kondratenko formula model approximates the two other models.

Software for Derbenev-Kondratenko formulas well-developed since A. Chao’s pioneering work 40 years ago.
  - See E. Gianfelice-Wendt’s talk at this conference.

Software for Bloch equation under development at UNM
  1. Applying method of averaging to Bloch equation to obtain effective Bloch equation
  2. Numerically solving effective Bloch equation via spectral phase-space discretization and time stepping

Software for SDEs under development at UNM
  1. Integration of SDEs
  2. Stochastic Collocation
Polarization vector of bunch at accelerator azimuth $\theta$:

$$\vec{P}(\theta) = P_{\text{DK}}(\theta)\langle \vec{n} \rangle(\theta)$$  \hspace{1cm} (1)

where:

1. $P_{\text{DK}}(\theta) = P_{\text{DK}}(+\infty)(1-e^{-\theta/\tau_{\text{DK}}}) + P_{\text{DK}}(0)e^{-\theta/\tau_{\text{DK}}}$
2. $\tau_{\text{DK}}, P_{\text{DK}}(+\infty)$ given by Derbenev-Kondratenko-formulas
3. $\langle \vec{n} \rangle(\theta) \equiv$ phase space average of invariant spin field $\vec{n}(\theta, y)$

All three effects taken into account by the parameters $\tau_{\text{DK}}$ and $P_{\text{DK}}(+\infty)$

The depolarizing part of $\tau_{\text{DK}}$ is often computed via Monte-Carlo simulation as for example in SITROS or SLICKTRACK.

Unresolved spin resonance issues via correction terms to $\tau_{\text{DK}}$ and $P_{\text{DK}}(+\infty)$ \(^5\)

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Polarization vector of bunch at time $t$:

$$\vec{P}(t) \equiv \int dz \vec{\eta}(t, z)$$  \hspace{1cm} (2)$$

where:

- $\vec{\eta}(t, z) \equiv$ polarization density of bunch $\propto$ spin angular momentum density
- $z \equiv (\vec{r}, \vec{p})$

Key facts:
- No differential equation for $\vec{P}(t)$
- But differential equation for $\vec{\eta}(t, z)$ namely Bloch equation
- Fokker-Planck equation for orbital phase space density $f$:
  \[
  \partial_t f = L_{FP}(t, z)f. \tag{3}
  \]

- Bloch equation for polarization density:
  \[
  \partial_t \vec{\eta} = L_{FP}(t, z)\vec{\eta} + [\Omega(t, z) + \Lambda(t, z)]\vec{\eta} \\
  + [1 + \nabla_p \cdot \vec{p}] \lambda(t, z) \frac{1}{m\gamma} \frac{\vec{p} \times \dot{\vec{v}}}{|\dot{\vec{v}}|} f(t, z), \tag{4}
  \]

1. $L_{FP}(t, z) := -\nabla_{\vec{r}} \cdot \frac{1}{m\gamma} \vec{p} - \nabla_{\vec{p}} \cdot [e\vec{E}(t, \vec{r}) + \frac{e}{m\gamma} (\vec{p} \times \vec{B}(t, \vec{r})) \\
  + \vec{F}_{rad}(t, z) + \vec{Q}_{rad}(t, z)] + \frac{1}{2} \sum_{i,j=1}^{3} \partial_{p_i} \partial_{p_j} \mathcal{E}_{ij}(t, z)$

2. $\Lambda(t, z) := -\lambda(t, z) \frac{5\sqrt{3}}{8} [I_{3 \times 3} - \frac{2}{9m^2\gamma^2} \vec{p} \vec{p}^T]$

3. $\lambda(t, z) := \hbar \frac{|e|^5}{m^8\gamma} |\vec{p} \times \vec{B}(t, \vec{r})|^3$
\[
\frac{d\vec{r}}{dt} = \frac{1}{m\gamma} \vec{p}, \quad \frac{d\vec{p}}{dt} = e\vec{E}(t, \vec{r}) + \frac{e}{m\gamma} (\vec{p} \times \vec{B}(t, \vec{r})) + \vec{F}_{rad}(t, z) + \vec{Q}_{rad}(t, z) + \vec{B}^{orb}(t, z) \xi(t),
\]

where \(\xi(t)\) = white noise and \(z \equiv (\vec{r}, \vec{p})\)

\[
\frac{d\vec{s}}{dt} = [\Omega(t, z) + \Lambda(t, z)]\vec{s} + \vec{D}^{spin}(t, z) + \vec{B}^{kin}(t, z) \xi(t)
\]

Length of \(\vec{s}\) not conserved \(\implies\) \(\vec{s}\) not spin vector

However \(\vec{s}\) is local ensemble average of spin vector \(\implies\)

\[
\vec{P}(t) \equiv \langle \vec{s}(t) \rangle
\]

Main insight: Newly discovered (6) \(^6\) is BKS equation \(^7\) plus noise -


Note:

\[
\vec{D}^{spin}(t, z) := \frac{1}{m\gamma} \lambda(t, z) \frac{\vec{p} \times \dot{\vec{v}}}{|\dot{\vec{v}}|}, \tag{8}
\]

\[
\vec{B}^{kin}(t, z) := -\frac{1}{m\gamma} \frac{\vec{p} \times \dot{\vec{v}}}{|\dot{\vec{v}}|} \sqrt{\frac{24\sqrt{3}}{55}} \lambda(t, z). \tag{9}
\]
Equivalence of SDEs and Bloch equation

- F-P equation for the \((z, \vec{s})\) process evolves (joint) probability density \(W = W(t, z, \vec{s})\)
- \(W = W(t, z, \vec{s})\) is related to \(f\) and \(\vec{\eta}\) via

\[
f(t, z) = \int d\vec{s} W(t, z, \vec{s}) , \quad \vec{\eta}(t, z) = \int_{\mathbb{R}^3} d\vec{s} \vec{s} W(t, z, \vec{s})
\]

\((10)\)

Polarization vector is expectation value of \(\vec{s} \Rightarrow \)

\[
\vec{P}(t) \equiv < \vec{s}(t) > \equiv \int dr dp d\vec{s} \vec{s} W(t, z, \vec{s})
\]

\((11)\)

\(\Rightarrow\) local polarization vector \(\vec{\eta}_f = \text{conditional expectation of } \vec{s} \text{ given } z\)

- F-P equation for \(W \Rightarrow f\) and \(\vec{\eta}\) evolve according to orbital F-P equation and Bloch equation
- SDE model intuitively simpler and easier to analyze than Bloch equation model
Transform lab frame SDEs to beam frame SDEs and obtain beam frame Bloch equation.

6D beam frame Bloch equation too difficult for numerics.

Apply method of averaging to beam frame SDEs treating synchrotron radiation and spin-orbit coupling from $\Omega$ as perturbation.

$\Rightarrow$ Effective Bloch equation numerically solved via spectral phase-space discretization and time stepping.\(^8\)

Future work

- Algorithm for Bloch equation model
  1. Numerically solving effective Bloch equation
  2. Alternative approaches: Machine Learning or Gram-Charlier Method

- Algorithms for SDEs
  1. Integration of SDEs
  2. Stochastic Collocation

- Implementing algorithms into Bmad \(^9\)

- Issue to be resolved: Do the Bloch equation and SDE models give insights into the polarization of FCC-ee and CEPC which are missing in the Derbenev-Kondratenko formula model?

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