Two new approaches of estimating the polarization in high energy electron storage rings [1]

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Outline-1

- Issue: Is polarization possible in FCC-ee and CEPC?
- Topic of talk: Three mathematical models and two new approaches to estimate the polarization in high energy electron storage rings
- Polarization is viewed as a balance of three factors:
  1. Sokolov-Ternov effect + Baier-Katkov correction
  2. Spin diffusion
  3. Kinetic polarization effect
- These three factors are modeled mathematically in three ways:
  1. Derbenev-Kondratenko formulas [2]
  2. Bloch equation for polarization density [3]
  3. Stochastic differential equations (SDEs) for orbit and spin [4]

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[2] Ya.S. Derbenev, A.M. Kondratenko, *Sov. Phys. JETP*, vol. 37, p. 968, 1973.

[3] Ya.S. Derbenev, A.M. Kondratenko, *Sov. Phys. Dokl.*, vol. 19, p. 438, 1975.

[4] K. Heinemann et al. Invited paper for ICAP18. To be published in Int. J. Mod. Phys. A, 2019.

Outline-2

- We believe that the Derbenev-Kondratenko formula model approximates the two other models
- Software for Derbenev-Kondratenko formulas well-developed since A.Chao's pioneering work 40 years ago
  - See E. Gianfelice-Wendt's talk at this conference.
- Software for Bloch equation under development at UNM
  1. Applying method of averaging to Bloch equation to obtain effective Bloch equation
  2. Numerically solving effective Bloch equation via spectral phase-space discretization and time stepping
- Software for SDEs under development at UNM
  1. Integration of SDEs
  2. Stochastic Collocation

Derbenev-Kondratenko-formula model

- Polarization vector of bunch at accelerator azimuth θ:

$$\vec{P}(\theta) = P_{\mathrm{DK}}(\theta)\langle\vec{n}\rangle(\theta) \tag{1}$$

where:

1. $P_{\mathrm{DK}}(\theta) = P_{\mathrm{DK}}(+\infty)(1 - e^{-\theta/\tau_{\mathrm{DK}}}) + P_{\mathrm{DK}}(0)e^{-\theta/\tau_{\mathrm{DK}}}$
2. $\tau_{\mathrm{DK}}, P_{\mathrm{DK}}(+\infty)$ given by Derbenev-Kondratenko-formulas
3. $\langle\vec{n}\rangle(\theta) \equiv$ phase space average of invariant spin field $\vec{n}(\theta, y)$

- All three effects taken into account by the parameters τ_{DK} and $P_{\mathrm{DK}}(+\infty)$
- The depolarizing part of τ_{DK} is often computed via Monte-Carlo simulation as for example in SITROS or SLICKTRACK.
- Unresolved spin resonance issues via correction terms to τ_{DK} and $P_{\mathrm{DK}}(+\infty)$ [5]

[5] See, e.g., Z. Duan, M. Bai, D.P. Barber, Q. Qin, *A Monte-Carlo simulation of the equilibrium beam polarization in ultra-high energy electron (positron) storage rings*, Nucl. Instr. Meth. A793 (2015), pp.81-91. Available also at arXiv.

Bloch equation in lab frame-1

- Polarization vector of bunch at time t:

$$\vec{P}(t) \equiv \int dz \vec{\eta}(t,z) \tag{2}$$

where:

  - $\vec{\eta}(t,z) \equiv$ polarization density of bunch $\propto$ spin angular momentum density
  - $z \equiv (\vec{r}, \vec{p})$

- Key facts:
  - No differential equation for $\vec{P}(t)$
  - But differential equation for $\vec{\eta}(t,z)$ namely Bloch equation

Bloch equation in lab frame-2

- Fokker-Planck equation for orbital phase space density f:

$$\partial_t f = L_{FP}(t,z) f \ . \tag{3}$$

- Bloch equation for polarization density:

$$\partial_t \vec{\eta} = L_{\mathrm{FP}}(t,z)\vec{\eta} + [\Omega(t,z) + \Lambda(t,z)]\vec{\eta} \\ + [1 + \nabla_{\vec{p}} \cdot \vec{p}]\lambda(t,z)\frac{1}{m\gamma}\frac{\vec{p}\times\dot{\vec{v}}}{|\dot{\vec{v}}|} f(t,z) \ , \tag{4}$$

1. $L_{FP}(t,z) := -\nabla_{\vec{r}} \cdot \frac{1}{m\gamma}\vec{p} - \nabla_{\vec{p}} \cdot [e\vec{E}(t,\vec{r}) + \frac{e}{m\gamma}(\vec{p}\times\vec{B}(t,\vec{r})) + \vec{F}_{rad}(t,z) + \vec{Q}_{rad}(t,z)] + \frac{1}{2}\sum_{i,j=1}^{3} \partial_{p_i}\partial_{p_j}\mathcal{E}_{ij}(t,z)$
2. $\Lambda(t,z) := -\lambda(t,z)\frac{5\sqrt{3}}{8}[I_{3\times 3} - \frac{2}{9m^2\gamma^2}\vec{p}\vec{p}^T]$
3. $\lambda(t,z) := \hbar\frac{|e|^5}{m^8\gamma}|\vec{p}\times\vec{B}(t,\vec{r})|^3$

SDEs in lab frame-1

- $$\frac{d\vec{r}}{dt} = \frac{1}{m\gamma}\vec{p}\,, \quad \frac{d\vec{p}}{dt} = e\vec{E}(t,\vec{r}) + \frac{e}{m\gamma}(\vec{p}\times\vec{B}(t,\vec{r})) + \vec{F}_{rad}(t,z) + \vec{Q}_{rad}(t,z) + \vec{\mathcal{B}}^{orb}(t,z)\xi(t)\ , \tag{5}$$

  where $\xi(t)$ = white noise and $z \equiv (\vec{r},\vec{p})$
- $$\frac{d\vec{s}}{dt} = [\Omega(t,z) + \Lambda(t,z)]\vec{s} + \vec{\mathcal{D}}^{spin}(t,z) + \vec{\mathcal{B}}^{kin}(t,z)\xi(t) \tag{6}$$
- Length of $\vec{s}$ not conserved $\Longrightarrow$ $\vec{s}$ not spin vector
- However $\vec{s}$ is local ensemble average of spin vector $\Longrightarrow$

  $$\vec{P}(t) \equiv < \vec{s}(t) > \tag{7}$$

- Main insight: Newly discovered (6) [6] is BKS equation [7] plus noise - a noisy damped and driven oscillator

[6] K. Heinemann et al. Invited paper for ICAP18. To be published in Int. J. Mod. Phys. A, 2019.

[7] V.N. Baier, V.M. Katkov, V.M. Strakhovenko, *Sov. Phys. JETP*, vol. 31, p. 908, 1970.

SDEs in lab frame-2

- Note:

$$\vec{\mathcal{D}}^{spin}(t,z) := \frac{1}{m\gamma}\lambda(t,z)\frac{\vec{p}\times\dot{\vec{v}}}{|\dot{\vec{v}}|}\ , \tag{8}$$

$$\vec{\mathcal{B}}^{kin}(t,z) := -\frac{1}{m\gamma}\frac{\vec{p}\times\dot{\vec{v}}}{|\dot{\vec{v}}|}\sqrt{\frac{24\sqrt{3}}{55}\lambda(t,z)}\ . \tag{9}$$

Equivalence of SDEs and Bloch equation

- F-P equation for the $(z, \vec{s})$ process evolves (joint) probability density $W = W(t, z, \vec{s})$
- $W = W(t, z, \vec{s})$ is related to f and $\vec{\eta}$ via

$$f(t,z) = \int d\vec{s}\, W(t,z,\vec{s})\ , \quad \vec{\eta}(t,z) = \int_{\mathbb{R}^3} d\vec{s}\, \vec{s}\, W(t,z,\vec{s}) \tag{10}$$

Polarization vector is expectation value of $\vec{s} \Longrightarrow$

$$\vec{P}(t) \equiv < \vec{s}(t) > \equiv \int drdpd\vec{s}\, \vec{s}\, W(t,z,\vec{s}) \tag{11}$$

$\Longrightarrow$ local polarization vector $\frac{\vec{\eta}}{f} =$ conditional expectation of $\vec{s}$ given z

- F-P equation for $W \Longrightarrow f$ and $\vec{\eta}$ evolve according to orbital F-P equation and Bloch equation
- SDE model intuitively simpler and easier to analyze than Bloch equation model

Effective Bloch equation in beam frame

- Transform lab frame SDEs to beam frame SDEs and obtain beam frame Bloch equation
- 6D beam frame Bloch equation too difficult for numerics
- Apply method of averaging to beam frame SDEs treating synchrotron radiation and spin-orbit coupling from Ω as perturbation
- $\Longrightarrow$ Effective Bloch equation numerically solved via spectral phase-space discretization and time stepping [8]

[8] K. Heinemann et al. Invited paper for ICAP18. To be published in Int. J. Mod. Phys. A, 2019.

Future work

- Algorithm for Bloch equation model
  1. Numerically solving effective Bloch equation
  2. Alternative approaches: Machine Learning or Gram-Charlier Method
- Algorithms for SDEs
  1. Integration of SDEs
  2. Stochastic Collocation
- Implementing algorithms into Bmad [9]
- Issue to be resolved: Do the Bloch equation and SDE models give insights into the polarization of FCC-ee and CEPC which are missing in the Derbenev-Kondratenko formula model?

[9] D. Sagan, *Bmad, a subroutine library for relativistic charged-particle dynamics*. See: `https://www.classe.cornell.edu/bmad`