The $e^+e^- \to WW$ process: QED exponentiation

S. Jadach, W. Płaczek, M. Skrzypek

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Introduction

- Exponentiation of QED radiation and interferences from unstable intermediate particles is regarded as an unsolved problem. It goes beyond Yennie-Frautchie-Suura 1961 scheme for stable external particles.
- But do off-shell internal particles radiate infrared photons?? Technically no, but they are quasi-stable. There is a clear time-space separation between production and decay.
- For $\sum k^0 < \Gamma$ we have normal YFS61 behaviour – internal radiation is suppressed. For $\sum k^0 > \Gamma$ Breit-Wigners should start to feel the real and virtual radiation (recoil).

FCC-based motivation: *W* mass

- At LEP2 *W* mass was measured from direct reconstruction.
- At FCCee the threshold scan is preferred.
- The threshold cross section at LEP2 was measured with the precision 0.5% – 2%.
- At FCCee we expect 3×10^7 events which gives statistical error for cross section of the order of 0.02% i.e. $\Delta M_W = 0.3$ MeV

Missing factor of 100 in precision requires new calculations

Precision calculations for WW cross section

At LEP2 a pragmatic/hybrid approach was used:

- complete four-fermion $e^+e^- \to 4f$ was used at the Born level and
- $\mathcal{O}(\alpha)$ corrections were calculated only for the doubly resonant process $e^+e^- \to WW \to 4f$. (two independent calculations implemented in `KoralW+YFSWW3` and `RacoonWW` Monte Carlo codes).

At FCCee similar pragmatic approach but one perturbative order higher should be precise enough:

- $e^+e^- \to 4f$ at $\mathcal{O}(\alpha)$ (already exists) and
- signal process $e^+e^- \to WW \to 4f$ at $\mathcal{O}(\alpha^2)$ (the challenge).

Exponentiation of soft W-based interferences in resonant graphs is

- a step towards complete $\mathcal{O}(\alpha^2)$ corrections to $e^+e^- \to WW \to 4f$
- an estimate of missing $\mathcal{O}(\alpha^3)$ and higher corrections
- a practical all-order Monte Carlo algorithm

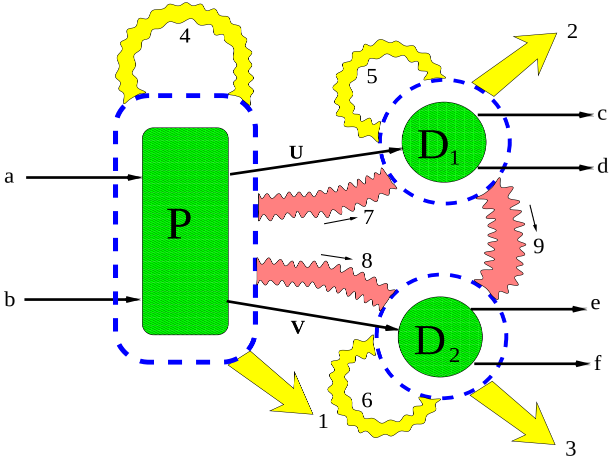
Existing partial solutions and Monte Carlo implementations

- The P-D and D-D interferences have been exponentiated and implemented into MC `KKMC` for the case of Z boson production. Of course Z does not emit photons.
- Exponentiation of the emission in the WW production ($ee \to WW$) has been done in `YFSWW3` MC.
- Exponentiation of the W decay ($W \to f_1 f_2$) is implemented for the single W process in the `WINHAC` MC.

The missing piece are the interferences P-D and D-D to the WW graphs. We will included them **to all orders in soft approximation** within extended YFS scheme presented here.

Exponentiation for charged resonances

All virtual and real emissions, in soft limit

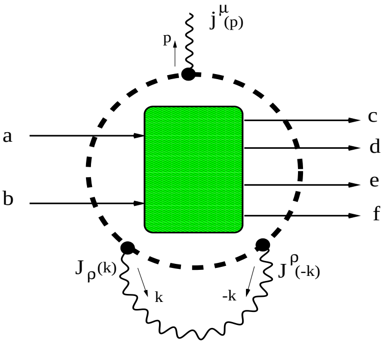


The above is our aim! How to get there?

Classical YFS resummation

- IR emission happens only from external legs (proved by YFS!)
- For each leg use identity:
  $$\sum_{perm.} \frac{1}{pk_1(pk_1+pk_2)(pk_1+pk_2+pk_3)\ldots(pk_1+pk_2+\ldots pk_n)} = \frac{1}{pk_1 pk_2 \ldots pk_n}$$
- For more external legs replace sum over leftover permutations by sum over partitions (2 legs here)
  $$\sum_{\substack{l_a,l_c=0\\ l_a+l_c=n}}^{n} \sum_{\pi/\pi_a/\pi_b/\pi_c}^{n!/l_a!/l_c!} = \sum_{\wp=(a,c)^n}^{2^n} .$$
- Turn sum over partitions into product e.g. (two legs, two photons)
  $$\frac{p_a^{\mu_1}}{p_a k_1}\frac{p_a^{\mu_2}}{p_a k_2} - \frac{p_a^{\mu_1}}{p_a k_1}\frac{p_c^{\mu_2}}{p_c k_2} - \frac{p_c^{\mu_1}}{p_c k_1}\frac{p_a^{\mu_2}}{p_a k_2} + \frac{p_c^{\mu_1}}{p_c k_1}\frac{p_c^{\mu_2}}{p_c k_2} = \left(\frac{p_a^{\mu_1}}{p_a k_1} - \frac{p_c^{\mu_1}}{p_c k_1}\right)\left(\frac{p_a^{\mu_2}}{p_a k_2} - \frac{p_c^{\mu_2}}{p_c k_2}\right).$$
- And the N-real-emission resummed formula is ...
  $$\mathcal{M}_N^{(0)\mu_1,\ldots,\mu_N}(k_1,\ldots,k_N) \simeq \prod_{i=1}^{N}\left(\frac{2p_a^{\mu_i}}{2p_a k_i} - \frac{2p_c^{\mu_i}}{2p_c k_i}\right).$$
- Squaring and integrating over k_i with $1/N!$ Bose-Einstein symmetry factor leads to desired exponential form.

Notation: EM real and virtual current



$$j^\mu(k) = ie \sum_{X=a,b,c,d,e,f} Q_X\theta_X \frac{2p_X^\mu}{2p_X k}$$

$$J^\mu(k) = \sum_{X=a,b,c,d,e,f} \hat{J}_X^\mu(k),$$

$$\hat{J}_X^\mu(k) \equiv Q_X\theta_X \frac{2p_X^\mu\theta_X + k^\mu}{k^2 + 2p_X k\theta_X + i\varepsilon}$$

Virtual lines are pair-contracted giving S-factors:

$$S(k) = J(k) \circ J(k) = \sum_{\substack{X=a,b,c,d,e,f\\ Y=a,b,c,d,e,f}} J_X(k) \circ J_Y(k),$$

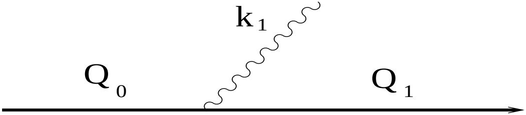
where Q_X is charge, $\theta = +1, -1$ for initial, final state and

$$J_X(k) \circ J_Y(k) \equiv J_X(k) \cdot J_Y(-k), \text{ for } X \neq Y,$$

$$J_X(k) \circ J_X(k) \equiv J_X(k) \cdot J_X(k).$$ (Exactly as in YFS61)

Factoring photon emission

Single emission from the internal W line:



Noticing that $Q_0^2 - Q_1^2 = 2k_1 Q_0 - k_1^2 = 2k_1 Q_1 + k_1^2$ we may write:

$$\frac{1}{(Q_0^2 - M^2)(Q_1^2 - M^2)} = \frac{1}{(2k_1 Q_0 - k_1^2)(Q_1^2 - M^2)} - \frac{1}{(Q_0^2 - M^2)(2k_1 Q_1 + k_1^2)}$$

where M is complex mass of W.

It looks like sum of two on-shell emission factors times pole term.

LHS: IR-finite!

RHS: Difference of two IR-divergent terms! Recoil included!

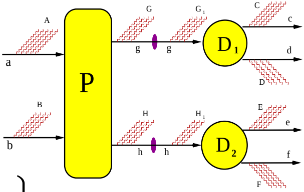
Closer look at 1-real-photon case

$$\mathcal{M}_1^{(0)\,\mu_1}(k) \simeq$$

$$\frac{1}{p_{cd}^2-M^2}\;\frac{1}{p_{ef}^2-M^2}\left\{ Q_a\frac{2p_a^\mu}{2p_a k} + Q_b\frac{2p_b^\mu}{2p_b k} - Q_g\frac{2p_g^\mu}{2p_g k} - Q_h\frac{2p_h^\mu}{2p_h k} \right\}$$

$$+\frac{1}{(p_{cd}+k)^2-M^2}\;\frac{1}{p_{ef}^2-M^2}\left\{ Q_g\frac{2p_g^\mu}{2p_g k} - Q_c\frac{2p_c^\mu}{2p_c k} - Q_d\frac{2p_d^\mu}{2p_d k} \right\}$$

$$+\frac{1}{p_{cd}^2-M^2}\;\frac{1}{(p_{ef}+k)^2-M^2}\left\{ Q_h\frac{2p_h^\mu}{2p_h k} - Q_e\frac{2p_e^\mu}{2p_e k} - Q_f\frac{2p_d^\mu}{2p_f k} \right\}$$



$$= \frac{1}{p_{cd}^2-M^2}\;\frac{1}{p_{ef}^2-M^2}\left\{ j_P^\mu + \frac{p_{cd}^2-M^2}{(p_{cd}+k)^2-M^2}\; j_{D_1}^\mu + \frac{p_{ef}^2-M^2}{(p_{ef}+k)^2-M^2}\; j_{D_2}^\mu \right\}$$

$$= \sum_{\wp=(P,D_1,D_2)}^{3} \frac{1}{p_g^2-M_W^2}\,\frac{1}{p_h^2-M_W^2}\; j_\wp^\mu, \qquad p_g(\wp) = p_{cd}+K_{D_1},\;\; p_h(\wp)=p_{ef}+K_{D_2},\;\; K_X=\sum_{i\in X} k_i$$

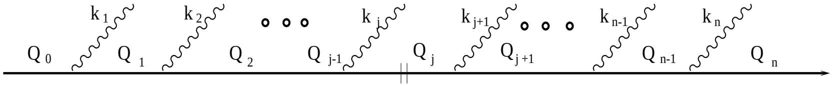
For $k^0 < \Gamma$: Normal YFS small limit, Emission from W's cancels out!

For $k^0 > \Gamma$: Each Intermediade W is present twice (4+3+3=10 sources).

Energy shift in W propagator properly coherently accounted for.

Three gauge-invariant currents: for production and 2 decays.

Multiple emission from the internal W line



In the soft photon limit we find general formula

$$\sum_{permut.} \frac{1}{(Q_0^2-M^2)(Q_1^2-M^2)\ldots(Q_n^2-M^2)} =$$

$$= \sum_{\wp=(P,D)^n} \prod_{\wp_i=P} \frac{1}{(Q_\wp+k_i)^2-Q_\wp^2} \times \frac{1}{Q_\wp^2-M^2} \times \prod_{\wp_k=D} \frac{1}{(Q_\wp-k_i)^2-Q_\wp^2},$$

where

$$Q_\wp = Q_0 - \sum_{\wp_i=P} k_i = Q_n + \sum_{\wp_i=D} k_i.$$

It looks like sum of 2^n on-shell emission factors times pole term!

The numerators of the bosonic line also factorize

$$D_W(p)V(p,k,p-k)_\rho D_W(p) \overset{k\to 0,p^2\to M_W^2}{=} D_W(p)(-2p_\rho)$$

$V(p,k,p-k)_\rho$ = $W\gamma W$ vertex; $D_W(p)$ = numerator of W propagator.

We obtain self-repeating structure

N-real-photon case $\quad e(p_a)\bar{\nu}_e(p_b) \to W \to \mu(p_c)\bar{\nu}_\mu(p_d)$

$$\mathcal{M}_N^{(0)\,\mu_1,\ldots,\mu_N}(k_1,\ldots,k_N) \simeq$$

$$\simeq \sum_{\substack{l_a,l_c,n=0\\ l_a+l_c+n=N}}^{N} \sum_{\substack{l_g,l_h=0\\ l_g+l_h=n}}^{n} \sum_{\pi}^{N!} \Bigg[\left(\frac{2p_a^{\mu_{\pi_1}}}{2p_a k_{\pi_1}} \frac{2p_a^{\mu_{\pi_2}}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2}} \cdots \frac{2p_a^{\mu_{\pi_{l_a}}}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_{l_a}}} \right)$$

$$\left(\frac{-2Q_{\pi_0}^{\mu_{\pi_{l_a+1}}}}{2Q_{\pi_{l_g}} k_{\pi_{l_a+1}}} \frac{-2Q_{\pi_0}^{\mu_{\pi_{l_a+2}}}}{2Q_{\pi_{l_g}} k_{\pi_{l_a+1}} + 2Q_{\pi_{l_g}} k_{\pi_{l_a+2}}} \cdots \frac{-2Q_{\pi_0}^{\mu_{\pi_{l_a+l_g}}}}{2Q_{\pi_{l_g}} k_{\pi_{l_a+1}} + 2Q_{\pi_{l_g}} k_{\pi_{l_a+2}} + \cdots + 2Q_{\pi_{l_g}} k_{\pi_{l_a+l_g}}} \right)$$

$$\frac{D_W(Q_{\pi_{l_g}})}{Q_{\pi_{l_g}}^2 - M_W^2} \left(\frac{2Q_{\pi_0}^{\mu_{\pi_{l_a+l_g+1}}}}{2Q_{\pi_{l_g}} k_{\pi_{l_a+l_g+1}}} \cdots \frac{2Q_{\pi_0}^{\mu_{\pi_{l_a+l_g+l_h}}}}{2Q_{\pi_{l_g}} k_{\pi_{l_a+l_g+1}} + 2Q_{\pi_{l_g}} k_{\pi_{l_a+l_g+2}} + \cdots + 2Q_{\pi_{l_g}} k_{\pi_{l_a+l_g+l_h}}} \right)$$

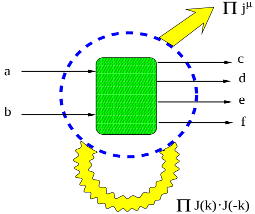
$$\left(\frac{-2p_c^{\mu_{\pi_{l_a+n+1}}}}{2p_c k_{\pi_{l_a+n+1}}} \frac{-2p_c^{\mu_{\pi_{l_a+n+2}}}}{2p_c k_{\pi_{l_a+n+1}} + 2p_c k_{\pi_{l_a+n+2}}} \cdots \frac{-2p_c^{\mu_{\pi_{l_a+n+l_c}}}}{2p_c k_{\pi_{l_a+n+1}} + 2p_c k_{\pi_{l_a+n+2}} + \cdots + 2p_c k_{\pi_{l_a+n+l_c}}} \right) \Bigg]$$

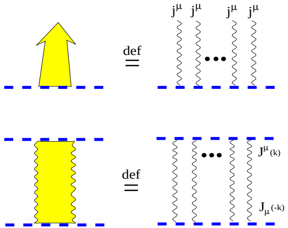
Blue lines describe standard YFS emission, magenta lines – emission from W-boson.
Both have identical structure – standard resummation can be performed

$$\mathcal{M}_N^{(0)} \simeq \sum_{\wp=(P,D)^N}^{2^N} \frac{D_W(Q_g)}{Q_g^2 - M_W^2} \prod_{i=1}^{N} j_{\wp_i}^{\mu_i}, \qquad j_P^{\mu_i} = \frac{2p_a^{\mu_i}}{2p_a k_i} - \frac{2Q_g^{\mu_i}}{2Q_g k_i}, \quad j_D^{\mu_i} = \frac{2Q_g^{\mu_i}}{2Q_g k_i} - \frac{2p_c^{\mu_i}}{2p_c k_i}.$$

$\sum_{\wp=(P,D)^N}^{2^N}$ is a sum over partitions of photons between production and decay, $Q_g = p_{ab} + K_P$

Standard Yennie-Frautschi-Suura-1961, 6 external legs





$$M^{\mu_1\mu_2\cdots\mu_m}(k_1,k_2,\ldots,k_m) =$$

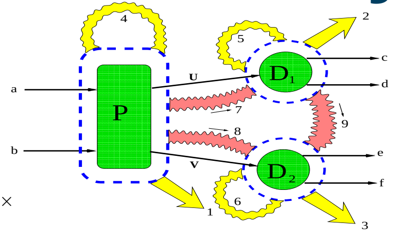
$$= \mathcal{M} \prod_{l=1}^{m} j^\mu(k_l) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int \frac{i}{(2\pi)^3} \frac{d^4k_i}{k_i^2-\lambda^2+i\varepsilon} J^\mu(k_i)\circ J_\mu(k_i)$$

$$= \mathcal{M} \prod_{l=1}^{m} j^\mu(k_l)\; e^{\alpha B_6},$$

$$B_6 = \int \frac{i}{(2\pi)^3} \frac{d^4k}{k^2-\lambda^2+i\varepsilon} J(k)\circ J(k).$$

NEW!!! 6 external legs + 2 internal lines (resonaces)

For a given assignment (permutation) of real photons to P, D_1 and D_2 we have for matrix element:



$$M_{n_1 n_2 n_3}^{\mu_{1_1}\cdots\mu_{3n_3}}(\{k\}) = \mathcal{M}_0 \prod_{i_1=1}^{n_1} j_P^{\mu_{i_1}}(k_{i_1}) \prod_{i_2=1}^{n_2} j_{D_1}^{\mu_{i_2}}(k_{i_2}) \prod_{i_3=1}^{n_3} j_{D_2}^{\mu_{i_3}}(k_{i_3}) \times$$

$$\sum_{n_4=0}^{\infty} \frac{1}{n_4!} \prod_{i_4=1}^{n_4} \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_4}}{k_{i_4}^2 - m_\gamma^2 + i\varepsilon} J_P(k_{i_4}) \circ J_P(k_{i_4}) \sum_{n_5=0}^{\infty} \frac{1}{n_5!} \prod_{i_5=1}^{n_5} \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_5}}{k_{i_5}^2 - m_\gamma^2 + i\varepsilon} J_{D_1}(k_{i_5}) \circ J_{D_1}(k_{i_5})$$

$$\sum_{n_6=0}^{\infty} \frac{1}{n_6!} \prod_{i_6=1}^{n_6} \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_6}}{k_{i_6}^2 - m_\gamma^2 + i\varepsilon} J_{D_2}(k_{i_6}) \circ J_{D_2}(k_{i_6}) \sum_{n_7=0}^{\infty} \frac{1}{n_7!} \prod_{i_7=1}^{n_7} 2 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_7}}{k_{i_7}^2 - m_\gamma^2 + i\varepsilon} J_P(k_{i_7}) \circ J_{D_1}(k_{i_7})$$

$$\sum_{n_8=0}^{\infty} \frac{1}{n_8!} \prod_{i_8=1}^{n_8} 2 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_8}}{k_{i_8}^2 - m_\gamma^2 + i\varepsilon} J_P(k_{i_8}) \circ J_{D_2}(k_{i_8}) \sum_{n_9=0}^{\infty} \frac{1}{n_9!} \prod_{i_9=1}^{n_9} 2 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_9}}{k_{i_9}^2 - m_\gamma^2 + i\varepsilon} J_{D_1}(k_{i_9}) \circ J_{D_2}(k_{i_9})$$

$$\frac{1}{(p_{cd} + K_2 - K_7 + K_9)^2 - M^2} \frac{1}{(p_{ef} + K_3 - K_8 - K_9) - M^2},$$

Sum up for P, D_1 and D_1 as in YFS61

$$
= \mathcal{M}_0 \prod_{i_1=1}^{n_1} j_P^{\mu_{i_1}}(k_{i_1}) \prod_{i_2=1}^{n_2} j_{D_1}^{\mu_{i_2}}(k_{i_2}) \prod_{i_3=1}^{n_3} j_{D_2}^{\mu_{i_3}}(k_{i_3})
$$

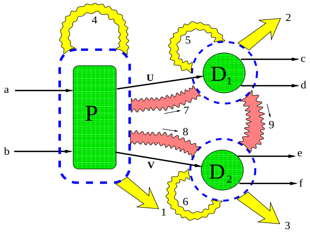
$$
e^{\alpha B_P}\; e^{\alpha B_{D_1}}\; e^{\alpha B_{D_2}}
$$

$$
\sum_{n_7=0}^{\infty} \frac{1}{n_7!} \prod_{i_7=1}^{n_7} 2 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_7}}{k_{i_7}^2 - m_\gamma^2} J_P(k_{i_7}) \circ J_{D_1}(k_{i_7})
$$

$$
\sum_{n_8=0}^{\infty} \frac{1}{n_8!} \prod_{i_8=1}^{n_8} 2 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_8}}{k_{i_8}^2 - m_\gamma^2} J_P(k_{i_8}) \circ J_{D_2}(k_{i_8})
$$

$$
\sum_{n_9=0}^{\infty} \frac{1}{n_9!} \prod_{i_9=1}^{n_9} 2 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_9}}{k_{i_9}^2 - m_\gamma^2} J_{D_1}(k_{i_9}) \circ J_{D_2}(k_{i_9})
$$

$$
\frac{1}{(U_2 - K_7 + K_9)^2 - M^2} \; \frac{1}{(V_3 - K_8 - K_9) - M^2},
$$



where $\alpha B_X = \int \frac{i}{(2\pi)^3} \frac{d^4 k}{k^2 - m_\gamma^2 + i\varepsilon} J_X(k) \circ J_X(k)$, $X = P, D_1, D_2$.

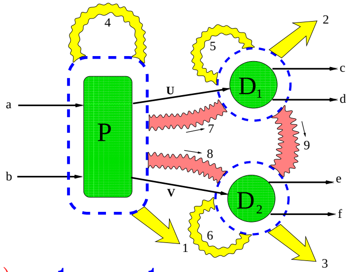
Now tricky point:

In the soft photon limit:

$$\frac{1}{(U_2-K_7+K_9)^2-M^2} \simeq \frac{1}{U_2^2-M^2-2U_2K_7+2U_2K_9}$$
$$= \frac{1}{U_2^2-M^2}\,\frac{1}{1-\sum_{i_7}\frac{2U_2k_{i_7}}{U_2^2-M^2}+\sum_{i_9}\frac{2U_2k_{i_9}}{U_2^2-M^2}}$$
$$= \frac{1}{U_2^2-M^2}\prod_{i_7}\frac{1}{1-\frac{2U_2k_{i_7}}{U_2^2-M^2}}\prod_{i_9}\frac{1}{1+\frac{2U_2k_{i_9}}{U_2^2-M^2}}$$
$$\simeq \frac{1}{U_2^2-M^2}\prod_{i_7}\frac{U_2^2-M^2}{(U_2-k_{i_7})^2-M^2}\prod_{i_9}\frac{U_2^2-M^2}{(U_2+k_{i_9})^2-M^2}$$

leading to final CEEX result, see next slide...

CEEX for narrow resonances



$$M^{\mu_1\dots\mu_n}(k_1,k_2,\dots,k_n) =$$
$$= \sum_{\wp\in(P,D_1,D_2)^n} \mathfrak{M}_0 \prod_{i=1}^{n} j^{\mu_i}_{\wp_i}(k_i)\; e^{\alpha B^{\rm CEEX}_{10}(U_\wp,V_\wp)}\; \frac{1}{U^2_\wp - M^2}\; \frac{1}{V^2_\wp - M^2},$$

where

$$U_\wp = p_{cd} + \sum_{\wp_i=D1} k_i, \quad V_\wp = p_{ef} + \sum_{\wp_i=D2} k_i,$$

$$\alpha B^{\rm CEEX}_{10}(U,V) = \alpha B_P + \alpha B_{D_1} + \alpha B_{D_2} + 2\alpha B_{P\otimes D_1}(U) + 2\alpha B_{P\otimes D_1}(U) + 2\alpha B_{D_1\otimes D_2}(U,V),$$

$$\alpha B_{P\otimes D_1}(U) = \int \frac{i}{(2\pi)^3}\, \frac{d^4k}{k^2-m_\gamma^2+i\varepsilon}\, J_P(k)\circ J_{D_1}(k)\, \frac{U^2-M^2}{(U-k)^2-M^2},$$

$$\alpha B_{P\otimes D_1}(V) = \int \frac{i}{(2\pi)^3}\, \frac{d^4k}{k^2-m_\gamma^2+i\varepsilon}\, J_P(k)\circ J_{D_2}(k)\, \frac{V^2-M^2}{(V-k)^2-M^2},$$

$$\alpha B_{D_1\otimes D_2}(U,V) = \int \frac{i}{(2\pi)^3}\, \frac{d^4k}{k^2-m_\gamma^2+i\varepsilon}\, J_{D_1}(k)\circ J_{D_2}(k)\, \frac{U^2-M^2}{(U+k)^2-M^2}\; \frac{V^2-M^2}{(V-k)^2-M^2}.$$

Summary of new exponentiation scheme

Interferences between production and two decays of W's can be implemented to infinite order in diffrential distributions

- It is extension of YFS61 scheme at the amplitude level (CEEX)
- It is exact in the soft photon limit
- Recoil in W propagators is properly described
- $\mathcal{O}(\alpha)$ corrs. can be added to the finite, non-IR, β functions
- EEX-type scheme (ready for `YFSWW3`) without P-D, D-D interferences can be derived from general formula by dropping non-diagonal products of currents
- Works for single-W as well (relevant for LHC)
- The CEEX case of Z-boson is already implemented in `KKMC` (as reweighted EEX)

Fine print

- Our derivation of the virtual form factors has been sketchy.
- We have not discussed the issues related to the definition and resummation of mass and width of the resonance nor the UV renormalisation.
- Our approach exploited the similarity between virtual and real form factors guaranteed by the IR cancellations.