Macro↔micro: improving magnetic-field models

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This talk will hopefully not be of the/my conventional form:

- ‘This is what I’ve done...
- ...and this is why you should all care’

Instead the aim is to look at some places where:

- Micro input can improve macro models
- Macro input can improve micro models

of neutron-star magnetic fields.

Apologies in advance for my (mis)understanding of the micro side...
Models of NS magnetic fields broadly fall into two categories:

### Steady-state/evolution in elastic crust alone
- Ions locked in place (crust assumed able to absorb *any* stress)
- Only electrons move
- $B = 0$ ‘type-I superconducting’ (!) inner boundary
- Papers by Geppert, Pons, Vigano, Cumming, Gourgouliatos, ...

### Steady-state in core and (fluid) crust
- Entire star is effectively fluid
- Can have a crust, but must be *unstressed* (equilibrium state is fluid)
- Force balance at crust-core boundary can lead to sharp features
- Papers by me, Ciolfi, Fujisawa, Eriguchi, ...

Crust-core boundary treatment important for both – will focus on latter case.
Typical global model of NS magnetic field geometry is a *twisted-torus*: toroidal field fills region of closed poloidal field lines.

Vector sum of poloidal+toroidal $\rightarrow$ coiled equatorial field lines. Field geometry is quite general and follows from:

- $\nabla \cdot \mathbf{B} = 0$
- no exterior electric currents
- axisymmetry.
at crust-core boundary, impose force balance
- this implies magnetic-force balance if everything else smooth
- crust is always normal, magnetic force $\propto B^2$
- if core normal, force $\propto B^2$ and field smooth
- if core superconducting, force $\propto HB$ where $H \sim 10^{15} \text{ G}$

So, in the latter case transition can be abrupt for $B \ll 10^{15} \text{ G}$.
**Question:** what happens to the global $B$ with a better, microphysical, treatment of the crust-core boundary?

- conductivity in pasta phases
- anchoring fluxtubes at the boundary
- current sheets?
- symmetry energy, localised instabilities etc (Coimbra group...)
Macro→micro: how strong can the field be?

Many studies concern microphysics at $B \gtrsim 10^{17}$ G. Makes theoretical sense to probe effects in extreme limits, but...

**Question**: how strong can $B$ really be? What is a ‘realistic’ geometry?

### Upper limits

- **Hard upper limit**: a mythical ‘mega-magnetar’.
  
  \[ P = P_{\text{mag}} \sim B^2 \] balances gravity \[ \implies B \sim 10^{18} \text{ G} \]

- for $B \gtrsim 10^{16}$ G, superconductivity* broken \cite{Glampedakis+2011, Sinha&Sedrakian2015}

- for $B \gtrsim 10^{15}$ G, elastic crust* readily fails \cite{Lander+2015,Lander2016}

- for $B \gtrsim 10^{16}$ G, field generation mechanisms at birth saturate

*possible key field stabilisation mechanism - see next
Macro→micro: what geometry can the field have?

Ignore the genesis of $\mathbf{B}$. What allows for a stable magnetic field?

Main villain: the Tayler instability

- plasma kink instability in spherical star (Tayler, Markey&Tayler, Wright, 1973)
- pure-poloidal fields (e.g. those from the Lorene code) unstable in blue shaded region
- pure-toroidal fields unstable in red shaded region
- vigorous insuppressible dynamical instability, causes global field rearrangement
- timescale $\sim 0.01 \text{ s at } 10^{15} \text{ G } (10^{-5} \text{ s at } 10^{18} \text{ G!})$


→ **room for improvement** in microscopic models...
two places for fruitful micro-macro collaboration:

- **microphysics** of crust-core boundary likely very important in global eqm, especially for superconducting cores
- more realistic **macrophysical** field geometries likely important for microphysics (beyond normal matter, poloidal fields, etc)

- plenty of other issues: fluxtubes at $T \ll T_c$, fluxtubes at $B \sim H$, NS ocean/surface, ...
- bigger goals: interpret magnetar QPOs, understand X-ray burst and flare energy reservoir, model large glitches, ...
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Appendix: equilibrium equations

No dynamo in mature NSs \(\Rightarrow\) field regeneration not possible.
Natural assumption: \(\mathbf{B}\) is in a dynamical equilibrium. Will also assume:
- core neutrons are superfluid (reasonable after a few hundred years)
- the elastic crust is unstressed (not so reasonable) \(\Rightarrow\) crust is ‘fluid’
- rigid rotation is a trivial extension: \(\Phi \mapsto \Phi_{\text{grav}} + \Phi_{\text{rot}}\)

One Euler equation per fluid:
\[
\nabla \tilde{\mu}_p + \nabla \Phi - \mathbf{F}_{\text{mag}}/\rho_p = 0,
\]
\[
\nabla \tilde{\mu}_n + \nabla \Phi = 0.
\]

Equation of state is a double polytrope:
\[
\tilde{\mu}_p = \tilde{\mu}_p(\rho_p), \tilde{\mu}_n = \tilde{\mu}_n(\rho_n).
\]

The two fluids only couple through gravity:
\[
\nabla^2 \Phi = 4\pi G(\rho_p + \rho_n),
\]
and we always need to satisfy \(\nabla \cdot \mathbf{B} = 0\). The magnetic force \(\mathbf{F}_{\text{mag}}\) will change though...

\[\begin{array}{l}
\chi_p(r = 0) = 0.15, \\
\rho/\rho_c = 0.03 \text{ at crust base}, \\
\Gamma_{\text{core}} \approx 2.4, \\
\Gamma_{\text{crust}} \approx 1.6.
\end{array}\]
Appendix: magnetic force

- Since the lattice of fluxtubes is microscopic and regular, we can take a sensible macroscopic average.
- This yields the supercon magnetic force, physically a fluxtube tension \((\text{Easson \& Pethick 1977; Glampedakis, Andersson, Samuelsson 2011})\)

Normal

In normal MHD, \(F_{\text{mag}}\) is the familiar Lorentz force:

\[
F_{\text{mag}} = \frac{1}{4\pi} (\nabla \times B) \times B.
\]

Some algebraic tricks lead to a single PDE, which is fairly convenient to solve.

Superconducting

For superconducting matter we have instead:

\[
F_{\text{mag}} = \frac{1}{4\pi} \left( (\nabla \times H_{c1}) \times B - \rho_p \nabla \left( B \frac{\partial H_{c1}}{\partial \rho_p} \right) \right).
\]

There are now two magnetic fields: a ‘global’ one \(B\) and a ‘local’ one \(H_{c1}\).
Appendix: equilibrium equation (normal protons)

Now assume star is axisymmetric → magnetic field may now be rewritten in terms of a streamfunction \( u \), so that \( \nabla \cdot \mathbf{B} = 0 \) is automatically satisfied:

\[
\mathbf{B} = \frac{1}{\omega} \nabla u \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi.
\]

The Grad-Shafranov equation \((\text{Grad \\ Rubin } 1958, \text{ Shafranov } 1958)\)

After some algebra we arrive at a single PDE for the magnetic field:

\[
\frac{\partial^2 u}{\partial \omega^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\omega} \frac{\partial u}{\partial \omega} = -4\pi \omega^2 \rho_p \frac{dM}{du} - f_N \frac{df_N}{du},
\]

\(M(u)\) is related to the magnetic force through \( \mathbf{F}_{mag} = \rho_p \nabla M \) and \( f_N(u) \) to the toroidal component.

Note one peculiarity: \( u \) appears on both sides of the equation! → natural to solve with iterative methods
Even when protons form a type-II superconductor, can perform a similar derivation as for Grad-Shafranov equation.

One key step fails from the normal-matter derivation; the magnetic-force function $M$ is no longer a function of $u$. In addition, factors of the magnetic-field magnitude $B$ appear. The result is:

\[
\frac{\partial^2 u}{\partial \varpi^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\varpi} \frac{\partial u}{\partial \varpi} = \frac{\nabla \Pi \cdot \nabla u}{\Pi} - \varpi^2 \rho_p \frac{dy}{du} - \Pi^2 f \frac{df}{du},
\]

where $\Pi \equiv B/\rho_p$; $y(u) \sim M + B$ is related to the magnetic force and $f(u)$ is related to the toroidal component.