Introduction to Localization (lecture 2)

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SUSY on curved space and Localisation

We will see how the idea of localisation applies to the path integral of a supersymmetric quantum field theory on a compact curved manifold $\mathcal{M}$.

- Compact space provides IR cutoff, making path integral better defined
- Supersymmetric localisation reduces it to a finite-dimensional integral

$$Z_{\mathcal{M}}[J] = \left\langle e^{-\int J\mathcal{O}} \right\rangle = \int [\mathcal{D}X]\; e^{-S[X]-\int J\mathcal{O}}$$

J is a supersymmetric source, coupled to a supersymmetric observable $\mathcal{O}$.

The dependence on $\mathcal{M}$ is hidden in $S[X]$ and the notion of supersymmetry.

Plan

1. Rigid SUSY on curved space
2. Supersymmetric localisation
3. Example: 3d $\mathcal{N} = 2$ gauge theories on S_b^3

Rigid SUSY on curved space

The problem of defining rigid SUSY on curved space

Supersymmetric QFT on flat space $(\mathbb{R}^d, g^{(0)}_{\mu\nu} = \eta_{\mu\nu})$**:**

- Flat space SUSY algebra ➜ SUSY transformations $\delta^{(0)} X$
- $\mathcal{L}^{(0)}$ SUSY Lagrangian ➜ $\delta^{(0)} \mathcal{L}^{(0)} = \partial_\mu (\dots)^\mu$

$$\downarrow$$

Supersymmetric QFT on curved space $(\mathcal{M}_d, g_{\mu\nu})$**:**

- Curved space SUSY algebra ➜ SUSY transformations δX
- $\mathcal{L}$ SUSY Lagrangian ➜ $\delta \mathcal{L} = \nabla_\mu (\dots)^\mu$

?

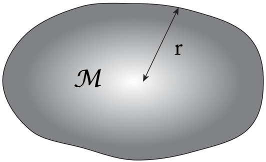
We would like to know:

1. For which flat space SUSY algebras and $(\mathcal{M}_d, g_{\mu\nu})$ this is possible
2. What are δX and $\mathcal{L}(X, \partial_\mu X)$

Approach 1: Trial and error

$$\delta = \delta^{(0)}\Big|_{\substack{\eta\to g\\ \partial\to\nabla}} + \sum_{n\geq 1}\frac{1}{r^n}\delta^{(n)}$$

$$\mathcal{L} = \mathcal{L}^{(0)}\Big|_{\substack{\eta\to g\\ \partial\to\nabla}} + \sum_{n\geq 1}\frac{1}{r^n}\mathcal{L}^{(n)}$$



until SUSY algebra closes and $\delta\mathcal{L} = \nabla_\mu(\dots)^\mu$.

Drawbacks:

- No guarantee it will work
- Case by case
- When it succeeds, expansions stop at $n = 1$ and $n = 2$ resp. Why?

Approach 2: Background supergravity [Festuccia, Seiberg 2011]

see also [Karlhede, Roček 1988; Johansen 1995; Adams, Jockers, Kumar, Lapan 2011]

- Nonlinearly couple supersymmetric FT to an *off-shell* supersymmetric background for supergravity multiplet $(g_{\mu\nu}, \psi_{\mu\alpha}, \mathrm{aux})$.
- Rigid limit of supergravity: gravity multiplet becomes non-dynamical.
- Only require that the background is supersymmetric:

Generalised Killing spinor equations

$$\psi_{\mu\alpha} = 0 \,, \qquad \delta_\zeta \psi_{\mu\alpha} = 0$$

Advantages:

- Model independent: only input is flat space SUSY algebra.
- $\delta_{SuGra}\big|_{\mathrm{bg}} X_{SFT} = \delta X_{SFT} \,, \quad \mathcal{L}_{SFT+SuGra}\big|_{\mathrm{bg}} = \mathcal{L}_{SFT}$.
- $1/r$ expansion above due to auxiliary fields.
- Supersymmetric backgrounds $\mathrm{bg} = (\mathcal{M}_d, g_{\mu\nu}, \mathrm{aux}, \zeta)$ can be classified.

3d $\mathcal{N} = 2$ SUSY with $U(1)_R$ symmetry

SUSY algebra on $\mathbb{R}^3$:

$$\{Q_\alpha, \widetilde{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z$$

$$\{Q_\alpha, Q_\beta\} = 0 \qquad \{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\} = 0$$

$$[R, Q_\alpha] = -Q_\alpha \qquad [R, \widetilde{Q}_\alpha] = +\widetilde{Q}_\alpha$$

$$[Z, Q_\alpha] = [Z, \widetilde{Q}_\alpha] = [Z, R] = 0$$

[Dumitrescu, Seiberg 2011]

Supercurrent $\mathcal{R}$-multiplet:	$T^{\mu\nu}$	$S^{\mu\alpha}$	$\widetilde{S}^{\mu\alpha}$	$j^\mu_{(R)}$	$j^\mu_{(Z)}$	$i\epsilon^{\mu\nu\rho}\partial_\rho J_{(Z)}$
New min'l SUGRA multiplet:	$h_{\mu\nu}$	$\psi_{\mu\alpha}$	$\widetilde{\psi}_{\mu\alpha}$	A_μ	C_μ	$B_{\mu\nu}$

3d version of [Sohnius, West 1981/82]

$$C_\mu \longleftrightarrow V^\mu = -i\epsilon^{\mu\nu\rho}\partial_\mu C_\rho \qquad B_{\mu\nu} \longleftrightarrow H = \frac{i}{2}\epsilon^{\mu\nu\rho}\partial_\mu B_{\nu\rho}$$

$$\delta\mathcal{L}^{lin}_{min} = -T^{\mu\nu} h_{\mu\nu} - \frac{1}{2} S^\mu \psi_\mu + \frac{1}{2}\widetilde{S}^\mu \widetilde{\psi}_\mu + j^\mu_{(R)}(A_\mu - \frac{3}{2} V_\mu) + j^\mu_{(Z)} C_\mu + J_{(Z)} H$$

3d $\mathcal{N} = 2$ SUSY with $U(1)_R$ symmetry on $\mathcal{M}_3$

[Klare, Tomasiello, Zaffaroni 2012; Closset, Dumitrescu, Festuccia, Komargodski 2012]

$\delta_\zeta \psi_{\mu\alpha}$, $\delta_{\widetilde{\zeta}} \widetilde{\psi}_{\mu\alpha}$ in the rigid limit can be inferred from linear theory, diffeo + local R invariance and dimensional analysis, without knowing the full SuGra.

$$\delta_\zeta \psi_\mu = 2(\nabla_\mu - iA_\mu)\zeta + H\gamma_\mu\zeta + 2iV_\mu\zeta + \epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta + (\dots)$$
$$\delta_{\widetilde{\zeta}} \widetilde{\psi}_\mu = 2(\nabla_\mu + iA_\mu)\widetilde{\zeta} + H\gamma_\mu\widetilde{\zeta} - 2iV_\mu\widetilde{\zeta} - \epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta} + (\dots)\ ,$$

(Generalised) Killing spinor equations

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{H}{2}\gamma_\mu\zeta - iV_\mu\zeta - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta$$
$$(\nabla_\mu + iA_\mu)\widetilde{\zeta} = -\frac{H}{2}\gamma_\mu\widetilde{\zeta} + iV_\mu\widetilde{\zeta} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta}\ .$$

Supersymmetric background:

$(\mathcal{M}_3, g_{\mu\nu}, A_\mu, V_\mu, H)$ allowing solutions $(\zeta, \widetilde{\zeta}) \neq 0$ of GKSE.

Curved space supersymmetry algebra

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\phi_{(r,z)} = -2i\left(\mathcal{L}'_K + \zeta\widetilde{\zeta}(z - rH)\right)\phi_{(r,z)}$$

$$\{\delta_\zeta, \delta_\eta\}\phi_{(r,z)} = 0 \qquad\qquad \{\delta_{\widetilde{\zeta}}, \delta_{\widetilde{\eta}}\}\phi_{(r,z)} = 0$$

where $\mathcal{L}'_K$ is a fully covariant Lie derivative along the Killing vector $K^\mu = \zeta\gamma^\mu\widetilde{\zeta}$,

$$\mathcal{L}'_K\varphi_{(r,z)} = \left(K^\mu D_\mu + \tfrac{i}{2}(D_\mu K_\nu)S^{\mu\nu}\right)\varphi_{(r,z)} \ ,$$
$$D_\mu\varphi_{(r,z)} = \left(\nabla_\mu - ir(A_\mu - \tfrac{1}{2}V_\mu) - izC_\mu\right)\varphi_{(r,z)}$$

the totally covariant derivative of a field $\varphi_{(r,z)}$ of R-charge r and Z-charge z.

The representation of this SUSY algebra on a general multiplet is known.

We will be mostly interested in vector and chiral multiplets.

Vector multiplet V

SUSY transformations:

$$\begin{aligned}
\delta a_\mu &= -i\big(\zeta\gamma_\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma_\mu\lambda\big) \\
\delta\sigma &= -\zeta\widetilde{\lambda} + \widetilde{\zeta}\lambda \\
\delta\lambda &= +\zeta\,(D + iH\sigma) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\zeta f_{\mu\nu} - \gamma^\mu\zeta\,(iD_\mu\sigma - V_\mu\sigma) \\
\delta\widetilde{\lambda} &= -\widetilde{\zeta}\,(D + iH\sigma) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\widetilde{\zeta} f_{\mu\nu} + \gamma^\mu\widetilde{\zeta}\,(iD_\mu\sigma + V_\mu\sigma) \\
\delta D &= D_\mu\big(\zeta\gamma^\mu\widetilde{\lambda} - \widetilde{\zeta}\gamma^\mu\lambda\big) - iV_\mu\big(\zeta\gamma^\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma^\mu\lambda\big) - H\big(\zeta\widetilde{\lambda} - \widetilde{\zeta}\lambda\big)
\end{aligned}$$

SUSY Lagrangians:

$$\begin{aligned}
\mathcal{L}_{YM} &= \frac{1}{g_{YM}^2}\,\mathrm{Tr}\Big(\frac{1}{2} f_{\mu\nu}f^{\mu\nu} + D_\mu\sigma D^\mu\sigma + (D + iH\sigma)^2 + i\sigma\varepsilon^{\mu\nu\rho}V_\mu f_{\nu\rho} - V^\mu V_\mu\sigma^2 \\
&\qquad - 2i\widetilde{\lambda}\gamma^\mu(D_\mu + \tfrac{i}{2}V_\mu)\lambda - 2i\widetilde{\lambda}[\sigma,\lambda] + iH\widetilde{\lambda}\lambda\Big) \\
\mathcal{L}_{CS} &= i\frac{k}{4\pi}\,\mathrm{Tr}\left(\varepsilon^{\mu\nu\rho}(a_\mu\partial_\nu a_\rho + i\frac{2}{3}a_\mu a_\nu a_\rho) + 2D\sigma + 2\widetilde{\lambda}\lambda\right) \\
\mathcal{L}_{FI} &= -i\frac{\xi}{2\pi}\,\mathrm{Tr}\,(D - iH\sigma - iV^\mu a_\mu)
\end{aligned}$$

Chiral multiplet $\Phi_{(r,z)}$

SUSY transformations:

$$\begin{aligned}
\delta\phi &= \sqrt{2}\zeta\psi \\
\delta\psi &= \sqrt{2}\zeta F - \sqrt{2}i\,(z - \sigma - rH)\,\widetilde{\zeta}\phi - \sqrt{2}i\gamma^\mu\widetilde{\zeta}D_\mu\phi \\
\delta F &= \sqrt{2}i\,(z - \sigma - (r-2)H)\,\widetilde{\zeta}\psi + 2i\widetilde{\zeta}\widetilde{\lambda}\phi
\end{aligned}$$

SUSY Lagrangians:

$$\begin{aligned}
\mathcal{L}_{mat} &= D^\mu\widetilde{\phi}D_\mu\phi - i\widetilde{\psi}\gamma^\mu D_\mu\psi - \widetilde{F}F - i\widetilde{\phi}V^\mu D_\mu\phi \\
&+ \widetilde{\phi}\Big(- i(D + iH\sigma) + (z - \sigma - rH)^2 + 2H(z-\sigma) + \frac{r}{2}(\frac{1}{2}R + V^\mu V_\mu - H^2)\Big)\phi \\
&+ i\widetilde{\psi}\Big(z - \sigma - (r - \frac{1}{2})H\Big)\psi - \frac{1}{2}\widetilde{\psi}\gamma^\mu V_\mu\psi + \sqrt{2}i(\widetilde{\phi}\lambda\psi + \phi\widetilde{\lambda}\widetilde{\psi}) \\
\mathcal{L}_W &= F_{W(\Phi)} + \widetilde{F}_{\widetilde{W}(\widetilde{\Phi})} = \left(F\frac{\partial W}{\partial\phi} + \psi\psi\frac{\partial^2 W}{\partial\phi^2}\right) + \left(\widetilde{F}\frac{\partial\widetilde{W}}{\partial\widetilde{\phi}} + \widetilde{\psi}\widetilde{\psi}\frac{\partial^2\widetilde{W}}{\partial\widetilde{\phi}^2}\right)
\end{aligned}$$

Compact supersymmetric backgrounds

1 supercharge ζ: $\mathcal{M}_3$ has a transversely holomorphic foliation

Coordinates $(\tau, z, \bar{z})$: $\tau' = \tau + t(z,\bar{z})$, $z' = f(z)$.

Metric: $ds^2 = (d\tau + h(\tau,z,\bar{z})dz + \bar{h}(\tau,z,\bar{z})d\bar{z})^2 + c(\tau,z,\bar{z})^2 dz d\bar{z}$

ζ determines all background fields, up to invariance of GKSE.

2 supercharges ζ, $\widetilde{\zeta}$: $\mathcal{M}_3$ is a Seifert manifold $(S^1 \hookrightarrow M_3 \to \Sigma)$

Metric: $ds^2 = \Omega(z,\bar{z})^2 (d\psi + h(z,\bar{z})dz + \bar{h}(z,\bar{z})d\bar{z})^2 + c(z,\bar{z})^2 dz d\bar{z}$

4 supercharges ζ_1, ζ_2, $\widetilde{\zeta}_1$, $\widetilde{\zeta}_2$:

- T^3
- Round $S^2 \times S^1$ with $H = 0$, $A = V = \pm \frac{i}{R_{S^1}} d\tau$ [Imamura, S. Yokoyama 2011]
- (Squashed) S^3 with $SU(2) \times U(1)$ isometry: [Imamura, D. Yokoyama 2011]

$$ds^2 = R^2 \left((\mu^1)^2 + (\mu^2)^2 + h^2 (\mu^3)^2 \right) , \quad H = \frac{ih}{R} , \quad A = V = 2\sqrt{h^2 - 1}\mu^3$$

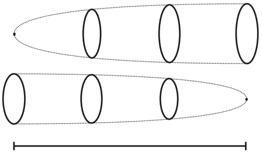
The 3d ellipsoid S_b^3

$$b^2|z_1|^2 + b^{-2}|z_2|^2 = R^2$$

$$z_1 = Rb^{-1}\sin\vartheta\ e^{i\varphi_1}$$
$$z_2 = Rb\cos\vartheta\ e^{i\varphi_2}$$

S^3 topology

$U(1)^2$ isometry



Background:

$$ds^2 = R^2\left(b^2\sin^2\vartheta\ d\varphi_1^2 + b^{-2}\cos^2\vartheta\ d\varphi_2^2 + f(\vartheta)^2 d\vartheta^2\right)$$

$$H = -\frac{i}{Rf(\vartheta)}\ , \quad 2A = \left(1 - \frac{b}{f(\vartheta)}\right)d\varphi_1 + \left(1 - \frac{b^{-1}}{f(\vartheta)}\right)d\varphi_2$$

$$f(\vartheta) = (b^{-2}\sin^2\vartheta + b^2\cos^2\vartheta)^{1/2}$$

$$\zeta = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i}{2}(\varphi_1+\varphi_2+\vartheta)} \\ e^{\frac{i}{2}(\varphi_1+\varphi_2-\vartheta)} \end{pmatrix}\ , \quad \tilde{\zeta} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{-\frac{i}{2}(\varphi_1+\varphi_2-\vartheta)} \\ e^{-\frac{i}{2}(\varphi_1+\varphi_2+\vartheta)} \end{pmatrix}$$

Supersymmetric localisation

The path integral of a supersymmetric QFT

Consider a supersymmetric QFT with fields $X \in \mathcal{F}$:

- Supercharge $\mathcal{Q}$: $\qquad \mathcal{Q}^2 = \mathcal{H}$
- Action $S[X]$: $\qquad \mathcal{Q}S[X] = 0$
- Supersymmetric observable $\mathcal{O}$: $\qquad \mathcal{Q}\mathcal{O} = 0$

We wish to compute

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]}$$

Note that [Witten 1988]

$$\langle \mathcal{Q}\mathcal{O}' \rangle = \int_{\mathcal{F}} [\mathcal{D}X]\ (\mathcal{Q}\mathcal{O}')e^{-S[X]} = \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{Q}\left(\mathcal{O}' e^{-S[X]}\right) = 0 \,,$$

therefore expectation values only depend on the $\mathcal{Q}$-cohomology class:

$$\langle \mathcal{O} + \mathcal{Q}\mathcal{O}' \rangle = \langle \mathcal{O} \rangle$$

Localisation argument 1

Assume the QFT has a symmetry group G which acts *freely* on field space $\mathcal{F}$. For a G-invariant operator $\mathcal{O}$,

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]} = \mathrm{Vol}(G) \cdot \int_{\mathcal{F}/G} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]} \ .$$

If G is generated by a fermionic charge $\mathcal{Q}$, then $\mathrm{Vol}(G) \propto \int d\theta\ 1 = 0$.

A supercharge $\mathcal{Q}$ does not act freely. Fixed points form the

BPS locus $\qquad \mathcal{F}_{\mathcal{Q}} = \{[X] \in \mathcal{F} \mid \text{fermions} = 0,\ \ \mathcal{Q}(\text{fermions}) = 0\} \ .$

If $\mathcal{F}_{\mathcal{Q},\varepsilon}$ is an infinitesimal tubular neighbourhood of $\mathcal{F}_{\mathcal{Q}}$ of size ε,

$$\langle \mathcal{O} \rangle = \lim_{\varepsilon \to 0} \left(\int_{\mathcal{F} \setminus \mathcal{F}_{\mathcal{Q},\varepsilon}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]} + \int_{\mathcal{F}_{\mathcal{Q},\varepsilon}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]} \right) = \lim_{\varepsilon \to 0} \int_{\mathcal{F}_{\mathcal{Q},\varepsilon}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]}$$

Hence the path integral over field space $\mathcal{F}$ localises to the BPS locus $\mathcal{F}_{\mathcal{Q}}$.

Localisation argument 2

We can exploit the fact that the expectation value of a $\mathcal{Q}$-closed observable $\mathcal{O}$ only depends on its $\mathcal{Q}$-cohomology class $[\mathcal{O}]$. Change representative:

$$\langle \mathcal{O} \rangle = \langle \mathcal{O}\, e^{-t\mathcal{Q}\mathcal{V}[X]} \rangle = \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{O}\, e^{-S[X]-t\mathcal{Q}\mathcal{V}[X]} \qquad \forall\, t,\ \mathcal{V}[X]\ \text{ s.t. }\ \mathcal{Q}^2\mathcal{V}[X] = 0\ .$$

We will assume that $\mathrm{Re}\mathcal{Q}\mathcal{V}[X]|_{bos}$ is positive semi-definite and consider $t \geq 0$.

$$\langle \mathcal{O} \rangle = \lim_{t\to+\infty} \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]-t\mathcal{Q}\mathcal{V}[X]}\ .$$

- $t \to +\infty$: integral dominated by the saddle points of the

Localising action

$$S_{loc}[X] = \mathcal{Q}\mathcal{V}[X]\ ,$$

Localisation locus

$$\mathcal{F}_{loc} = \{X_0 \in \mathcal{F} | \frac{\delta S_{loc}[X]}{\delta X}\Big|_{X_0} = 0\}\ .$$

Exact semiclassical approximation in $\hbar_{aux} = 1/t$, around saddles X_0 of S_{loc}:

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X$$

$$S[X] + tS_{loc}[X] \xrightarrow[t\to+\infty]{} S[X_0] + \frac{1}{2}\iint \frac{\delta^2 S_{loc}[X]}{\delta X^2}\bigg|_{X_0} (\delta X)^2$$

Integrating out the transverse fluctuations δX, one obtains

Localisation formula

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}_{loc}} [\mathcal{D}X_0]\ \mathcal{O}|_{X_0}\ e^{-S[X_0]} \frac{1}{\text{Sdet}\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right]} .$$

E.g., for the standard choice (here $\Psi = \{\text{fermions}\}$) [Pestun 2007]

$$\mathcal{V}_P = (\mathcal{Q}\Psi, \Psi) \quad \Longrightarrow \quad S_{loc}|_{bos} = (\mathcal{Q}\Psi, \mathcal{Q}\Psi) \geq 0 ,$$

the path integral over $\mathcal{F}$ localizes to $\mathcal{F}_{loc} = \mathcal{F}_{\mathcal{Q}}$, the BPS locus.

The 1-loop determinant Z_{1-loop}

$$Z_{1-loop} = \text{Sdet}\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right] = \left(\frac{\det K_{ferm}}{\det K_{bos}}\right)^{1/2} ,$$

where the kinetic operators for bosonic and fermionic fluctuations K_{bos}, K_{ferm} are some modified Laplace and Dirac operators.

General observations:

- Computing their spectrum can be difficult.
- Many cancellations because SUSY pairs bosons and fermions.
- One can localise to fixed points of $\mathcal{Q}^2$ in spacetime.

The computation is best done by organising fields cohomologically (*i.e.* in multiplets of $\mathcal{Q}$) and applying index theorems. [Pestun 2007]

1. Reorganise fields in $\mathcal{Q}$-multiplets $\{X\} = \{\phi,\ \psi' = \mathcal{Q}\phi,\ \psi,\ \phi' = \mathcal{Q}\psi\}$:

$$\mathcal{Q}\phi = \psi' , \qquad \mathcal{Q}\psi' = \mathcal{Q}^2\phi .$$
$$\mathcal{Q}\psi = \phi' , \qquad \mathcal{Q}\phi' = \mathcal{Q}^2\psi .$$

For simplicity, [Hosomichi 2015]

$$\mathcal{V}_H = (\phi, \mathcal{Q}\phi) + (\psi, \mathcal{Q}\psi)$$
$$\mathcal{Q}\mathcal{V}_H = (\psi', \psi') + (\phi, \mathcal{Q}^2\phi) + (\phi', \phi') - (\psi, \mathcal{Q}^2\psi)$$

$$Z_{1-loop} = \left(\frac{\det_\psi \mathcal{Q}^2}{\det_\phi \mathcal{Q}^2} \right)^{1/2}$$

2. If there is a differential operator $\mathcal{D}$ that commutes with $\mathcal{Q}^2$,

$$\mathcal{D} : \begin{array}{ccc} \Gamma(E_0) & \to & \Gamma(E_1) \\ \Cup & & \Cup \\ \phi & & \psi \end{array} \qquad \mathcal{D}^\dagger : \begin{array}{ccc} \Gamma(E_1) & \to & \Gamma(E_0) \\ \Cup & & \Cup \\ \psi & & \phi \end{array}$$

then

$$Z_{1-loop} = \left(\frac{\det_{\mathrm{coker}\mathcal{D}} \mathcal{Q}^2}{\det_{\ker \mathcal{D}} \mathcal{Q}^2} \right)^{1/2} .$$

$\leftarrow$ unpaired ψ
$\leftarrow$ unpaired ϕ

The 1-loop determinant can be deduced from the $\mathcal{Q}^2$-equivariant index of $\mathcal{D}$

$$\text{Ind}(\mathcal{D}; e^{\mathcal{Q}^2}) := \text{tr}_{\ker \mathcal{D}}(e^{\mathcal{Q}^2}) - \text{tr}_{\text{coker}\mathcal{D}}(e^{\mathcal{Q}^2}) = \sum_j d_j \, e^{h_j}$$

as

$$Z_{1-loop} = \left(\frac{\det_{\text{coker}\mathcal{D}} \mathcal{Q}^2}{\det_{\ker \mathcal{D}} \mathcal{Q}^2} \right)^{1/2} = \prod_j h_j^{-d_j/2} \, .$$

If $\mathcal{D}$ is transversally elliptic, which ensures that d_j are finite, the equivariant index can be computed by the Atiyah-Bott fixed point formula *e.g.* [Atiyah 1974]

$$\text{Ind}(\mathcal{D}; e^{\mathcal{Q}^2}) = \sum_{p | e^{\mathcal{Q}^2} \cdot p = p} \frac{\text{tr}_{E_0(p)} \, e^{\mathcal{Q}^2} - \text{tr}_{E_1(p)} \, e^{\mathcal{Q}^2}}{\det_{T\mathcal{M}(p)}(1 - e^{\mathcal{Q}^2})} \, .$$

This reduces the computation of Z_{1-loop} to determining the local action of $\mathcal{Q}^2$ around fixed points in field space $\mathcal{F}$ and in spacetime $\mathcal{M}$.

Localisation of $3d$ $\mathcal{N}=2$ gauge theories on S_b^3

[Hama, Hosomichi, Lee 2011], building on [Kapustin, Willett, Yaakov '09; Jafferis '10; HHL '10]

$$Z[\widehat{V}] = \int [\mathcal{D}V][\mathcal{D}\Phi][\mathcal{D}\widetilde{\Phi}]\, e^{-(S_{YM}[V]+S_{CS}[V]+S_{FI}[V]+S_{mat}[\Phi,\widetilde{\Phi},V,\widehat{V}]+S_W[\Phi,\widetilde{\Phi}])}$$

$\widehat{V}$: background vector multiplet (global symmetry)

- Localising supercharge: $\mathcal{Q} = \delta_\zeta + \delta_{\widetilde{\zeta}}$
- Localising action: $S_{loc} = \mathcal{Q}\mathcal{V}_P\,, \qquad \mathcal{V}_P = \sum_{\Psi\in\{\lambda,\widetilde{\lambda},\psi,\widetilde{\psi}\}} (\mathcal{Q}\Psi,\Psi)$
- Localisation locus $\mathcal{F}_\mathcal{Q}$: $D=-iH\sigma,\quad a_\mu=0,\quad \sigma=\text{const.}$ $\quad \phi=\widetilde{\phi}=F=\widetilde{F}=0$
- Classical action: $S[X_0] = -ik\pi\,\mathrm{tr}(R\sigma)^2 + 2\pi i(\xi R)\,\mathrm{tr}(R\sigma)$

  [S_{YM}, S_{mat}, S_W are $\mathcal{Q}$-exact] $\qquad Z_{class} = e^{ik\pi\,\mathrm{tr}(R\sigma)^2 - 2\pi i(\xi R)\,\mathrm{tr}(R\sigma)}$

- Diagonalise $\sigma = \sigma^i H_i$: $$|J| = \prod_{\alpha\in\Delta_+} \alpha(R\sigma)^2$$
- 1-loop det of $\Phi_{(r,z)}$: $$Z^{\Phi}_{1-loop} = \prod_{m,n=0}^{\infty} \frac{(m+1)b+(n+1)b^{-1}+iRz_{\mathbb{C}}}{mb+nb^{-1}-iRz_{\mathbb{C}}} \text{``}=\text{''}\Gamma_h(Rz_{\mathbb{C}})$$ $$Rz_{\mathbb{C}} = Rz + i\frac{b+b^{-1}}{2}r\,, \qquad z = \rho(\sigma)+\widehat{\rho}(\widehat{\sigma})$$
- 1-loop det of V: $$Z^{V}_{1-loop} = \prod_{\alpha\in\Delta_+} \frac{4\sinh(\pi b\alpha(R\sigma))\sinh(\pi b^{-1}\alpha(R\sigma))}{\alpha(R\sigma)^2}$$

Coulomb branch localisation formula $(R=1)$

$$Z_{S^3_b}(\widehat{\sigma};k,\xi,r) = \frac{1}{|\mathcal{W}_G|}\int \prod_{i=1}^{\mathrm{rk}(G)} d\sigma_i\, Z_{class}(\sigma;\xi,k)\, Z_{1-loop}(\sigma,\widehat{\sigma},r)\,.$$

This result generalizes to any background with S^3 topology: $b\in\mathbb{C}$ is the modulus of the transversely holomorphic foliation on S^3.

[Closset, Dumitrescu, Festuccia, Komargodski 2013; Alday, Martelli, Richmond, Sparks 2013]

An alternative: Higgs branch localisation

[Benini, SC '12] in 2d; [Fujitsuka, Honda, Yoshida '13; Benini, Peelaers '13] in 3d; ...

Localising action:

$$S'_{loc} = \mathcal{Q}\left(\mathcal{V}_P + \mathcal{V}_{Higgs}\right)$$

$$\mathcal{V}_{Higgs} = \int d^3x \sqrt{g}\, \mathrm{tr}\left(\frac{\widetilde{\zeta}\lambda - \zeta\widetilde{\lambda}}{2i} M(\phi, \widetilde{\phi})\right)$$

$$M(\phi, \widetilde{\phi}) = \sum_\alpha \phi^\alpha \phi^\dagger_\alpha - \widehat{\xi}\,, \qquad \widehat{\xi} = \sum_{i \in \mathrm{Cartan}(g)} \widehat{\xi}^i h_i$$

When the "fake FI parameter" $\widehat{\xi} \to \infty$ (in an appropriate direction):

- Coulomb branch saddles are suppressed
- Higgs branch saddles controlled by M (plus zero size vortices) dominate.

Higgs branch localisation formula

$$Z = \sum_{\text{Higgs vacua}} Z_{class}\, Z'_{1-loop}\, Z_v^{(NP)}\, Z_{av}^{(SP)}$$

proving the factorisation of Z observed in [Pasquetti '11], [Beem, Dimofte, Pasquetti '12].

Bonus tracks: two applications

Partition function and field theory dualities

The partition function $Z_{\mathcal{M}}(\widehat{V};\lambda)$ computed exactly by localisation allows detailed tests of field theory dualities. If theory A is dual to theory B, then

$$Z_{\mathcal{M}}^{(A)}(\widehat{V}^{(A)};\lambda^{(A)}) = Z_{\mathcal{M}}^{(B)}(\widehat{V}^{(B)};\lambda^{(B)})$$

with a duality map

$$\widehat{V}_a^{(A)} = \sum_b c_a{}^b \widehat{V}_b^{(B)}$$
$$\lambda^{(A)} = f(\lambda^{(B)}) \,.$$

These tests have been performed for a variety of theories:
[Dolan, Osborn '08; Spiridonov, Vartanov '08-'12; Kapustin, Willett, Yaakov '10; Willett, Yaakov '11; Benini, Closset, SC '11; Benini, SC '12; Doroud, Gomis, Le Floch, Lee '12; . . .]

- Identities between integrals of special functions
- Useful to determine the duality map
- Can be extended to supersymmetric operators

The free energy of 3d $\mathcal{N}=2$ SCFTs

A particularly interesting quantity for a 3d CFT:

Free energy $$F = -\log Z_{S^3}|_{finite}$$

3d analogue of c and a central charges: it counts degrees of freedom.

- It can be computed exactly using localisation for $\mathcal{N} \geq 2$ theories.
- Using the large-N limit of the matrix model for Z_{S^3}, it was shown that M2-brane theories have $F \propto N^{3/2}$. [Drukker, Mariño, Putrov 2010]
- **F-maximisation**: the $\mathcal{N}=2$ superconformal R-symmetry

$$R(t) = R_0 + \sum_a t_a Q^a$$

maximizes $\mathrm{Re}(\mathrm{F}(\mathrm{t}))$. [Jafferis 2010; Closset, Dumitrescu, Festuccia, Seiberg 2012]

- **F-theorem**: F decreases along RG-flows. [Jafferis, Klebanov, Pufu, Safdi 2011] [Casini, Huerta 2012]

Conclusions

- Localisation has led to many exact results for broad classes of supersymmetric QFTs on curved manifolds of various dimensions.

- What about 4d $\mathcal{N} = 1$?
  - $S^3 \times S^1$ ✓
  - $\Sigma_g \times T^2$ ✓
  - S^4 ✗

- I have overlooked many interesting results: (Sorry!)
  - "Superconformal" indices: $Z_{S^{d-1} \times S^1} = \mathrm{Tr}\,(-1)^F e^{-\beta' \{Q,S\} - \beta \sum_a v_a F^a}$
  - Twisted partition functions
  - Localisation on manifolds with boundaries
  - Inclusion of order/disorder operators
  - Even localisation of supersymmetric quantum theories of gravity!

Localisation will remain an important tool in the future:

- Very concrete approach to make exact calculations in SUSY QFT.
- Probes generic (strongly coupled) regimes of parameter/moduli space.
- Allows to address fundamental questions in QFT and String Theory.
- Many open questions, likely new directions are waiting to be explored.

Thank you for your attention!