Classification of Supersymmetric Solutions II

Jan Gutowski

with G. Papadopoulos and Ulf Gran

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University of Surrey

Bicocca-Surrey School

Outline for Lectures 3+4

- The Homogeneity theorem for $N > 16$ solutions in D=11 and type II D=10 supergravity [arXiv:1208.0553]
- $N = 29, 30, 31$ solutions in IIB supergravity [arXiv:0710.1829]
- Global properties of horizons/AdS [arXiv:1303.0869]

The Homogeneity Theorem [Hustler, O'Farrill]

Let $\mathcal{S}$ be the space of D=11, IIB/A spinors. Given $\epsilon_1, \epsilon_2 \in \mathcal{S}$, let

$$\varphi(\epsilon_1, \epsilon_2) = D(\epsilon_1, \Gamma^M \epsilon_2)\partial_M$$

D is a non-degenerate gauge-invariant inner product,

$$D : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$$

In IIB supergravity

$$D(\epsilon_1, \epsilon_2) = \mathrm{Re}\langle \epsilon_1, \Gamma_0 \epsilon_2 \rangle$$

φ is a *squaring operator*, which associates to every pair of spinors a tangent vector, pointwise on the spacetime.

φ is:

- Symmetric: $\varphi(\epsilon_1, \epsilon_2) = \varphi(\epsilon_2, \epsilon_1)$.
  This implies

$$\varphi(\epsilon_1, \epsilon_2) = \frac{1}{2}\Big(\varphi(\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_2) - \varphi(\epsilon_1, \epsilon_1) - \varphi(\epsilon_2, \epsilon_2)\Big)$$

  so φ is fixed by its diagonal values.
- $\varphi(\epsilon, \epsilon)$ is either timelike or null [JG, Papadopoulos]

Suppose W is the space of Killing spinors, and $\dim(W) > 16$.

If M is the spacetime, and $p \in M$, the map

$$\varphi|_W : W \times W \to T_p(M)$$

is surjective iff the perpendicular component of its image is trivial.

Equivalently,

$$\varphi|_W : W \times W \to T_p(M)$$

is surjective iff the only tangent vector V such that

$$V^M \varphi(\epsilon_1, \epsilon_2)_M = 0, \qquad \forall \epsilon_1, \epsilon_2 \in W$$

is $V = 0$.

Suppose that

$$V^M \varphi(\epsilon_1, \epsilon_2)_M = 0, \qquad \forall \epsilon_1, \epsilon_2 \in W$$

Then such V must be null. To prove this, the above condition is equivalent to requiring

$$\Gamma^M V_M : W \to W^\perp$$

where

$$W^\perp = \{\epsilon \in \mathcal{S} : D(\epsilon, \chi) = 0, \forall \chi \in W\}$$

We know that $\dim(W) + \dim(W^\perp) = 32$ and $\dim(W) > 16$.

This implies that $\dim(W^\perp) < 16$. As we have

$$\Gamma^M V_M : W \to W^\perp$$

this implies by the rank-nullity theorem that $\mathrm{Ker}(\Gamma^M V_M) \neq \{0\}$.

However

$$\left(\Gamma^M V_M\right)^2 = V^M V_M$$

so in order to have $\mathrm{Ker}(\Gamma^M V_M) \neq \{0\}$ we must have $V^2 = 0$.

If the spacetime were Euclidean, this would imply $V = 0$ and so $\varphi|_W$ maps surjectively pointwise onto the tangent space.

But the supergravity is Lorentzian, so if V is nonzero, it must be null.

Choose a basis for the tangent space

$$\{\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_i\}$$

with $V \propto \mathbf{e}_+$.

We need $\varphi(\epsilon, \epsilon)$ to be both orthogonal to $\mathbf{e}_+$ and be non-spacelike.

Hence

$$\varphi(\epsilon, \epsilon) = \lambda(\epsilon)\mathbf{e}_+$$

for some function $\lambda : W \to \mathbb{R}$, and so

$$\begin{aligned} \varphi(\epsilon_1, \epsilon_2) &= \frac{1}{2}\Big(\varphi(\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_2) - \varphi(\epsilon_1, \epsilon_1) - \varphi(\epsilon_2, \epsilon_2)\Big) \\ &= \frac{1}{2}\Big(\lambda(\epsilon_1 + \epsilon_2) - \lambda(\epsilon_1) - \lambda(\epsilon_2)\Big)\mathbf{e}_+ \end{aligned}$$

It follows that both the image of $\varphi|_W$ and the perpendicular component of the image of $\varphi|_W$ are spanned by $\mathbf{e}_+$.

This is impossible, so we must have $V = 0$, i.e. $\varphi|_W$ is surjective.

Note: in $D = 10, D = 11$ supergravity theories, if ϵ_1, ϵ_2 are Killing spinors, then the KSEs imply that $\varphi(\epsilon_1, \epsilon_2)$ is an isometry which preserves all the bosonic fields of the theory.

So $N > 16$ supergravity solutions are all *locally transitive* - at every point there is a local frame consisting of Killing vectors which preserve all the supergravity fields.

Application to Warped Product AdS Solutions

Consider $N > 16$ warped product solutions $AdS_n \times_w M$ in D=10 type II or D=11 supergravity.

M is some compact and smooth Euclidean internal space.

- The spacetime spinors can always be written in terms of spinors σ_+ defined on M
- These spinors satisfy generalized Lichnerowicz theorems on M
- For $N > 16$ solutions, there are sufficiently many σ_+ spinors in order for the homogeneity theorem to apply for M.
- In all cases, the warp factor is constant, and M must be a homogeneous manifold.
- All such M have been fully classified (at least for $\dim(M) \leq 9$) by S. Klaus: (*Einfachzusammenh angende kompakte homogene Raume bis zur Dimension 9*).

IIB Supergravity and Killing Spinors

The bosonic fields of IIB supergravity are the spacetime metric g, the axion σ and dilaton ϕ , two three-form field strengths $G^\alpha = dA^\alpha$ $(\alpha = 1, 2)$, and a self-dual five-form field strength F

The axion and dilaton give rise to a complex 1-form P [Schwarz, West].

The 3-forms are combined to give a complex 3-form G.

To achieve this, introduce a $SU(1,1)$ matrix $U = (V_+^\alpha, V_-^\alpha)$, $\alpha = 1, 2$ such that

$$V_-^\alpha V_+^\beta - V_-^\beta V_+^\alpha = \epsilon^{\alpha\beta} \ , \quad (V_-^1)^* = V_+^2, \qquad (V_-^2)^* = V_+^1$$

$\epsilon^{12} = 1 = \epsilon_{12}$.

The $V_\pm^\alpha$ are related to the axion and dilaton by

$$\frac{V_-^2}{V_-^1} = \frac{1 + i(\sigma + ie^{-\phi})}{1 - i(\sigma + ie^{-\phi})} \ .$$

Then P and G are defined by

$$P_M = -\epsilon_{\alpha\beta} V_+^\alpha \partial_M V_+^\beta, \quad G_{MNR} = -\epsilon_{\alpha\beta} V_+^\alpha G_{MNR}^\beta$$

The gravitino Killing spinor equation is

$$\tilde{\nabla}_M \epsilon + \frac{i}{48}\Gamma^{N_1 \dots N_4}\epsilon F_{N_1 \dots N_4 M} - \frac{1}{96}(\Gamma_M{}^{N_1 N_2 N_3} G_{N_1 N_2 N_3} \\ -9\Gamma^{N_1 N_2} G_{M N_1 N_2})(C * \epsilon) = 0$$

where

$$\tilde{\nabla}_M = \partial_M - \frac{i}{2} Q_M + \frac{1}{4}\Omega_{M,AB}\Gamma^{AB}$$

is the standard covariant derivative twisted with $U(1)$ connection Q_M, given in terms of the $SU(1,1)$ scalars by

$$Q_M = -i\epsilon_{\alpha\beta} V_-^\alpha \partial_M V_+^\beta$$

and Ω is the spin connection.

There is also an algebraic condition

$$P_M \Gamma^M (C * \epsilon) + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0$$

The Killing spinor ϵ is a complex Weyl spinor constructed from two copies of the same Majorana-Weyl representation Δ^+_{16}:

$$\epsilon = \psi_1 + i\psi_2$$

Majorana-Weyl spinors ψ satisfy

$$\psi = C * \psi$$

C is the charge conjugation matrix with the property that

$$C * \Gamma_M = \Gamma_M C *$$

A basis can be chosen in which $C = \Gamma_{6789}$.

Spinors as Forms

- Let $e_1, \ldots, e_5$ be a locally defined orthonormal basis of $\mathbb{R}^5$.
- Take U to be the span over $\mathbb{R}$ of $e_1, \ldots, e_5$.
- The space of Dirac spinors is ${}^c\Delta = \Lambda^*(U \otimes \mathbb{C})$ (the complexified space of all forms on U).
- ${}^c\Delta$ decomposes into even forms ${}^c\Delta^+$ and odd forms ${}^c\Delta^-$, which are the complex Weyl representations of $Spin(9,1)$.

- The gamma matrices are represented on ${}^c\Delta$ as

$$\begin{array}{rcll} \Gamma_0\eta & = & -e_5\wedge\eta + e_5\lrcorner\eta & \\ \Gamma_5\eta & = & e_5\wedge\eta + e_5\lrcorner\eta & \\ \Gamma_i\eta & = & e_i\wedge\eta + e_i\lrcorner\eta & i=1,\dots,4 \\ \Gamma_{5+i}\eta & = & ie_i\wedge\eta - ie_i\lrcorner\eta & i=1,\dots,4 \end{array}$$

- These gamma matrices are chosen so that Γ_j for $j=1,\dots,9$ are hermitian and Γ_0 is anti-hermitian with respect to the inner product

$$< z^a e_a, w^b e_b >= \sum_{a=1}^{5}(z^a)^* w^a \ ,$$

This inner product can be extended from $U\otimes\mathbb{C}$ to ${}^c\Delta$.

We take Majorana-Weyl spinors $\psi \in \Delta_{16}^{+}$ fixed by $\psi = C * \psi$.

A basis (over $\mathbb{R}$) of Δ_{16}^{+} is given by

$$\{1 + e_{1234}, \quad i\big(1 - e_{1234}\big), \quad e_{ij} - \frac{1}{2}\epsilon_{ijpq}e_{pq}, \quad i\big(e_{ij} + \frac{1}{2}\epsilon_{ijpq}e_{pq}\big)$$
$$e_{j5} + \frac{1}{6}\epsilon_{jmnp}e_{mnp5}, \quad i\big(e_{j5} - \frac{1}{6}\epsilon_{jmnp}e_{mnp5}\big)\}$$

for $i, j, p, q, m, n = 1, 2, 3, 4$.

A IIB Killing spinor $\epsilon \in {}^{c}\Delta^{+}$ is given by

$$\epsilon = \psi_1 + i\psi_2$$

for $\psi_1, \psi_2 \in \Delta_{16}^{+}$.

- There is a $Spin(9,1)$ invariant inner product defined on ${}^c\Delta$ defined by

$$B(\epsilon_1, \epsilon_2) = < \Gamma_0 C * \epsilon_1, \epsilon_2 >$$

  B is skew-symmetric in ϵ_1, ϵ_2.

  B vanishes when restricted to ${}^c\Delta^+$ or ${}^c\Delta^-$.

- This defines a non-degenerate pairing $\mathcal{B} : {}^c\Delta^+ \otimes {}^c\Delta^- \to \mathbb{R}$ given by

$$\mathcal{B}(\epsilon, \xi) = \mathrm{Re}\ B(\epsilon, \xi)$$

Canonical forms of spinors

$Spin(9,1)$ has one type of orbit with stability subgroup $Spin(7) \ltimes \mathbb{R}^8$ in Δ^+_{16} [Figueroa-O'Farrill, Bryant].

To see this, decompose Δ^+_{16} as

$$\Delta^+_{16} = \mathbb{R} < 1 + e_{1234} > + \Lambda^1(\mathbb{R}^7) + \Delta_8 \ ,$$

$\mathbb{R} < 1 + e_{1234} >$ is the singlet generated by $1 + e_{1234}$

$\Lambda^1(\mathbb{R}^7)$ is the vector representation of $Spin(7)$ spanned by Majorana spinors associated with 2-forms in the directions e_1, e_2, e_3, e_4 and by $i(1 - e_{1234})$.

Δ_8 is the spin representation of $Spin(7)$ spanned by the remaining Majorana spinors of type $e_5 \wedge \eta'$ where η' is generated by odd forms in the directions e_1, e_2, e_3, e_4.

$Spin(7)$ acts transitively on the S^7 in Δ_8, with stability subgroup G_2, and G_2 acts transitively on the S^6 in $\Lambda^1(\mathbb{R}^7)$ with stability subgroup $SU(3)$ [Salamon]

Using these transitive actions, one can show that a single Majorana-Weyl spinor lies in the orbit of $1 + e_{1234}$. This spinor is $Spin(7) \ltimes \mathbb{R}^8$ invariant.

To see this, write the spinor ψ_1 as

$$\psi_1 = a(1 + e_{1234}) + \theta_1 + \theta_2 \ ,$$

with $a \in \mathbb{R}$, $\theta_1 \in \Lambda^1(\mathbb{R}^7)$ and $\theta_2 \in \Delta_8$

There are several cases to consider.

If $a \neq 0$, $\theta_2 = 0$, using the transitive action of $G_2 \subset Spin(7)$ on the S^6 in $\Lambda^1(\mathbb{R}^7)$, make a gauge transformation so that $\theta_1 = ib(1 - e_{1234})$, and hence

$$\psi_1 = a(1 + e_{1234}) + ib(1 - e_{1234}) = \sqrt{a^2 + b^2} e^{\arctan(\frac{b}{a})\Gamma_{16}} (1 + e_{1234})$$

So ψ_1 lies in the same orbit as $1 + e_{1234}$.

The other cases, for which $a \neq 0$ and $\theta_2 \neq 0$; and $a = 0$ can be dealt with similarly.

Having fixed ψ_1 to be proportional to $1 + e_{1234}$ using $Spin(9,1)$ gauge transformations, it remains to consider ψ_2.

By using $Spin(7)$ gauge transformations, which leave ψ_1 invariant, one can write

$$\psi_2 = b_1(1 + e_{1234}) + ib_2(1 - e_{1234}) + b_3(e_{15} + e_{2345})$$

There are various cases

i) $b_3 \neq 0$. Then we have (taking $\Gamma_+ = \frac{1}{\sqrt{2}}(\Gamma_5 + \Gamma_0)$):

$$\psi_2 = e^{-\frac{b_1}{2b_3}\Gamma_+\Gamma_6 + \frac{b_2}{2b_3}\Gamma_+\Gamma_1} b_3(e_{15} + e_{2345})$$

so using a $\mathbb{R}^8 \in Spin(7) \ltimes \mathbb{R}^8$ gauge transformation one can take

$$\psi_2 = g(e_{15} + e_{2345})$$

The stability subgroup of $Spin(9,1)$ which leaves ψ_1 and ψ_2 invariant is G_2.

ii) If $b_3 = 0$ then

$$\psi_2 = g_1(1 + e_{1234}) + ig_2(1 - e_{1234})$$

and the stability subgroup is $SU(4) \ltimes \mathbb{R}^8$

iii) If $b_2 = b_3 = 0$ then

$$\psi_2 = g(1 + e_{1234})$$

and the stability subgroup is $Spin(7) \ltimes \mathbb{R}^8$.

$N = 31$ Solutions: Algebraic Conditions

Suppose that there exists a solution with exactly (and no more than) 31 linearly independent Killing spinors over $\mathbb{R}$.

Consider the algebraic condition

$$P_M \Gamma^M (C * \epsilon^r) + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon^r = 0$$

where ϵ^r are Killing spinors for $r = 1, \ldots, 31$.

The space of Killing spinors is orthogonal to a single *normal spinor*, $\nu \in \Delta_c^-$ with respect to the $Spin(9,1)$ invariant inner product $\mathcal{B}$. Using $Spin(9,1)$ gauge transformations, this normal spinor can be brought into one of 3 canonical forms:

$$\begin{aligned} Spin(7) \ltimes \mathbb{R}^8 : &\qquad \nu = (n + im)(e_5 + e_{12345})\,, \\ SU(4) \ltimes \mathbb{R}^8 : &\qquad \nu = (n - \ell + im) e_5 + (n + \ell + im) e_{12345}\,, \\ G_2 : &\qquad \nu = n(e_5 + e_{12345}) + im(e_1 + e_{234})\,, \end{aligned}$$

In general, one can write

$$\epsilon^r = \sum_{i=1}^{32} f^r{}_i \eta^i$$

where $f^r{}_i$ are real, η^p for $p = 1, \ldots, 16$ is a basis for Δ_{16}^+ and $\eta^{16+p} = i\eta^p$.

The matrix with components $f^r{}_i$ is of rank 31.

The functions $f^r{}_i$ are related by the orthogonality condition $\mathcal{B}(\epsilon^r, \nu) = 0$

For example, take the case for which $\nu = (n + im)(e_5 + e_{12345})$: set

$$\epsilon^r = f^r{}_1(1 + e_{1234}) + f^r{}_{17} i(1 + e_{1234}) + f^r{}_k \eta^k$$

where η^k the remaining (even form spinor) basis elements orthogonal (w.r.t the Dirac inner product $\langle , \rangle$) to $1 + e_{1234}, i(1 + e_{1234})$.

Then the orthogonality relation $\mathcal{B}(\epsilon^r, \nu) = 0$ implies

$$n{f^r}_1 - m{f^r}_{17} = 0$$

and so, taking without loss of generality $n \neq 0$; one finds

$$\epsilon^r = \frac{{f^r}_{17}}{n}(m + in)(1 + e_{1234}) + {f^r}_k \eta^k$$

Substituting this back into the algebraic Killing spinor equation gives

$$P_M \Gamma^M C*[(m+in)(1+e_{1234})] + \frac{1}{24} G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} (m+in)(1+e_{1234}) = 0$$

and

$$P_M \Gamma^M \eta^p = 0, \qquad G_{M_1 M_2 M_3} \Gamma^{M_1 M_2 M_3} \eta^p = 0, \quad p = 2, \ldots, 16$$

Analogous equations are obtained for $SU(4) \ltimes \mathbb{R}^8$ and G_2 invariant normals.

In all cases, the conditions $P_M \Gamma^M \eta^p = 0$ fix $P = 0$.

This means that the algebraic Killing spinor equation is linear over $\mathbb{C}$, so if there is a background with $N = 31$ linearly independent solutions of the algebraic Killing spinor equation, then this equation must have 32 linearly independent solutions.

This in turn fixes $G = 0$. However, if $G = 0$ then the gravitino Killing spinor equation also becomes linear over $\mathbb{C}$.

In this case, if the gravitino Killing spinor equation has 31 linearly independent solutions, it must have 32 solutions also. So the background is maximally supersymmetric.

$N = 30$ Solutions: Algebraic Conditions

Having excluded $N = 31$ solutions, consider $N = 30$.

To simplify the analysis, we use the homogeneity theorem of Figueroa O'Farrill, Hackett-Jones and Moutsopoulos.

This states that all solutions with $N > 16$ linearly independent Killing spinors are homogeneous, and hence have $P = 0$.

So, for $N = 30$ solutions, the algebraic Killing spinor equation becomes linear over $\mathbb{C}$:

$$\frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0$$

To analyse the case of $N = 30$ solutions, note that the spinors satisfying the algebraic KSE are all orthogonal to a normal spinor $\nu \in \Delta_c^-$ with respect to the inner product B.

This can be brought into canonical form using gauge transformations.

$$\begin{aligned} Spin(7) \ltimes \mathbb{R}^8 : \qquad & \nu = (n + im)(e_5 + e_{12345})\,, \\ SU(4) \ltimes \mathbb{R}^8 : \qquad & \nu = (n - \ell + im)e_5 + (n + \ell + im)e_{12345}\,, \\ G_2 : \qquad & \nu = n(e_5 + e_{12345}) + im(e_1 + e_{234})\,, \end{aligned}$$

The solutions to the algebraic Killing spinor equation are

$$\epsilon^r = \sum_{s=1}^{15} z^r{}_s \eta^s \ ,$$

where $\{\eta^i\}$ is a basis of Δ_c^+ normal to ν (with respect to B), and z is an invertible 15×15 matrix of spacetime dependent complex functions.

There are three cases to consider, corresponding to the types of normal spinor ν.

In all cases, one can choose the basis $\{\eta^i\}$ to have 13 (very simple) common elements, which are orthogonal to ν: $e_{pq}, e_{15pq}, e_{1p}, e_{1q}$ for $p = 2, 3, 4$ and $e_{15} - e_{2345}$.
The remaining two basis elements are case-dependent

$$
\begin{aligned}
Spin(7) \ltimes \mathbb{R}^8 : &\qquad 1 - e_{1234}, e_{15} + e_{2345}\,, \\
SU(4) \ltimes \mathbb{R}^8 : &\qquad e_{15} + e_{2345}, (n - \ell + im)1 - (n + \ell + im)e_{1234}\,, \\
G_2 : &\qquad 1 - e_{1234}, m(1 + e_{1234}) + in(e_{15} + e_{2345})
\end{aligned}
$$

In all cases, evaluating the algebraic Killing spinor equation on the basis $\{\eta^i\}$ produces sufficient conditions to fix $G = 0$.

Integrability Conditions for N=30 Solutions

It remains to consider the integrability conditions of the Killing spinor equations for solutions with $G = P = 0$.

The curvature $\mathcal{R} = [\mathcal{D}, \mathcal{D}]$ of the covariant connection $\mathcal{D}$ of IIB supergravity can be expanded as

$$\mathcal{R}_{MN} = \frac{1}{2}(T^2_{MN})_{PQ}\Gamma^{PQ} + \frac{1}{4!}(T^4_{MN})_{Q_1\dots Q_4}\Gamma^{Q_1\dots Q_4} \ ,$$

where

$$\begin{aligned}
(T^2_{MN})_{P_1P_2} &= \tfrac{1}{4}R_{MN,P_1P_2} - \tfrac{1}{12}F_{M[P_1}{}^{Q_1Q_2Q_3}F_{|N|P_2]Q_1Q_2Q_3} \ , \\
(T^4_{MN})_{P_1\dots P_4} &= \tfrac{i}{2}D_{[M}F_{N]P_1\dots P_4} + \tfrac{1}{2}F_{MNQ_1Q_2[P_1}F_{P_2P_3P_4]}{}^{Q_1Q_2}
\end{aligned}$$

The T^2 and T^4 tensors satisfy various algebraic conditions, following from the Bianchi identities and field equations:

$$
\begin{aligned}
(T^2_{MN})_{P_1P_2} &= (T^2_{P_1P_2})_{MN} \ , \\
(T^2_{M[P_1})_{P_2P_3]} &= 0 \ , \\
(T^2_{MN})_P{}^N &= 0 \ , \\
(T^4_{[P_1P_2})_{P_3P_4P_5P_6]} &= 0 \\
(T^4_{MN})_{P_1P_2P_3}{}^N &= 0 \ , \\
(T^4_{M[P_1})_{P_2P_3P_4P_5]} &= -\frac{1}{5!}\epsilon_{P_1P_2P_3P_4P_5}{}^{Q_1Q_2Q_3Q_4Q_5}(T^4_{M[Q_1})_{Q_2Q_3Q_4Q_5]} \ .
\end{aligned}
$$

And $(T^4{}_{P_1(M})_{N)P_2P_3P_4}$ is totally antisymmetric in P_1, P_2, P_3, P_4.

Analysis of Conditions

The integrability conditions of the gravitino Killing spinor equations

$$\mathcal{R}\epsilon^r = 0$$

To analyse this condition, note that $\mathcal{R}\epsilon^r = 0$, implies

$$\mathcal{R}_{MN,ab'} = u_{MN,r}\eta^r_a \nu_{b'} + u_{MN}\chi_a \nu_{b'}$$

where u are complex valued, and $\{\eta^r, \chi\}$ is a basis for Δ_c^+ in which $\{\eta^r\}$ is a basis for the space of Killing spinors.

a are indices on *even* Weyl spinors, b' are indices on *odd* Weyl spinors.

Indices on (even products) of Gamma matrices are lowered as

$$(\Gamma^{A_1 A_2 \ldots A_{2k}})_{ab'} = -i(\Gamma_{06789})_{cb'}(\Gamma^{A_1 A_2 \ldots A_{2k}})_a{}^c$$

We also have the formula

$$\psi_a \nu_{b'} = -\frac{1}{16} \sum_{k=0}^{2} \frac{1}{(2k)!} B(\psi, \Gamma_{A_1 A_2 \ldots A_{2k}} \nu)(\Gamma^{A_1 A_2 \ldots A_{2k}})_{ab'} \ ,$$

for any positive chirality spinor ψ.

Note that $\mathcal{R}_{MN}$ consists of 2-Γ and 4-Γ terms whose trace over the even Weyl spinor indices vanishes,

$$(\Gamma_{MN})_a{}^a, \quad (\Gamma_{MNPQ})_a{}^a$$

Requiring that $\mathcal{R}_{MN,a}{}^a = 0$ imposes

$$u_{MN} B(\chi, \nu) = 0$$

which eliminates the contribution to $\mathcal{R}_{MN,ab'}$ from u_{MN}.

Hence we are left with

$$\begin{aligned}
\mathcal{R}_{MN,ab'} &= u_{MN,r}\eta^r_a \nu_{b'} \\
&= -\frac{1}{16} u_{MN,r} \sum_{k=1}^{2} \frac{1}{(2k)!} B(\eta^r, \Gamma_{A_1 A_2 \ldots A_{2k}} \nu)(\Gamma^{A_1 A_2 \ldots A_{2k}})_{ab'}
\end{aligned}$$

which in turn relates T^2, T^4 to $u_{MN,r}$ via

$$\begin{aligned}
(T^2_{MN})_{A_1 A_2} &= -\frac{1}{16} u_{MN,r} B(\eta^r, \Gamma_{A_1 A_2} \nu) \\
(T^4_{MN})_{A_1 A_2 A_3 A_4} &= -\frac{1}{16} u_{MN,r} B(\eta^r, \Gamma_{A_1 A_2 A_3 A_4} \nu)
\end{aligned}$$

The method is then as follows

- Determine all components of T^2 and T^4 in terms of $u_{MN,r}$
- Translate the T^2 and T^4 conditions into conditions on u
- After some moderately unpleasant linear algebra, one finds that these are sufficient to fix $u_{MN,r} = 0$.
- This then implies that $T^2 = 0$, $T^4 = 0$.
- However these are equivalent (together with $P = 0, G = 0$) to the conditions on maximally supersymmetric backgrounds.

So all $N = 30$ solutions are (locally) maximally supersymmetric.

$N = 29$ Solutions

Solutions with exactly $N = 29$ linearly independent Killing spinors are excluded as follows:

- As $P = 0$, the algebraic Killing spinor eqns are linear over $\mathbb{C}$.
- So a background with $N = 29$ linearly independent solutions to the algebraic Killing spinor equation must have at least 30 solutions to this equation.
- By the $N = 30$ analysis, this is sufficient to fix $G = 0$
- As $G = 0$, the gravitino Killing spinor equation is linear over $\mathbb{C}$, and so an exactly $N = 29$ solution is excluded.

Conclusions

There are no solutions of IIB supergravity with exactly $N = 29$, $N = 30$ or $N = 31$ linearly independent Killing spinors

What about solutions with $N = 28$ supersymmetries? A non-trivial example is known - the plane wave geometry of Bena and Roiban.

In fact in order to have a solution with exactly 28 linearly independent Killing spinors, one is *forced* to take $G \neq 0$.

Analysis of the Killing spinor equation integrability conditions with $G \neq 0$ is *much* more complicated!

The gravitino integrability conditions are

$$\mathcal{S}\epsilon + \mathcal{T}(C\epsilon)^* = 0$$

where

$$\begin{aligned}
\mathcal{T} \quad = \quad & -\frac{\kappa}{96}(\Gamma_{[N}{}^{L_1L_2L_3}D_{M]}G_{L_1L_2L_3} + 9\Gamma^{L_1L_2}D_{[N}G_{M]L_1L_2}) \\
& + \frac{i\kappa^2}{32}(\frac{1}{3}F_{NM}{}^{L_1L_2L_3}G_{L_1L_2L_3} + \Gamma^{L_1L_2}F_{[N|L_1L_2}{}^{Q_1Q_2}G_{|M]Q_1Q_2} \\
& \quad + \frac{1}{3}\Gamma_{[N}{}^{Q}F_{M]Q}{}^{L_1L_2L_3}G_{L_1L_2L_3} - \frac{1}{2}\Gamma^{L_1\ldots L_4}F_{NML_1L_2}{}^{Q}G_{L_3L_4Q} \\
& \quad + \frac{1}{2}\Gamma_{[N}{}^{L_1L_2L_3}F_{M]L_1L_2}{}^{Q_1Q_2}G_{L_3Q_1Q_2} + \frac{1}{4}\Gamma^{L_1\ldots L_4}F_{L_1\ldots L_4}{}^{Q}G_{NMQ} \\
& \quad - \frac{1}{2}\Gamma_{[N|}{}^{L_1L_2L_3}F_{L_1L_2L_3}{}^{Q_1Q_2}G_{|M]Q_1Q_2}) \,.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{S} \quad = \quad & \frac{1}{8} R_{NM}{}^{L_1 L_2} \Gamma_{L_1 L_2} - \frac{1}{2} P_{[N} P^{\star}_{M]} + \frac{i\kappa}{48} \Gamma^{L_1 \ldots L_4} D_{[N} F_{M] L_1 \ldots L_4} \\
& + \frac{\kappa^2}{24} \big(-\Gamma^{L_1 L_2} F_{[N|L_1}{}^{Q_1 Q_2 Q_3} F_{|M] L_2 Q_1 Q_2 Q_3} + \frac{1}{2} \Gamma^{L_1 \ldots L_4} F_{NM L_1}{}^{Q_1 Q_2} F_{L_2 L_3 L_4 Q_1 Q_2} \\
& \qquad + \frac{1}{2} \Gamma_{[N}{}^{L_1 L_2 L_3} F_{M] L_1}{}^{Q_1 Q_2 Q_3} F_{L_2 L_3 Q_1 Q_2 Q_3} \big) \\
& + \frac{\kappa^2}{32} \big(-\frac{1}{2} G_{[N}{}^{L_1 L_2} G^{\star}_{M] L_1 L_2} + \frac{1}{48} \Gamma_{NM} G^{L_1 L_2 L_3} G^{\star}_{L_1 L_2 L_3} \\
& \quad - \frac{1}{4} \Gamma_{[N}{}^{L_1} G_{M]}{}^{L_2 L_3} G^{\star}_{L_1 L_2 L_3} + \frac{1}{8} \Gamma_{[N|}{}^{Q} G_{Q}{}^{L_1 L_2} G^{\star}_{|M] L_1 L_2} \\
& \quad + \frac{3}{16} \Gamma^{L_1 L_2} G_{NM}{}^{L_3} G^{\star}_{L_1 L_2 L_3} - \Gamma^{L_1 L_2} G_{[N|L_1}{}^{Q} G^{\star}_{|M] L_2 Q} \\
& \quad - \frac{3}{16} \Gamma^{L_1 L_2} G_{L_1 L_2}{}^{Q} G^{\star}_{NMQ} + \frac{1}{16} \Gamma_{NM}{}^{L_1 L_2} G_{L_1}{}^{Q_1 Q_2} G^{\star}_{L_2 Q_1 Q_2} \\
& \quad - \frac{1}{16} \Gamma^{L_1 \ldots L_4} G_{L_1 L_2 L_3} G^{\star}_{NM L_4} + \frac{1}{8} \Gamma_{[N|}{}^{L_1 L_2 L_3} G_{L_1 L_2}{}^{Q} G^{\star}_{|M] L_3 Q} \\
& \quad + \frac{1}{4} \Gamma^{L_1 \ldots L_4} G_{[N|L_1 L_2} G^{\star}_{|M] L_3 L_4} + \frac{1}{16} \Gamma^{L_1 \ldots L_4} G_{NM L_1} G^{\star}_{L_2 L_3 L_4} \\
& \quad + \frac{1}{4} \Gamma_{[N|}{}^{L_1 L_2 L_3} G_{|M] L_1}{}^{Q} G^{\star}_{L_2 L_3 Q} + \frac{1}{24} \Gamma_{[N|}{}^{L_1 \ldots L_5} G_{|M] L_1 L_2} G^{\star}_{L_3 L_4 L_5} \\
& \quad - \frac{1}{48} \Gamma_{[N|}{}^{L_1 \ldots L_5} G_{L_1 L_2 L_3} G^{\star}_{|M] L_4 L_5} - \frac{1}{32} \Gamma_{NM}{}^{L_1 \ldots L_4} G_{L_1 L_2}{}^{Q} G^{\star}_{L_3 L_4 Q} \\
& \quad - \frac{1}{288} \Gamma_{NM}{}^{L_1 \ldots L_6} G_{L_1 L_2 L_3} G^{\star}_{L_4 L_5 L_6} \big)
\end{aligned}
$$

One can show that the Bena and Roiban plane wave is the unique solution with $N=28$ supersymmetries:

$$
\begin{aligned}
ds^2 &= 2dw(dv - (\frac{9}{8} + 2h^2)\delta_{ij}x^i x^j dw) + \delta_{ij}dx^i dx^j \\
G &= -2\sqrt{2}ie^{i\phi}dw \wedge (dx^{15} + dx^{26} + dx^{37} + dx^{48}) \\
F &= 2hdw \wedge (dx^{1256} - dx^{3478})
\end{aligned}
$$

All homogeneous solutions with $N > 16$ linearly independent Killing vectors could (in principle) be classified using similar methods.

N.B. It has also been shown [Gran, JG, Papadopoulos, Roest], that there are no $N=31$ or $N=30$ solutions in D=11 supergravity.

<u>The Classical Lichnerowicz Theorem</u>

Suppose that N is a spin compact manifold. Then the following identity holds:

$$\int_N \langle \Gamma^i \nabla_i \epsilon, \Gamma^j \nabla_j \epsilon \rangle = \int_N \langle \nabla_i \epsilon, \nabla^i \epsilon \rangle + \frac{1}{4} \int_N R \langle \epsilon, \epsilon \rangle$$

where ∇ is the Levi-Civita connection, $\langle , \rangle$ is the Dirac inner product, R is the Ricci scalar.

Note: this uses $\Gamma_i = (\Gamma_i)^\dagger$, and also

$$\Gamma^{ij} \nabla_i \nabla_j \epsilon = -\frac{1}{4} R \epsilon$$

An alternative approach: if ϵ satisfies the Dirac equation $\Gamma^i \nabla_i \epsilon = 0$ then

$$\nabla^i \nabla_i \parallel \epsilon \parallel^2 = \frac{1}{2} R \parallel \epsilon \parallel^2 + 2\langle \nabla_i \epsilon, \nabla^i \epsilon \rangle$$

Consequences:

- If $R > 0$ then the Dirac operator has no zero modes.
- If $R = 0$ then the zero modes of the Dirac operator are also parallel.

Question: can we generalize the classical Lichnerowicz theorem to supergravity backgrounds with flux?

Gaussian Null Co-ordinates

We consider the event horizon of a supersymmetric extremal D=11 BH

We assume that the spacetime is stationary, and contains a regular black hole event horizon, which is a Killing horizon of a Killing vector W.

To analyse the near-horizon geometry, we introduce a co-odinate system adapted to the horizon.

These *Gaussian null co-ordinates* are higher dimensional generalization of the Eddington-Finkelstein co-ordinates

We assume the future event horizon $\mathcal{H}^+$ has a single connected component.

Let Σ be a Cauchy surface for the exterior of the black hole, with a boundary $\mathcal{S}$ on the future event horizon.

The Gaussian null co-ordinates are $\{u, r, y^I\}$.

[Isenberg $\&$ Moncrief, Friedrich, Racz, Wald]

Here $W = \frac{\partial}{\partial u}$ is the black hole Killing vector.

As W is an isometry, there is no dependence on u in the metric.

r denotes the radial distance away from the event horizon.

The event horizon is at $r = 0$, which is a null hypersurface.

The $\{y^I\}$, $I = 1, \ldots, 9$ are local co-ordinates on $\mathcal{S}$.

In the Guassian null co-ordinates the metric is

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + \gamma_{IJ}dy^I dy^J$$

$$\mathbf{e}^+ = du, \qquad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du, \quad \mathbf{e}^i = e^i{}_J dy^J$$

Here

$$h = h_I(r,y)dy^I, \quad \Delta = \Delta(r,y), \quad \gamma_{IJ} = \gamma_{IJ}(r,y)$$

are u-independent 1-form, scalar, and metric, on $\mathcal{S}$ which depend analytically on r.

The Near-Horizon Limit

Having obtained the metric

$$ds^2 = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + \gamma_{IJ}dy^I dy^J$$

with **near horizon data** $\{\Delta, h_I, \gamma_{IJ}\}$, which are u-independent and analytic in r, we can take the near-horizon limit [Reall et al.] by setting

$$r \to \ell r, \qquad u \to \ell^{-1}u$$

and then taking the limit $\ell \to 0$.

In this limit, the form of the metric is preserved, but now the near-horizon data depend only on y, and not on r.

The limit exists only for extremal black holes.

The limit decouples the bulk geometry from the near-horizon region.

The spatial cross-sections of the horizon $\mathcal{S}$, equipped with metric γ_{IJ}, are assumed to be smooth and compact without boundary.

Example: Warped Product AdS_2 Geometries

Take the near-horizon limit metric:

$$ds^2 = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + ds^2(\mathcal{S})$$

Choose the following near-horizon data:

$$h = d\Phi, \qquad \Delta = e^\Phi L$$

for constant L, where Φ is a smooth function on $\mathcal{S}$, and make the co-ordinate transformation

$$r = e^{-\Phi}\rho$$

The metric is then

$$ds^2 = 2e^{-\Phi}du(d\rho - \frac{1}{2}L^2\rho^2 du) + ds^2(\mathcal{S})$$

where

$$ds_2^2 = 2du(d\rho - \frac{1}{2}L^2\rho^2 du)$$

is the metric on AdS_2. So $AdS_2 \times_w \mathcal{S}$ is a near-horizon geometry.

$D = 11$ Supergravity

It is expected that there are many black hole solutions in M-Theory

The IIA Newton constant increases quadratically with string coupling.

As the IIA string coupling becomes large, the strength of the gravitational force increases and IIA matter collapses to black holes.

But the strong coupling limit of IIA string theory is conjectured to be M-Theory, which has as an effective theory D=11 supergravity.

The bosonic content of D=11 supergravity is the metric g and 4-form F.

We assume that all the components of F are independent of u and are analytic in r in the Gaussian null co-ordinates.

Then, on taking the near-horizon limit and imposing the Bianchi identity

$$dF = 0$$

one finds that

$$F = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + r\mathbf{e}^+ \wedge (dY - h \wedge Y) + X$$

where $Y \in \Lambda^2(\mathcal{S})$ and $X \in \Lambda^4(\mathcal{S})$ are u, r-independent 2 and 4-forms on the horizon sections.

The 4-form X must be closed as a consequence of the Bianchi identity:

$$dX = 0$$

The Field Equations

Before analyzing the conditions imposed by supersymmetry, we consider the gauge and Einstein field equations.

The D=11 gauge field equations are

$$d \star_{11} F - \frac{1}{2} F \wedge F = 0$$

These decompose into the following conditions on the 2-form Y and 4-form X on $\mathcal{S}$:

$$\begin{aligned} \tilde{\nabla}^i X_{i\ell_1\ell_2\ell_3} + 3\tilde{\nabla}_{[\ell_1} Y_{\ell_2\ell_3]} &= 3h_{[\ell_1} Y_{\ell_2\ell_3]} + h^i X_{i\ell_1\ell_2\ell_3} \\ &- \frac{1}{48} \epsilon_{\ell_1\ell_2\ell_3}{}^{q_1q_2q_3q_4q_5q_6} Y_{q_1q_2} X_{q_3q_4q_5q_6} \end{aligned}$$

and and

$$\tilde{\nabla}^j Y_{ji} - \frac{1}{1152} \epsilon_i{}^{q_1q_2q_3q_4q_5q_6q_7q_8} X_{q_1q_2q_3q_4} X_{q_5q_6q_7q_8} = 0$$

Here $\tilde{\nabla}$ is the Levi-Civita connection of the metric on $\mathcal{S}$.

The D=11 Einstein equations are:

$$R_{MN} = \frac{1}{12} F_{ML_1L_2L_3} F_N{}^{L_1L_2L_3} - \frac{1}{144} g_{MN} F_{L_1L_2L_3L_4} F^{L_1L_2L_3L_4} \ .$$

From the i, j frame component of the Einstein equations:

$$\begin{aligned} \tilde{R}_{ij} + \tilde{\nabla}_{(i} h_{j)} - \frac{1}{2} h_i h_j &= -\frac{1}{2} Y_{i\ell} Y_j{}^{\ell} + \frac{1}{12} X_{i\ell_1\ell_2\ell_3} X_j{}^{\ell_1\ell_2\ell_3} \\ &+ \delta_{ij} \left(\frac{1}{12} Y_{\ell_1\ell_2} Y^{\ell_1\ell_2} - \frac{1}{144} X_{\ell_1\ell_2\ell_3\ell_4} X^{\ell_1\ell_2\ell_3\ell_4} \right) , \end{aligned}$$

where $\tilde{R}_{ij}$ is the Ricci tensor of $\mathcal{S}$.

From the $+-$ component of the Einstein equations:

$$\tilde{\nabla}^i h_i = 2\Delta + h^2 - \frac{1}{3} Y_{\ell_1\ell_2} Y^{\ell_1\ell_2} - \frac{1}{72} X_{\ell_1\ell_2\ell_3\ell_4} X^{\ell_1\ell_2\ell_3\ell_4}$$

The $++$, $+i$ components of the Einstein equations are implied by these conditions.

Supersymmetric Near-Horizons

The Killing spinor equations of $D = 11$ supergravity are:

$$\nabla_M \epsilon + \left(-\frac{1}{288} \Gamma_M{}^{L_1 L_2 L_3 L_4} F_{L_1 L_2 L_3 L_4} + \frac{1}{36} F_{M L_1 L_2 L_3} \Gamma^{L_1 L_2 L_3} \right) \epsilon = 0$$

ϵ is a Majorana spinor

We solve the KSE first by integrating up the components in the light-cone directions $+$ and $-$.

This is possible, because all of the bosonic fields are u-independent and the dependence on r is explicit.

Having done this, we evaluate the remaining components of the KSE along the horizon section directions.

To solve the KSE along the light-cone directions, decompose the Killing spinor into positive and negative (light-cone) chirality parts:

$$\epsilon = \epsilon_+ + \epsilon_- \ , \quad \Gamma_\pm \epsilon_\pm = 0$$

On integrating up the light-cone components of the KSE, this gives

$$\epsilon_+ = \eta_+, \qquad \epsilon_- = \eta_- + r\Gamma_- \Theta_+ \eta_+$$

and

$$\eta_+ = \phi_+ + u\Gamma_+ \Theta_- \phi_-, \qquad \eta_- = \phi_-$$

where

$$\Theta_\pm = \left(\frac{1}{4} h_i \Gamma^i + \frac{1}{288} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{12} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right)$$

The spinors $\phi_\pm = \phi_\pm(y)$ do not depend on u, r.

Some algebraic conditions are also imposed by the "+", "-" KSE components

$$\Big(\frac{1}{2}\Delta - \frac{1}{8}dh_{ij}\Gamma^{ij} + \frac{1}{72}d_h Y_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}$$

$$+ \quad 2\big(\frac{1}{4}h_i\Gamma^i - \frac{1}{288}X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{12}Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2}\big)\Theta_+\Big)\eta_+ = 0$$

$$\Big(\frac{1}{4}\Delta h_i\Gamma^i - \frac{1}{4}\partial_i\Delta\Gamma^i + \big(-\frac{1}{8}dh_{ij}\Gamma^{ij} - \frac{1}{24}d_h Y_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}\big)\,\Theta_+\Big)\eta_+ = 0$$

$$\Big(-\frac{1}{2}\Delta - \frac{1}{8}dh_{ij}\Gamma^{ij} + \frac{1}{24}d_h Y_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}$$

$$+ \quad 2\big(-\frac{1}{4}h_n\Gamma^n + \frac{1}{288}X_{n_1n_2n_3n_4}\Gamma^{n_1n_2n_3n_4} + \frac{1}{12}Y_{n_1n_2}\Gamma^{n_1n_2}\big)\Theta_-\Big)\phi_- = 0$$

However, these are actually redundant...!

The remaining "spatial" components of the KSE (i.e. along the directions of $\mathcal{S}$) imply

$$\tilde{\nabla}_i \eta_+ + \Big(-\frac{1}{4} h_i - \frac{1}{288} \Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4} X_{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{36} X_{i\ell_1\ell_2\ell_3} \Gamma^{\ell_1\ell_2\ell_3} + \frac{1}{24} \Gamma_i{}^{\ell_1\ell_2} Y_{\ell_1\ell_2} - \frac{1}{6} Y_{ij} \Gamma^j \Big) \eta_+ = 0$$

and

$$\tilde{\nabla}_i \phi_- + \Big(\frac{1}{4} h_i - \frac{1}{288} \Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4} X_{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{36} X_{i\ell_1\ell_2\ell_3} \Gamma^{\ell_1\ell_2\ell_3} - \frac{1}{24} \Gamma_i{}^{\ell_1\ell_2} Y_{\ell_1\ell_2} + \frac{1}{6} Y_{ij} \Gamma^j \Big) \phi_- = 0$$

$\tilde{\nabla}$ is the supercovariant derivative on $\mathcal{S}$.

There is also an additional condition:

$$\tilde{\nabla}_i(\Theta_+\eta_+) + \left(-\frac{1}{2}h_i + \frac{1}{4}\Gamma_i{}^{\ell}h_\ell - \frac{1}{24}X_{i\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} + \frac{1}{8}\Gamma_i{}^{\ell_1\ell_2}Y_{\ell_1\ell_2}\right)(\Theta_+\eta_+)$$
$$+\left(\frac{1}{4}\Delta\Gamma_i - \frac{1}{16}\Gamma_i{}^{\ell_1\ell_2}dh_{\ell_1\ell_2} - \frac{3}{8}dh_{i\ell}\Gamma^\ell - \frac{1}{48}d_hY_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}\Gamma_i\right)\eta_+ = 0 \ ,$$

Using the KSE for $\tilde{\nabla}\eta_+$, this can be rewritten as a purely algebraic condition.

Simplifying the Killing Spinor Equations

Using purely **local** calculations, making extensive use of the field equations, the Killing spinor equations can be reduced to:

$$\nabla_i^{(\pm)}\phi_\pm \equiv \tilde{\nabla}_i\phi_\pm + \Psi_i^{(\pm)}\phi_\pm = 0$$

Note: the $\pm$ in $\nabla_i^{(\pm)}$ corresponds to the lightcone chirality of the spinor $\phi_\pm$, and is *not* a spacetime index on the covariant derivative.

The $\Psi_i^{(\pm)}$ is an algebraic operator:

$$\begin{aligned}\Psi_i^{(\pm)} &= \mp\frac{1}{4}h_i - \frac{1}{288}\Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4}X_{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{36}X_{i\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} \\ &\pm \frac{1}{24}\Gamma_i{}^{\ell_1\ell_2}Y_{\ell_1\ell_2} \mp \frac{1}{6}Y_{ij}\Gamma^j \ .\end{aligned}$$

Key property of solutions:

If ϕ_- satisfies

$$\nabla_i^{(-)}\phi_- = 0$$

Then ϕ'_+ defined by

$$\phi'_+ \equiv \Gamma_+ \Theta_- \phi_-$$

automatically satisfies (again from a purely *local* analysis)

$$\nabla_i^{(+)}\phi'_+ = 0$$

In principle, this provides a way to generate ϕ_+ spinors from ϕ_- solutions!

But: We need to know more about $\mathrm{Ker}(\Theta_-)$ to understand the counting better...

Global Analysis I: Properties of $\mathrm{Ker}(\Theta_-)$

Suppose that $\phi_- \not\equiv 0$, and satisfies $\nabla_i^{(-)}\phi_- = 0$, $\Theta_-\phi_- = 0$.

Then one of the algebraic KSE conditions implies

$$\langle \phi_-, \Big(-\frac{1}{2}\Delta - \frac{1}{8} dh_{ij}\Gamma^{ij} + \frac{1}{24} d_h Y_{\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} \Big)\phi_- \rangle = 0$$

which then implies

$$\Delta = 0$$

Next, the condition $\nabla_i^{(-)}\phi_- = 0$ implies

$$\tilde{\nabla}_i \langle \phi_-, \phi_- \rangle = -\frac{1}{2} h_i \langle \phi_-, \phi_- \rangle + \langle \phi_-, \Big(\frac{1}{144} \Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4} X_{\ell_1\ell_2\ell_3\ell_4} - \frac{1}{3} Y_{ij}\Gamma^j \Big)\phi_- \rangle$$

Using the condition $\Theta_- \phi_- = 0$, this simplifies further to

$$\tilde{\nabla}_i \langle \phi_-, \phi_- \rangle = -h_i \langle \phi_-, \phi_- \rangle$$

As $\phi_- \not\equiv 0$, this implies

$$dh = 0$$

The "$++$" component of the Einstein equation then implies

$$dY - h \wedge Y = 0$$

Returning to the condition:

$$d \parallel \phi_- \parallel^2 = - \parallel \phi_- \parallel^2 h$$

Take the divergence of this and use the "+-" component of the Einstein equations to eliminate the $\tilde{\nabla}_i h^i$ term:

$$\tilde{\nabla}^2 \parallel \phi_- \parallel^2 = \left(\frac{1}{3} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} + \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} \right) \parallel \phi_- \parallel^2$$

On integrating both sides of this expression over $\mathcal{S}$, the contribution from the LHS vanishes, leaving

$$\int_{\mathcal{S}} \left(\frac{1}{3} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} + \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} \right) \parallel \phi_- \parallel^2$$

So

$$Y = 0, \qquad X = 0$$

Finally, substitute $Y = 0$, $X = 0$ back into the "+-" component of the Einstein equations:

$$\tilde{\nabla}^i h_i = h^2$$

Again, integrating both sides of this expression over $\mathcal{S}$ implies

$$h = 0$$

We have therefore found that if there is a nonzero ϕ_- spinor such that $\nabla_i^{(-)} \phi_- = 0$ with $\phi_- \in \mathrm{Ker}(\Theta_-)$, then

$$\Delta = 0, \quad h = 0, \quad Y = 0, \quad X = 0$$

In this case, the 4-form vanishes, and the non-zero vector field $W_i = \langle \phi_-, \Gamma_i \phi_- \rangle$ is parallel.

The spacetime is $\mathbb{R}^{1,1} \times S^1 \times M$ where M is a compact Ricci-flat 8-manifold.

We will not consider this case any further...

Global Analysis II: Horizon Dirac Equations

Given KSE (on $\mathcal{S}$) of the form $\nabla_i^{(\pm)}\phi_\pm = 0$, we define *horizon Dirac operators*

$$\mathcal{D}^{(\pm)} = \Gamma^i \nabla_i^{(\pm)} = \Gamma^i \tilde{\nabla}_i + \Psi^{(\pm)}$$

where

$$\Psi^{(\pm)} = \Gamma^i \Psi_i^{(\pm)} = \mp \frac{1}{4} h_\ell \Gamma^\ell + \frac{1}{96} X_{\ell_1\ell_2\ell_3\ell_4} \Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{8} Y_{\ell_1\ell_2} \Gamma^{\ell_1\ell_2} \ .$$

These Dirac operators, in addition to the Levi-Civita connection, also depend on the fluxes of $D = 11$ supergravity restricted to the horizon section $\mathcal{S}$.

Generalized Lichnerowicz Theorem for ϕ_+

Suppose that the spinor ϕ_+ satisfies the horizon Dirac equation

$$\mathcal{D}^{(+)}\phi_+ = 0$$

On, making (extensive) use of the bosonic field equations, it follows that

$$\tilde{\nabla}^2 \parallel \phi_+ \parallel^2 - h^i \tilde{\nabla}_i \parallel \phi_+ \parallel^2 = 2\langle \nabla^{(+)i}\phi_+, \nabla^{(+)}_i \phi_+ \rangle$$

We do not assume that $\nabla^{(+)}_i \phi_+ = 0$, or any of the other algebraic conditions on ϕ_+ at any point in this analysis...

On applying the Hopf maximum principle, assuming that $\mathcal{S}$ is smooth and compact without boundary, one finds

$$\parallel \phi_+ \parallel = const. \qquad \nabla^{(+)}_i \phi_+ = 0$$

Generalized Lichnerowicz Theorem for ϕ_-

Suppose that the spinor ϕ_- satisfies the horizon Dirac equation

$$\mathcal{D}^{(-)}\phi_- = 0$$

On making (extensive) use of the bosonic field equations, it follows that

$$\tilde{\nabla}^2 \parallel \phi_- \parallel^2 + \tilde{\nabla}^i \Big(\parallel \phi_- \parallel^2 h_i \Big) = 2\langle \nabla^{(-)i}\phi_-, \nabla_i^{(-)}\phi_- \rangle$$

On integrating both sides over $\mathcal{S}$, assuming that $\mathcal{S}$ is smooth and compact without boundary, one finds

$$\nabla_i^{(-)}\phi_- = 0$$

Note: in this case, $\parallel \phi_- \parallel$ need not be constant...

Index Theory: Supersymmetries of M-Horizons

The generalized Lichnerowicz theorems imply

$$\nabla_i^{(\pm)}\phi_\pm = 0 \Longleftrightarrow \mathcal{D}^{(\pm)}\phi_\pm = 0$$

We have decomposed the spin bundle S of $D = 11$ supergravity as

$$S = S_+ + S_-$$

using the projectors $\Gamma_\pm$.

Note that

$$\mathcal{D}^+ : \Gamma(S_+) \to \Gamma(S_+)$$
$$\mathcal{D}^{+\dagger} : \Gamma(S_+) \to \Gamma(S_+)$$

where $\Gamma(S_+)$ are smooth sections of S_+.

The operator $\mathcal{D}^{(+)}$ is defined on $\mathcal{S}$ which is an odd-dimensional manifold.

So it follows that the index of $\mathcal{D}^{(+)}$ vanishes [Atiyah].

Hence

$$\dim \ker \mathcal{D}^{(+)} = \dim \ker (\mathcal{D}^{(+)})^\dagger$$

We can also compute directly the adjoint operator:

$$(\mathcal{D}^{(+)})^\dagger = -\Gamma^i \tilde{\nabla}_i - \frac{1}{4} h_\ell \Gamma^\ell + \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2}$$

If we set $\phi'_+ = \Gamma_+ \phi_-$ then

$$(\mathcal{D}^{(+)})^\dagger \phi'_+ = \Gamma_+ \mathcal{D}^{(-)} \phi_-$$

This establishes a 1-1 correspondence between $\mathrm{Ker}(\mathcal{D}^{(+)\dagger})$ and $\mathrm{Ker}(\mathcal{D}^{(-)})$.

Hence

$$\dim\ \ker(\mathcal{D}^{(+)\dagger}) = \dim\ \ker(\mathcal{D}^{(-)})$$

We also have the index theory result

$$\dim \ker \mathcal{D}^{(+)} = \dim \ker (\mathcal{D}^{(+)})^{\dagger}$$

Combining these two results, the total amount of supersymmetry preserved is

$$N = \dim \ker \mathcal{D}^{(+)} + \mathrm{dimker} \mathcal{D}^{(-)} = 2 \dim \ker \mathcal{D}^{(-)}.$$

So the number of supersymmetries preserved is always even.

Symmetry Enhancement

A priori, near horizon geometries admit two Killing vector fields generated by $\frac{\partial}{\partial u}$ and $u\frac{\partial}{\partial u} - r\frac{\partial}{\partial r}$.

However, all known examples exhibit a larger symmetry algebra which always includes a $\mathfrak{sl}(2,\mathbb{R})$ subalgebra.

We shall prove that this is a generic property of M-horizons (and AdS_2 solutions) of M-horizons with non-trivial fluxes, and it arises as a direct consequence of supersymmetry.

This symmetry is dynamical in the sense that it emerges after using the field equations.

We begin by considering the general structure of the Killing spinors

The most general Killing spinor is:

$$\epsilon = \phi_+ + u\Gamma_+\Theta_-\phi_- + \phi_- + r\Gamma_-\Theta_+\phi_+ + ru\,\Gamma_-\Theta_+\Gamma_+\Theta_-\phi_-$$

We consider two Killing spinors, both generated by $\chi = \chi_-$.

- The first is generated by setting $(\phi_+, \phi_-) = (0, \chi)$.
- The second is generated by setting $(\phi_+, \phi_-) = (\Gamma_+\Theta_-\chi, 0)$

So

$$\epsilon_1 = \phi_- + u\phi_+ + ru\Gamma_-\Theta_+\phi_+ \ , \quad \epsilon_2 = \phi_+ + r\Gamma_-\Theta_+\phi_+$$

where we have set $\phi_- = \chi$, $\phi_+ = \Gamma_+\Theta_-\chi$.

For any two Killing spinors ζ_1 and ζ_2, the 1-form bilinear

$$K = \langle(\Gamma_+ - \Gamma_-)\zeta_1, \Gamma_A\zeta_2\rangle\, e^A$$

is dual to a Killing vector which also preserves the 4-form F.

The Killing spinors ϵ_1, ϵ_2 generate 3 Killing vectors:

$$\begin{aligned} K_1 = \langle (\Gamma_+ - \Gamma_-)\epsilon_1, \Gamma_A \epsilon_2 \rangle \, e^A &= (2r\langle \Gamma_+\phi_-, \Theta_+\phi_+ \rangle + r^2 u \Delta \parallel \phi_+ \parallel^2)\, \mathbf{e}^+ \\ &- 2u \parallel \phi_+ \parallel^2 \ \mathbf{e}^- + V_i \mathbf{e}^i \end{aligned}$$

$$K_2 = \langle (\Gamma_+ - \Gamma_-)\epsilon_2, \Gamma_A \epsilon_2 \rangle \, e^A = r^2 \Delta \parallel \phi_+ \parallel^2 \ \mathbf{e}^+ - 2 \parallel \phi_+ \parallel^2 \ \mathbf{e}^-$$

$$\begin{aligned} K_3 = \langle (\Gamma_+ - \Gamma_-)\epsilon_1, \Gamma_A \epsilon_1 \rangle \, e^A &= (2 \parallel \phi_- \parallel^2 + 4ru\langle \Gamma_+\phi_-, \Theta_+\phi_+ \rangle \\ &+ r^2 u^2 \Delta \parallel \phi_+ \parallel^2) \mathbf{e}^+ \\ &- 2u^2 \parallel \phi_+ \parallel^2 \ \mathbf{e}^- + 2u V_i \mathbf{e}^i \end{aligned}$$

where

$$V_i = \langle \Gamma_+\phi_-, \Gamma_i \phi_+ \rangle$$

To obtain these components for K_1, K_2, K_3 two identities are used:

- From one of the algebraic KSE conditions:

$$\Delta \parallel \phi_+ \parallel^2 = 4 \parallel \Theta_+ \phi_+ \parallel^2$$

- From the condition $\parallel \phi_+ \parallel = const.$ and $\nabla_i^{(+)} \phi_+ = 0$:

$$\langle \phi_+, \Gamma_i \Theta_+ \phi_+ \rangle = 0$$

The Geometry of $\mathcal{S}$

There are 2 cases, corresponding to $V \neq 0$ or $V = 0$.

If $V \neq 0$ then the conditions $\mathcal{L}_{K_a} g = 0$ and $\mathcal{L}_{K_a} F = 0$, $(a = 1, 2, 3)$ give

$$\tilde{\nabla}_{(i} V_{j)} = 0 \ , \quad \tilde{\mathcal{L}}_V h = 0 \ , \quad \tilde{\mathcal{L}}_V \Delta = 0 \ , \quad \tilde{\mathcal{L}}_V Y = 0 \ , \quad \tilde{\mathcal{L}}_V X = 0$$

So $\mathcal{S}$ admits an isometry generated by V, which leaves h, Δ, Y, X invariant.

Other identities imposed by the field equations and KSE are:

$$\begin{aligned} -2 \parallel \phi_+ \parallel^2 - h_i V^i + 2\langle \Gamma_+ \phi_-, \Theta_+ \phi_+ \rangle &= 0 \\ i_V (dh) + 2d\langle \Gamma_+ \phi_-, \Theta_+ \phi_+ \rangle &= 0 \\ 2\langle \Gamma_+ \phi_-, \Theta_+ \phi_+ \rangle - \Delta \parallel \phi_- \parallel^2 &= 0 \\ V + \parallel \phi_- \parallel^2 h + d \parallel \phi_- \parallel^2 &= 0 \end{aligned}$$

The geometry of $\mathcal{S}$ is restricted by the existence of a nowhere vanishing spinor ϕ_-.

The structure group reduces to $Spin(7)$.

The existence of additional spinor ϕ_+ reduces the structure group further.

There are various possibilities depending on the isotropy group associated with the second spinor- $Spin(7)$, $SU(4)$, G_2, $SU(3)$.

If $V=0$ then the group action generated by K_1, K_2, K_3 has 2-dimensional orbits.

The KSE/field equations imply that

$$\Delta \parallel \phi_- \parallel^2 = 2 \parallel \phi_+ \parallel^2 \quad , \quad h = \Delta^{-1} d\Delta$$

As h is exact, the solution is static.

On making a co-ordinate transformation $r = \Delta \rho$, the geometry becomes a warped produce $AdS_2 \times_w \mathcal{S}$.

There are also further restrictions on the fluxes; the "$++$" Einstein equation implies

$$dY - h \wedge Y = 0$$

The $\mathfrak{sl}(2,\mathbb{R})$ Symmetry

The vector fields dual to the 1-form bilinears K_1, K_2, K_3 are:

$$\begin{aligned}
K_1 &= -2u \parallel \phi_+ \parallel^2 \partial_u + 2r \parallel \phi_+ \parallel^2 \partial_r + V^i \tilde{\partial}_i \\
K_2 &= -2 \parallel \phi_+ \parallel^2 \partial_u \\
K_3 &= -2u^2 \parallel \phi_+ \parallel^2 \partial_u + (2 \parallel \phi_- \parallel^2 + 4ru \parallel \phi_+ \parallel^2)\partial_r + 2uV^i \tilde{\partial}_i
\end{aligned}$$

These satisfy the $\mathfrak{sl}(2,\mathbb{R})$ commutation relations

$$\begin{aligned}
[K_1, K_2] &= 2 \parallel \phi_+ \parallel^2 K_2 \\
[K_2, K_3] &= -4 \parallel \phi_+ \parallel^2 K_1 \\
[K_3, K_1] &= 2 \parallel \phi_+ \parallel^2 K_3
\end{aligned}$$

Exercise: Check these commutation relations.

Other Applications: D=10 Horizons/AdS solutions

The same construction works for supersymmetric extreme black hole near-horizons in D=10 supergravity.

- Locally, KSE decompose on $\mathcal{S}$ into a pair of parallel transport equations for pairs of spinors $\phi_\pm$, together with a pair of algebraic conditions

$$\nabla^{(\pm)}\phi_\pm = 0, \qquad \mathcal{A}^{(\pm)}\phi_\pm = 0$$

- The generalized Lichnerowicz theorems hold. Globally, if $\phi_\pm$ satisfy the associated "horizon Dirac equations" then the KSE hold – **both parallel transport and algebraic conditions**.

  e.g. if $\mathcal{D}^{(+)}\phi_+ = 0$ then

$$\tilde{\nabla}^2 \parallel \phi_+ \parallel^2 - h^i \tilde{\nabla}_i \parallel \phi_+ \parallel^2 = 2\langle \nabla^{(+)i}\phi_+, \nabla^{(+)}_i \phi_+ \rangle + \parallel \mathcal{A}^{(+)}\phi_+ \parallel^2$$

  and the Hopf maximum principle implies $\nabla^{(+)}\phi_+ = 0$ and $\mathcal{A}^{(+)}\phi_+ = 0$

- There is again a 1-1 correspondence between the kernels of $\mathcal{D}^{(-)}$, $\mathcal{D}^{(+)\dagger}$: If we set $\phi'_+ = \Gamma_+ \phi_-$ then

$$(\mathcal{D}^{(+)})^\dagger \phi'_+ = \Gamma_+ \mathcal{D}^{(-)} \phi_-$$

- The total number of supersymmetries is

$$N = N_+ + N_-$$

where

$$N_\pm = \dim\, \mathrm{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}) = \dim\, \mathrm{Ker}(\mathcal{D}^{(\pm)})$$

because of the generalized Lichnerowicz theorems.

- We have

$$\mathrm{Index}(\mathcal{D}^{(+)}) = \dim\, \mathrm{Ker}(\mathcal{D}^{(+)}) - \dim\, \mathrm{Ker}(\mathcal{D}^{(+)\dagger}) = N_+ - N_-$$

because $N_- = \dim\, \mathrm{Ker}(\mathcal{D}^{(-)}) = \dim\, \mathrm{Ker}(\mathcal{D}^{(+)\dagger})$.

Hence

$$N = 2N_- + \mathrm{Index}(\mathcal{D}^{(+)}) = 2N_+ - \mathrm{Index}(\mathcal{D}^{(+)})$$

One subtlety: in D=10, $\mathcal{S}$ is 8-dimensional, so the index need not vanish!

If $N_- \neq 0$ then the solution admits a $\mathfrak{sl}(2, \mathbb{R})$ symmetry.

This construction generalizes to all warped product $AdS_n \times_w M$ solutions in D=10, D=11 supergravities, where M is some compact, smooth internal space without boundary.

In all cases, generalized Lichnerowicz theorems hold, and the entire content of the Killing spinor equations is equivalent to finding zero modes of Dirac operators defined on M.