# Iterative Non-iterative Integrals in QFT

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DESY, Zeuthen

based on: J. Ablinger et al., Journ. Math. Phys. (June 2018),
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# Introduction

One of the main and difficult issues in high energy physics is the calculation of involved multi-dimensional integrals.
In the following our attitude will be their analytic integration.
For quite some classes of integrals, particularly at lower order in the coupling constant, quite a series of analytic computational methods exist.
cf. e.g. [arXiv:1509.08324] for the algorithms.

- Hypergeometric functions.
- Mellin-Barnes representations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].

# Introduction

- Summation methods based on difference fields, implemented in the `Mathematica` program `Sigma` [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion of the summands in $\varepsilon$.
  - Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  - Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].
- Systems of Differential Equations.
- Almkvist-Zeilberger Theorem as Integration Method. [Multi-Integration]

Recent survey: JB, C. Schneider, DESY 18-045, Int. J. Mod. Phys. **A** (2018).
In the following we will concentrate on the method of Differential Equations since these are automatically obtained from the integration-by-parts identities representing all integrals by the so-called master integrals.
These may either be considered directly or in terms of difference equations obtained through a formal power-series ansatz or a Mellin transform.

# Introduction

Starting from the most simple cases and moving to gradually more and more involved (massive) topologies one observes:

- The lower order topologies correspond to differential or difference equation systems which are first order factorizable.
- Here, a wider class of solution methods exists. There are methods in both cases to constructively find all letters of the alphabet needed to express the solutions in terms of indefinitely nested sums or iterative integrals.
- Later also differential or difference equations occur which contain genuine higher than 1st order factors.
- The first example are ${}_2F_1$ solutions. In special cases these are also elliptic solutions.
- In the latter case one may represent the solutions in terms of meromorphic modular forms and in more special cases in terms of holomorphic modular forms and therefore in polynomials of Lambert-Eisenstein series (elliptic polylogarithms).

# Introduction

Project: 3-loop massive OMEs and DIS structure functions for lager $Q^2$.

- The ${}_2F_1$ solutions there appear to be the same or very closely related to those of the $\rho$ parameter.
- Perform a thorough study for the latter case first.
- There is a lot of particular order in all these structures (although they look accidentally very different).

# Some historical aspects: Nested Sums & Iterative Integrals

$\mathrm{Li}_2(x)$, 1696 shuffle, 1775 iter. integrals $\{1/(x-a_i)\}$, int. over $\{1/x, 1/(1-x)\}$,
1840 [1884] Nielsen, 1909

## Indefinitely nested sums:

$$S(N) = \sum_{k_1=1}^{N} s(k_1) \sum_{k_2=1}^{k_1} s(k_2) \ldots \sum_{k_m=1}^{k_{m-1}} s(k_m)$$

## Iterated integrals:

$$F(x) = \int_0^x dy_1 f_1(y_1) \int_0^{y_1} dy_2 f_2(y_2) \ldots \int_0^{y_{l-1}} dy_l f_l(y_l)$$

## Mellin transform:

$$\sum_{\alpha} c_\alpha S_\alpha(N) = \int_0^1 dx x^{N-1} F(x)$$

... much more to say about the historic development, cf. e.g. J. Ablinger, JB, C. Schneider, 1304.7071, 1310.5645

# Function Spaces

**Sums**

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

**Integrals**

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on CIS fct.

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

**Special Numbers**

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx \; {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

**shuffle, stuffle, and various structural relations** $\Longrightarrow$ algebras

Except the last line integrals, all other ones stem from 1st order factorizable equations.

# The world until $\sim$ 1997

Calculations up to $\sim$ 2 Loops massless and single mass:

- Express all results in terms of $\mathrm{Li}_n(z) = \int_0^z dx Li_{n-1}(x)/x, Li_0(x) = x/(1-x).$,
- and possibly $S_{p,n}(z) = (-1)^{n+p-1}/(p!(n-1)!) \int_0^1 dx \ln^{p-1}(x) \ln^n(1-xz)/x.$
- The argument $z = z(x)$ becomes a more and more complicated function.
- covering algebras of wider function spaces were widely unknown in physics, despite they were known in mathematics ...
- The complexity of expressions grew significantly, calling urgently for mathematical extensions.
- More complex argument structures do not easily allow an analytic Mellin inversion.
- Extremely long expressions are obtained, which would be much more compact, using adequate mathematical functions.
- Somewhen, new functions appeared: $\int_0^z dx \mathrm{Li}_3(x)/(1+x)$ not fitting into this frame.

# Spill-Off: New Function Classes and Algebras

- 1998: Harmonic Sums [Vermaseren; JB]
- 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- 2000,2003, 2009: Analytic continuation of harmonic sums, systematic algebraic reduction; structural relations [JB]
- 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Schneider]
- 2014-: Elliptic integrals with (involved) rational arguments.

**Particle Physics Generates NEW Mathematics.**

# Decoupling of Systems

- We consider linear systems of $N$ inhomogeneous differential equations and decouple them into a single scalar equation $+$ $(N-1)$ other determining equations.
- Usually one may use a series ansatz ($+ \ln^k(x)$ modulation)

$$f(x) = \sum_{k=1}^{\infty} a(k) x^k$$

and obtain

$$\sum_{k=0}^{m} p_k(N) F(N+k) = G(N)$$

.

- The latter equation is now tried to be solved using difference-field techniques.
- If the equation has successive 1st order solutions one ends up with a nested sums solution. All these cases have been algorithmized. [arXiv:1509.08324 [hep-ph]].
- This even applies for some cases ending up elliptic in $x$-space [arXiv:1310.5645 [math-ph]].

## A didactical example :

3 loop QCD corrections to the $\rho$-parameter

# Master integrals for the $\rho$-parameter @ $O(\alpha_s^3)$

Example : One usually has **no** Gaussian differential equation, but something like Heun or more general, i.e. with more than 3 singularities.

$$\frac{d^2}{dx^2} f_{8a}(x) + \frac{9 - 30x^2 + 5x^4}{x(x^2-1)(9-x^2)} \frac{d}{dx} f_{8a}(x) - \frac{8(-3+x^2)}{(9-x^2)(x^2-1)} f_{8a}(x) = I_{8a}(x)$$

Homogeneous solutions:

$$\begin{aligned}
\psi_{1a}^{(0)}(x) &= \sqrt{2\sqrt{3}\pi} \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; z\right] \\
\psi_{2a}^{(0)}(x) &= \sqrt{2\sqrt{3}\pi} \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; 1-z\right],
\end{aligned}$$

with

$$z = z(x) = \frac{x^2(x^2-9)^2}{(x^2+3)^3} \ .$$

Use contiguous relations first to get into the ball-park. $\Longrightarrow$ at least two differently indexed ${}_2F_1$'s are going to appear. All classical ${}_2F_1$ wisdom is always applied first.

# When can $_2F_1$-Solutions be mapped to Complete Elliptic Integrals?

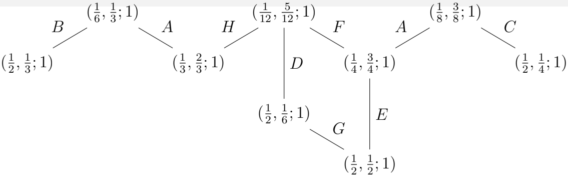



Figure 1: The transformation of special $_2F_1$ functions under the triangle group.

| $I$ | $d$ | $R$ | $f$ |
|---|---|---|---|
| $A$ | 2 | 1 | $4x(1-x)$ |
| $B$ | 2 | $(1-x)^{-1/6}$ | $\frac{1}{4}x^2/(x-1)$ |
| $C$ | 2 | $(1-x)^{-1/8}$ | $\frac{1}{4}x^2/(x-1)$ |
| $D$ | 2 | $(1-x)^{-1/12}$ | $\frac{1}{4}x^2/(x-1)$ |
| $E$ | 2 | $(1-x/2)^{-1/2}$ | $x^2/(x-2)^2$ |
| $F$ | 3 | $(1+3x)^{-1/4}$ | $27x(1-x)^2/(1+3x)^3$ |
| $G$ | 3 | $(1+\omega x)^{-1/2}$ | $1-(x+\omega)^3/(x+\overline{\omega})^3$ |
| $H$ | 4 | $(1-8x/9)^{-1/4}$ | $64x^3(1-x)/(9-8x)^3$ |

Table: The functions $R$ and $f$ for the different hypergeometric transformations of degree $d$; $\omega^2+\omega+1=0$.

$$_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = R(x)\, _2F_1\left[\begin{matrix} a', b' \\ c' \end{matrix}; f(x)\right]$$

# Master integrals for the $\rho$-parameter @ $O(\alpha_s^3)$

$$\frac{d^2}{dx^2} f_{8a}(x) + \frac{9-30x^2+5x^4}{x(x^2-1)(9-x^2)} \frac{d}{dx} f_{8a}(x) - \frac{8(-3+x^2)}{(9-x^2)(x^2-1)} f_{8a}(x) = I_{8a}(x)$$

Homogeneous solutions:

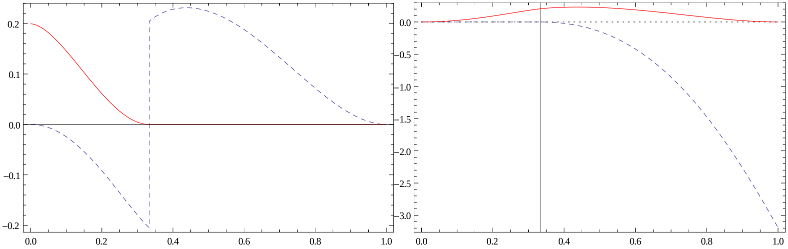
$$\begin{aligned}
\psi_3^{(0)}(x) &= -\frac{\sqrt{1-3x}\sqrt{x+1}}{2\sqrt{2\pi}} \left[ (x+1)\left(3x^2+1\right) \mathbf{E}(z) - (x-1)^2(3x+1)\mathbf{K}(z) \right] \\
\psi_4^{(0)}(x) &= -\frac{\sqrt{1-3x}\sqrt{x+1}}{2\sqrt{2\pi}} \left[ 8x^2\mathbf{K}(1-z) - (x+1)\left(3x^2+1\right)\mathbf{E}(1-z) \right], \\
z &= \frac{16x^3}{(x+1)^3(3x-1)} \quad \text{[This function is not at all random! (see later)].}
\end{aligned}$$

$\mathbf{K}, \mathbf{E}$ are the complete elliptic integrals of the 1st and 2nd kind.

$$\mathbf{K}(z) = \frac{2}{\pi} {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; z\right], \qquad \mathbf{E}(z) = \frac{2}{\pi} {}_2F_1\left[\begin{matrix}\frac{1}{2}, -\frac{1}{2} \\ 1\end{matrix}; z\right]$$

$I_{8a}$ contains rational functions of $x$ and HPLs.

# Solutions with a Singularity




## Inhomogeneous Solution

$$\psi(x) = \psi_3^{(0)}(x)\left[C_1 - \int dx \psi_4^{(0)}(x)\frac{N(x)}{W(x)}\right] + \psi_4^{(0)}(x)\left[C_2 - \int dx \psi_3^{(0)}(x)\frac{N(x)}{W(x)}\right]$$

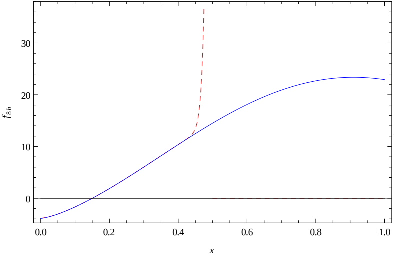
$C_{1,2}$ : from physical boundary conditions.

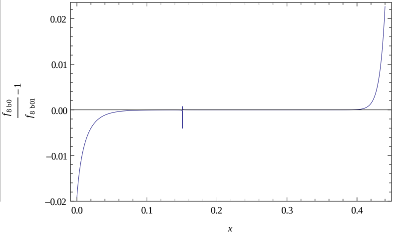
# Series Solution

$$
\begin{aligned}
f_{8a}(x) \quad = \quad & -\sqrt{3}\Bigg[\pi^3\left(\frac{35x^2}{108}-\frac{35x^4}{486}-\frac{35x^6}{4374}-\frac{35x^8}{13122}-\frac{70x^{10}}{59049}-\frac{665x^{12}}{1062882}\right)+\Bigg(12x^2-\frac{8x^4}{3}\\
&-\frac{8x^6}{27}-\frac{8x^8}{81}-\frac{32x^{10}}{729}-\frac{152x^{12}}{6561}\Bigg)\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-\frac{i\pi}{6}}}{\sqrt{3}}\right)\right]\Bigg]-\pi^2\Bigg(1+\frac{x^4}{9}-\frac{4x^6}{243}-\frac{46x^8}{6561}\\
&-\frac{214x^{10}}{59049}-\frac{5546x^{12}}{2657205}\Bigg)-\left(-\frac{3}{2}-\frac{x^4}{6}+\frac{2x^6}{81}+\frac{23x^8}{2187}+\frac{107x^{10}}{19683}+\frac{2773x^{12}}{885735}\right)\psi^{(1)}\left(\frac{1}{3}\right)\\
&-\sqrt{3}\pi\left(\frac{x^2}{4}-\frac{x^4}{18}-\frac{x^6}{162}-\frac{x^8}{486}-\frac{2x^{10}}{2187}-\frac{19x^{12}}{39366}\right)\ln^2(3)-\Bigg[33x^2-\frac{5x^4}{4}-\frac{11x^6}{54}\\
&-\frac{19x^8}{324}-\frac{751x^{10}}{29160}-\frac{2227x^{12}}{164025}+\pi^2\left(\frac{4x^2}{3}-\frac{8x^4}{27}-\frac{8x^6}{243}-\frac{8x^8}{729}-\frac{32x^{10}}{6561}-\frac{152x^{12}}{59049}\right)\\
&+\left(-2x^2+\frac{4x^4}{9}+\frac{4x^6}{81}+\frac{4x^8}{243}+\frac{16x^{10}}{2187}+\frac{76x^{12}}{19683}\right)\psi^{(1)}\left(\frac{1}{3}\right)\Bigg]\ln(x)+\frac{135}{16}+19x^2\\
&-\frac{43x^4}{48}-\frac{89x^6}{324}-\frac{1493x^8}{23328}-\frac{132503x^{10}}{5248800}-\frac{2924131x^{12}}{236196000}-\left(\frac{x^4}{2}-12x^2\right)\ln^2(x)\\
&-2x^2\ln^3(x)+O\left(x^{14}\ln(x)\right)
\end{aligned}
$$

The solution can be easily extended to accuracies of $O(10^{-30})$ using Mathematica or Maple.

# Solutions with a Singularity



## A New Class of Integrals in QFT:

$$
\begin{aligned}
\mathbb{H}_{a_1,\dots,a_{m-1};\{a_m;F_m(r(y_m))\},a_{m+1},\dots,a_q}(x) &= \int_0^x dy_1 f_{a_1}(y_1) \int_0^{y_1} dy_2 \dots \int_0^{y_{m-1}} dy_m f_{a_m}(y_m) \\
&\quad \times F_m[r(y_m)] H_{a_{m+1},\dots,a_q}(y_{m+1}), \\
F[r(y)] &= \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y],
\end{aligned}
$$

In general, this spans all solutions and the story would end here.
May be, most of the practical physicists, would led it end here anyway.
This type of solution applies to many more cases beyond ${}_2F_1$-solutions (if being properly generalized).

[JB, ICMS 2016, July 2016, Berlin]

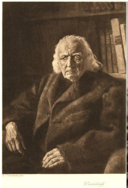
If one has no elliptic solution, one has to see, what else one has, and whether these cases are known mathematically as closed form solutions, with which properties etc. etc.

**In the elliptic case we proceed as follows.**

# Some historical aspects: Elliptic integrals and functions

A.-M. Legendre

C.G.J. Jacobi



K. Weierstraß

A. Cayley

J.H. Lambert

G. Eisenstein

R. Dedekind

- complete elliptic integrals and their inverse, Jacobi $\vartheta_i$ functions
- elliptic curves and Weierstraß $\wp$-function
- higher order Legendre-Jacobi transformations (Cayley's summary)
- Lambert-Eisenstein series
- Dedekind $\eta$-function

# Modular Functions and Modular Forms

Let $r = (r_\delta)_{\delta|N}$ be a finite sequence of integers indexed by the divisors $\delta$ of $N \in \mathbb{N}\backslash\{0\}$. The function $f_r(\tau)$

$$\begin{aligned} \tau &\in \mathbb{H} = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\} \\ f_r(\tau) &:= \prod_{d|N} \eta(d\tau)^{r_d}, \quad d, N \in \mathbb{N}\backslash\{0\}, \quad r_d \in \mathbb{Z}, \end{aligned}$$

is called $\eta$-ratio. Let

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in \mathbb{Z}, \ \det(M) = 1 \right\}.$$

$\mathrm{SL}_2(\mathbb{Z})$ is the modular group.
For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{C} \cup \infty$ one defines the Möbius transformation

$$gz \mapsto \frac{az + b}{cz + d}.$$

Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S, T \in \mathrm{SL}_2(\mathbb{Z}).$$

For $N \in \mathbb{N}\backslash\{0\}$ one considers the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$, defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), a \equiv d \equiv 1 \pmod{N}, \quad c \equiv 0 \; (\mathrm{mod}\; N) \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), a \equiv d \equiv 1 \pmod{N}, \quad b \equiv c \equiv 0 \pmod{N} \right\}$$

with $\mathrm{SL}_2(\mathbb{Z}) \supseteq \Gamma_0(N) \supseteq \Gamma_1(N) \supseteq \Gamma(N)$ and $\Gamma_0(N) \subseteq \Gamma_0(M),\ M|N$.

# Modular Functions and Modular Forms

If $N \in \mathbb{N}\backslash\{0\}$, then the index of $\Gamma_0(N)$ in $\Gamma_0(1)$ is

$$\mu_0(N) = [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

The product is over the prime divisors $p$ of $N$.
Let $x \in \mathbb{Z}\backslash\{0\}$. The analytic function $f : \mathbb{H} \to \mathbb{C}$ is a (holomorphic) modular form of weight $\mathrm{w} = k$ for $\Gamma_0(N)$ and character $a \mapsto \left(\frac{x}{a}\right)$ if

1. $$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{x}{a}\right)(cz+d)^k f(z), \quad \forall z \in \mathbb{H}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$
2. $f(z)$ is holomorphic in $\mathbb{H}$
3. $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$.

Here $\left(\frac{x}{a}\right)$ denotes the Jacobi symbol. A modular form is called a cusp form if it vanishes at the cusps.
For any congruence subgroup $G$ of $\mathrm{SL}_2(\mathbb{Z})$ a cusp of $G$ is an equivalence class in $\mathbb{Q} \cup \infty$ under the action of $G$.

# Modular Functions and Modular Forms

A meromorphic modular form $f$ for $\Gamma_0(N)$ and weight $\mathrm{w} = k$ obeys

1. $f(\gamma z) = (cz + d)^k f(z), \quad \forall z \in \mathbb{H}$ and $\forall \gamma \in \Gamma_0(N)$
2. $f$ is meromorphic in $\mathbb{H}$
3. $f$ is meromorphic at the cusps of $\Gamma_0(N)$.

The $q$ expansion of a modular function has the form

$$f^*(q) = \sum_{k=-N_0}^{\infty} a_k q^k, \quad \text{for some} \quad N_0 \in \mathbb{N}.$$

The set of functions $\mathcal{M}(k; N; x)$ for $\Gamma_0(N)$ and character $x$, defined above, forms a finite dimensional vector space over $\mathbb{C}$. In particular, for any non-zero function $f \in \mathcal{M}(k; N; x)$ we have

$$\mathrm{ord}(f) \leq b = \frac{k}{12}\mu_0(N),$$

The bound can be refined. The number of independent modular forms $f \in \mathcal{M}(k; N; x)$ is $\leq b$, allowing for a basis representation in finite terms.

For any $\eta$-ratio $f_r$ one can prove that there exists a minimal integer $l \in \mathbb{N}$, an integer $N \in \mathbb{N}$ and a character $x$ such that

$$\bar{f}_r(\tau) = \eta^l(\tau) f_r(\tau) \in \mathcal{M}(k; N; x)$$

is a holomorphic modular form. All quantities which are expanded in $q$-series below will be first brought into the above form. In some cases one has $l = 0$. This form is of importance to obtain Lambert-Eisenstein series, which can be rewritten in terms of elliptic polylogarithms.
$\eta$-ratios belonging to $\mathcal{M}(\mathrm{w}; N; 1)$:
If we refer to modular forms they are thought to be those of $\mathrm{SL}_2(\mathbb{Z})$, if not specified otherwise.

# $\eta$-Ratios

Map:

$$x \to q : q = \exp[-\pi \mathbf{K}(1-z(x))/\mathbf{K}(z(x)] := \exp[i\pi\tau], \quad |q| < 1$$

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^{2k})$$

$$\prod_{l=1}^{m} \eta^{n_l}(l\tau) = \frac{1}{\eta^k(\tau)} \mathcal{M}, \qquad n_l \in \mathbb{Z}$$

- Every $\eta$-ratio can be separated into a holomorphic modular form $\mathcal{M}$ and a factor $\eta^{-k}(\tau)$. [Algorithm 1]
- For the $\eta$-ratio $\mathcal{M}$ is given as a polynomial of (generalized) Lambert-Eisenstein series. [Algorithm 2]
- All $\mathcal{M}$ can be mapped into polynomials out of $\ln(q), \mathrm{Li}_0(q^j)$, and elliptic polylogarithms (of higher weight and also with indices depending on $q$).

# Elliptic Polylogarithms as a Frame

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^k}{k^n} \frac{y^l}{l^m} q^{kl}.$$

Weinzierl et al.:

$$\overline{E}_{n;m}(x;y;q) = \begin{cases} \frac{1}{i}[\mathrm{ELi}_{n;m}(x;y;q) - \mathrm{ELi}_{n;m}(x^{-1};y^{-1};q)], & n+m \text{ even} \\ \mathrm{ELi}_{n;m}(x;y;q) + \mathrm{ELi}_{n;m}(x^{-1};y^{-1};q), & n+m \text{ odd.} \end{cases}$$

Multiplication:

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;0,2o_2,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) = \mathrm{ELi}_{n_1;m_1}(x_1;y_1;q) \\ \mathrm{ELi}_{n_2,\ldots,n_l;m_2,\ldots,m_l;2o_2,\ldots,2o_{l-1}}(x_2,\ldots,x_l;y_2,\ldots,y_l;q),$$

$$\mathrm{ELi}_{n,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots y_l;q) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \sum_{k_1=1}^{\infty} \cdots \sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}} \cdots \frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}} \frac{y_l^{k_l}}{k_l^{m_l}} \\ \times \frac{q^{j_1 k_1 + \ldots + q_l k_l}}{\prod_{i=1}^{l-1} (j_i k_i + \ldots + j_l k_l)^{o_i}}, \; l > 0.$$

Synchronization: performed for Lambert-Eisenstein series $q^m \to q$.
Re-translate after this.

# Elliptic PolyLogarithms as a Frame

Integration:

$$\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2(o_1+1),2o_2,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q) = \int_0^q \frac{dq'}{q'}\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q').$$

Multiplication:

$$\overline{E}_{n_1,\dots,n_l;m_1,\dots,m_l;0,2o_2,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q) = \overline{E}_{n_1;m_1}(x_1;y_1;q)\,\overline{E}_{n_2,\dots,n_l;m_2,\dots,m_l;2o_2,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q)$$

$$\overline{E}_{n_1,\dots,n_l;m_1,\dots,m_l;2(o_1+1),2o_2,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q) = \int_0^q \frac{dq'}{q'}\overline{E}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q')$$

Ablinger et al.: Integration in generalized cases:

$$\int_0^q \frac{d\bar{q}}{\bar{q}}\mathrm{ELi}_{m,n}(x,q^a,q^b)\mathrm{ELi}_{m',n'}(x',q^{a'},q^{b'}) = \sum_{k=1}^\infty\sum_{l=1}^\infty\sum_{k'=1}^\infty\sum_{l'=1}^\infty \frac{x^k}{k^m}\frac{x'^k}{k'^{m'}}\frac{q^{al}}{l^n}\frac{q^{a'l'}}{l'^n} \times \frac{q^{bkl+b'k'l'}}{al+a'l'+bkl+bk'l'}.$$

# Elliptic Solutions and Analytic $q$-Series

Map:

$$x \to q : q = \exp[-\pi \mathbf{K}(1 - z(x))/\mathbf{K}(z(x)], \quad |q| < 1$$

- One attempts to calculate the integrals of the inhomogeneous solution in terms of $q$-series analytically.
- It is expected to write it in terms of products (and integrals over) elliptic polylogarithms [ and possibly other functions].
- Note that the corresponding results are rather deep multi-series!
- Inspiration from algebraic geometry.

Elliptic polylogarithm (as a partly suitable frame):

$$\mathrm{ELi}_{n,m}(x,y,q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}$$

Is it (and its generalizations) a modular form ?
$\Longrightarrow$ The central functions turn out to be more special ones.

# The Individual Steps: from IBPs to Closed Form $q$-Series

- Generate the master integrals, determine their hierarchy, and look whether you have only 1st order factorization or also 2nd order terms
- The latter can be trivial in case; check whether they persist in Mellin space
- If yes, analyze the 2nd order differential equation
- One usually finds a ${}_2F_1$-solution with rational argument $r(z)$, where $r(z)$ has additional singularities, i.e. the problem is of 2nd order, but has more than 3 singularities.
- Triangle group relations may be used to map the ${}_2F_1$ depending on the rational parameters a,b,c to the complete elliptic integrals or not.
- In the latter case return to the formalism on slide 21 and stop.
- If yes, one may walk along the $q$-series avenue.
- Different Levels of Complexity:
  - 1st order factorization in Mellin space:

$$\mathbf{M}[\mathbf{K}(1-z)](N) = \frac{2^{4N+1}}{(1+2N)^2\binom{2N}{N}^2}$$

$$\mathbf{M}[\mathbf{E}(1-z)](N) = \frac{2^{4N+2}}{(1+2N)^2(3+2N)\binom{2N}{N}^2}$$

# The Individual Steps: from IBPs to Closed Form $q$-Series

- Criteria by Herfurtner (1991), Movasati et al. (2009) are obeyed. $\Longrightarrow$ 2-loop sunrise and kite diagrams, cf. Weinzierl et al. 2014-17. Only $\mathbf{K}(r(z))$ and $\mathbf{K}'(r(z))$ contribute as elliptic integrals.
- Also $\mathbf{E}(r(z))$ and $\mathbf{E}'(r(z))$, square roots of quadratic forms etc. contribute (present case)
- Transform now: $x \to q$.
- The kinematic variable $x$:

$$
\begin{aligned}
k^2 &= \frac{-x^3}{(1+x)^3(1-3x)} = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)} \\
x &= \frac{\vartheta_2^2(q)}{3\vartheta_2^2(q^3)}, \quad \text{i.e.} \quad x \in [1, +\infty[
\end{aligned}
$$

by a cubic transformation (Legendre-Jacobi).
[see also Borwein,Borwein: AGM; and Broadhurst (2008).]

$$
x = \frac{1}{3}\frac{\eta^2(2\tau)\eta^2(3\tau)}{\eta^2(\tau)\eta^4(6\tau)}, \quad \text{singular}, \propto \frac{1}{q}
$$

It is a modular function.

# The Individual Steps: from IBPs to Closed Form $q$-Series

- Map to a holomorphic Modular Form, which can be represented by Lambert Series
  - How to find the $\eta$-ratio ? $\Longrightarrow$ Many are listed as sequences in Sloan's OEIS.
  - To find a modular form, situated in a corresponding finite-dimensional vector space $M_k$ one has to meet a series of conditions and usually split off a factor $1/\eta^k(\tau), k > 0$.
  - The remainder modular form is now a polynomial over $\mathbb{Q}$ of Lambert-Eisenstein series

$$\sum_{n=0}^{\infty} \frac{m^n q^{an+b}}{1 - q^{an+b}} .$$

Example:

$$\mathbf{K}(z(x)) = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{q^k}{1 + q^{2k}}$$

  - In this case, two $q$ series are equal, if both are modular forms, and agree in a series of $k$ first terms, where $k$ is predicted for each congruence sub-group of $\Gamma(N)$.

# The Individual Steps: from IBPs to Closed Form $q$-Series

- Map Lambert-Eisenstein Series into the frame of Elliptic Polylogarithms
- Examples:

$$
\begin{aligned}
\mathbf{K}(z) &= \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}} = \frac{\pi}{i}\sum_{k=1}^{\infty}\left[\mathrm{Li}_0\left(iq^k\right)-\mathrm{Li}_0\left(-iq^k\right)\right] \\
&= \frac{\pi}{4}\overline{E}_{0,0}(i,1,q), \\
q\frac{\vartheta_4'(q)}{\vartheta_4(q)} &= -\frac{1}{2}\left[\mathrm{ELi}_{-1;0}(1;1;q)+\mathrm{ELi}_{-1;0}(-1;1;q)\right] \\
&\quad + \left[\mathrm{ELi}_{0;0}(1;q^{-1};q)+\mathrm{ELi}_{0;0}(-1;q^{-1};q)\right] \\
&\quad - \left[\mathrm{ELi}_{-1;0}(1;q^{-1};q)+\mathrm{ELi}_{-1;0}(-1;q^{-1};q)\right].
\end{aligned}
$$

- New type of elliptic polylogarithm, e.g.: $\mathrm{ELi}_{-1;0}(-1;q^{-1};q), \quad y = y(q)!$
- Argument synchronization necessary: $-q \to q, \quad q^k \to q$ (cyclotomic).

# Elliptic Solutions and Analytic $q$-Series

- Terms to be translated:
  - rational functions in x
  - $\mathbf{K}, \mathbf{E}$
  - $\sqrt{(1-3x)(1+x)}$
  - $H_{\vec{a}}(x)$

Examples:

$$\begin{aligned}
\mathbf{E}(k^2) &= \mathbf{K}(k^2) + \frac{\pi^2 q}{\mathbf{K}(k^2)} \frac{d}{dq} \ln\left[\vartheta_4(q)\right] \\
\mathbf{E}'(k^2) &= \frac{\pi}{2\mathbf{K}(k^2)} \left[1 + 2\ln(q)\ q\frac{d}{dq} \ln\left[\vartheta_4(q)\right]\right].
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\mathbf{K}(k^2)} &= \frac{2}{\pi\eta^{12}(\tau)}\left\{\frac{5}{48}\left\{1 - 24\mathrm{ELi}_{0;-1}(1;1;q) - 4\Big[1 - \frac{3}{2}\Big[\mathrm{ELi}_{0;-1}(1;1;q) + \mathrm{ELi}_{0;-1}(1;i;q)\right.\right. \\
&\quad \left.\left.+\mathrm{ELi}_{0;-1}(1;-1;q) + \mathrm{ELi}_{0;-1}(1;-i;q)\Big]\Big]\right\}\left\{-1 + 4\Big[-\frac{1}{2}\Big[\mathrm{ELi}_{-2;0}(i;1/q;q)\right.\right. \\
&\quad +\mathrm{ELi}_{-2,0}(-i;1/q;q)\Big] + \Big[\mathrm{ELi}_{-1;0}(i;1/q;q) + \mathrm{ELi}_{-1;0}(-i;1/q;q)\Big] - \frac{1}{2}\Big[\mathrm{ELi}_{0,0}(i;1/q;q)
\end{aligned}$$

# Elliptic Solutions and Analytic $q$-Series

$$+\mathrm{ELi}_{0,0}(-i;1/q;q)\Big]\Big]\Bigg\} - \frac{1}{16}\Bigg\{5+4\Big[-\frac{1}{2}\Big[\mathrm{ELi}_{-4;0}(i;1/q;q)+\mathrm{ELi}_{-4;0}(-i;1/q;q)\Big]$$

$$+2\Big[\mathrm{ELi}_{-3;0}(i;1/q;q)+\mathrm{ELi}_{-3,0}(-i;1/q;q)\Big]-3\Big[\mathrm{ELi}_{-2;0}(i;1/q;q)$$

$$+\mathrm{ELi}_{-2,0}(-i;1/q;q)\Big]+2\Big[\mathrm{ELi}_{-1;0}(i;1/q;q)+\mathrm{ELi}_{-1;0}(-i;1/q;q)\Big]$$

$$-\frac{1}{2}\Big[\mathrm{ELi}_{0;0}(i;1/q;q)+\mathrm{ELi}_{0,0}(-i;1/q;q)\Big]\Big\}\Bigg\}$$

$$\begin{aligned} H_{-1}(x) = \ln(1+x) &= -\ln(3q) - \overline{E}_{0;-1;2}(-1;-1;q) + \overline{E}_{0;-1;2}(\rho_6;-1;q) \\ &\quad -\overline{E}_{0;-1;2}(\rho_3;-i;q) - \overline{E}_{0;-1;2}(\rho_3;i;q) \\ H_1(x) &= -H_{-1}(x)\big|_{q\to -q} + 2\pi i, etc.; \quad \rho_m = exp(2\pi i/m) \end{aligned}$$

$$I(q) = \frac{1}{\eta^k(\tau)} \cdot \mathbf{P}\left[\ln(q), \mathrm{Li}_0(q^m), \mathrm{ELi}_{k,l}(x,y,q), \mathrm{ELi}_{k',l'}(x,q^{-1},q)\right]$$

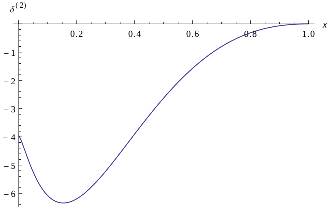
$$\int \frac{dq}{q} I(q)$$

is usually not an elliptic polylogarithm, due to the $\eta$-factor, but a higher transcendental function in $q$.

We are still in the unphysical region and have to map back to $x \in [0,1]$.

$$\Delta\rho = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}}\left(\delta^{(0)}(x) + \frac{\alpha_s}{\pi}\delta^{(1)}(x) + \left(\frac{\alpha_s}{\pi}\right)^2 \delta^{(2)}(x) + \mathcal{O}(\alpha_s^3)\right)$$

$$\begin{aligned}
\delta^{(2)}(x) \quad = \quad & \cdots + C_F\left(C_F - \frac{C_A}{2}\right)\Bigg[\frac{11-x^2}{12(1-x^2)^2} f_{8a}(x) + \frac{9-x^2}{3(1-x^2)^2} f_{9a}(x) + \frac{1}{12} f_{10a}(x) \\
& \qquad + \frac{5-39x^2}{36(1-x^2)^2} f_{8b}(x) + \frac{1-9x^2}{9(1-x^2)^2} f_{9b}(x) + \frac{x^2}{12} f_{10b}(x)\Bigg] \\
& + \frac{C_F T_F}{9(1-x^2)^3}\Bigg[(5x^4 - 28x^2 - 9) f_{8a}(x) + \frac{1-3x^2}{3x^2}(9x^4 + 9x^2 - 2) f_{8b}(x) \\
& \qquad + (9-x^2)(x^4 - 6x^2 - 3) f_{9a}(x) + \frac{1-9x^2}{3x^2}(3x^4 + 6x^2 - 1) f_{9b}(x)\Bigg]
\end{aligned}$$




For $x = 0$, this agrees with the result by Chetyrkin *et al* (and Avdeev *et al*), $\delta^{(2)}(0) = -3.9696$

In the case of one of the homogeneous solutions to the differential equation of $f_{8a}(x)$, namely

$$\psi_{1b}(x) = \frac{2}{\sqrt{3}} H(x)$$

with

$$H(x) = \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} \, {}_2F_1\left[ \begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix} ; \frac{x^2(x^2-9)^2}{(x^2+3)^3} \right]$$

Setting the kinematic variable

$$x = 3\frac{\eta_1^2 \eta_6^4}{\eta_2^4 \eta_3^2},$$

Broadhurst has found the following modular representation:

$$\begin{aligned} H\left(3\frac{\eta_1^2\eta_6^4}{\eta_2^4\eta_3^2}\right) &= \frac{1}{2}\left[\frac{\eta_1^{14}\eta_6^{10}}{\eta_2^{22}\eta_3^2} + \frac{\eta_1^6\eta_6^4}{\eta_2^{12}\eta_3^2}\left(\frac{\eta_1^4\eta_6^8}{\eta_2^8\eta_3^4} + \frac{1}{3}\right) q\frac{d}{dq}\right]\frac{\eta_2\eta_3}{\eta_1^3\eta_6^2} \\ &= q - 6q^2 + 24q^3 - 74q^4 + 195q^5 - 474q^6 + 1100q^7 + \mathcal{O}(q^8) \end{aligned}$$

where $\eta_k = \eta(k\tau)$.

# Conclusions

- We have automated the chain from IBPs to 2nd order solutions within the theory of differential equations [Before we had solved the 1st order factorizing cases for whatsoever basis of MIs.]
- General solution in the case not 1st order factorizing: Non-iterative iterative integrals $\mathbb{H}$.
- These solutions might be sufficient and are very precise numerically and the result has a compact representation.
- In the elliptic cases we were enforced to generalize to structures not yet appearing in the case of the sunrise/kite integrals.
- Modular forms need to become a manifest part of knowledge for particle physicist working on fundamental QFTs [String 'theory' needs it as well.]
- We can solve any $\eta$ ratio.

# Conclusions

- The general solution is given in terms of polynomials of elliptic polylogarithms, more precisely: Lambert-Eisenstein series and a few simpler functions in $q$-space
- Singularity treatment ?
- How to map back to the different physical regions ?
- We calculated 3-loop 2-mass corrections to the $\rho$ parameter analytically. Here differential equations appear which only factorize to second order.
- What are the minimal bases ?
  $\Longrightarrow$ An important mathematical research topic.
- Interesting observation: $q$-series for equal mass sunrise appeared in 1987 in a similar form in Beukers' 2nd proof of the irrationality of $\zeta_3$ in form of an Eichler-integral [Zagier].
- What comes next ? Abel integrals ? K3 surfaces (Kummer, Kähler, Kodaira), Calabi-Yau structures...?
- **Again a new and exciting territory for theoretical physics!**

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