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# Symbolic Summation and Integration Tools for Feynman Integrals


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SAGEX

# Symbolic integration

- method of hyperlogarithms (F. Brown)
- multivariate Almkvist-Zeilberger algorithm

# Multivariate Almkvist-Zeilberger algorithm (MultiIntegrate.m)

GIVEN an integral in the from

$$F(\varepsilon, N) = \int_0^1 dx_1 \cdots \int_0^1 dx_s f(\varepsilon, N, x_1, \ldots, x_s)$$

where $f$ is hyperexponential in $x_i$ for $i = 1, 2, \ldots, s$, i.e.,

$$\frac{D_{x_i} f(\ldots, x_i, \ldots)}{f(\ldots, x_i, \ldots)} \in \mathbb{K}(\varepsilon, N, x_1, \ldots, x_s).$$

$\mathbb{K}$ is a field containing the rational numbers.

M. Apagodu and D. Zeilberger, Adv. Appl. Math. (Special Regev Issue), **37** (2006) 139;
J. Ablinger, Ph.D. Thesis, J. Kepler University Linz, 2012, arXiv:1305.0687;
J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202(2016), arXiv:1509.08324.

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$$\frac{D_{x_i} f(\ldots, x_i, \ldots)}{f(\ldots, x_i, \ldots)} \in \mathbb{K}(\varepsilon, N, x_1, \ldots, x_s).$$

Compute a linear recurrence of the form

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_d(\varepsilon, N)F(\varepsilon, N+d) = 0$$

with polynomials $a_i(\varepsilon, N) \in \mathbb{K}[\varepsilon, N]$

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# Example: A master integral from Ladder and $V$-topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1}$$
$$(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

$$
\begin{aligned}
a_0(N,\varepsilon) = {} & (N+1)(N+2)\big(8\varepsilon^{10} + 104\varepsilon^9(N+3) + 4\varepsilon^8\big(96N^2+601N+887\big) \\
& + 4\varepsilon^7\big(12N^3+414N^2+1583N+1393\big) \\
& - 8\varepsilon^6\big(264N^4+2436N^3+8643N^2+14518N+9947\big) \\
& - 16\varepsilon^5\big(156N^5+1690N^4+6847N^3+12661N^2+9537N+717\big) \\
& + 32\varepsilon^4\big(68N^6+1158N^5+8155N^4+30114N^3+61712N^2+67616N+31693\big) \\
& + 64\varepsilon^3\big(40N^7+560N^6+2755N^5+3729N^4-14194N^3-61920N^2-89140N-46600\big) \\
& - 128\varepsilon^2(N+2)\big(12N^7+254N^6+2249N^5+10758N^4+30173N^3+50610N^2 \\
& + 49122N+22706\big) \\
& + 256\varepsilon(N+2)^2(N+3)(N+4)\big(44N^4+501N^3+2044N^2+3455N+1976\big) \\
& - 512(N+1)(N+2)^3(N+3)^2(N+4)\big(6N^2+47N+95\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1(N,\varepsilon) = (N+2)\big(&-22\varepsilon^{11} - 2\varepsilon^{10}(157N+435) - \varepsilon^9\big(1500N^2+8611N+11745\big)\\
&- \varepsilon^8\big(2548N^3+22936N^2+63597N+54229\big)\\
&+ 4\varepsilon^7\big(266N^4+1857N^3+6065N^2+14351N+15987\big)\\
&+ 8\varepsilon^6\big(994N^5+12961N^4+67246N^3+174692N^2+226821N+116092\big)\\
&+ 16\varepsilon^5\big(336N^6+5348N^5+33569N^4+104918N^3+165290N^2+108259N+6100\big)\\
&- 16\varepsilon^4\big(404N^7+7578N^6+61778N^5+284762N^4+802660N^3+1382074N^2\\
&+ 1340455N+560287\big)\\
&- 64\varepsilon^3\big(94N^8+1823N^7+14305N^6+55870N^5+96299N^4-37256N^3\\
&- 447044N^2-704959N-379338\big)\\
&+ 128\varepsilon^2(N+3)\big(30N^8+715N^7+7667N^6+48253N^5+194086N^4+507439N^3\\
&+ 835393N^2+785327N+320382\big)\\
&- 256\varepsilon(N+2)(N+3)^2\big(107N^6+2070N^5+16342N^4+67226N^3+151557N^2\\
&+ 176932N+83196\big)\\
&+ 256(N+2)^3(N+3)^3(N+4)\big(30N^3+331N^2+1193N+1386\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_2(N,\varepsilon) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17N+45) + 2\varepsilon^{10}\big(620N^2+3553N+4795\big) \\
&+ 2\varepsilon^9\big(1504N^3+14190N^2+41901N+38907\big) \\
&+ 4\varepsilon^8\big(172N^4+4983N^3+30942N^2+69119N+50850\big) \\
&- 4\varepsilon^7\big(1996N^5+24056N^4+113313N^3+269119N^2+337198N+185290\big) \\
&- 16\varepsilon^6\big(450N^6+8210N^5+59749N^4+227386N^3+486841N^2+563176N+275664\big) \\
&+ 16\varepsilon^5\big(340N^7+4314N^6+19137N^5+25532N^4-55105N^3-206516N^2-191528N \\
&- 23458\big) \\
&+ 32\varepsilon^4\big(140N^8+2940N^7+26550N^6+139926N^5+493839N^4+1240186N^3 \\
&+ 2161699N^2+2304248N+1100084\big) \\
&+ 64\varepsilon^3\big(4N^9+506N^8+8651N^7+63510N^6+236215N^5+395334N^4-105413N^3 \\
&- 1551017N^2-2362944N-1217770\big) \\
&- 128\varepsilon^2(N+3)\big(12N^9+314N^8+3782N^7+29105N^6+160727N^5+640273N^4 \\
&+ 1750874N^3+3052505N^2+3017094N+1276604\big) \\
&+ 256\varepsilon(N+2)(N+3)^2(N+4)\big(26N^6+825N^5+8967N^4+46529N^3+125411N^2 \\
&+ 168628N+88652\big) \\
&- 512(N+1)(N+2)^2(N+3)^3(N+4)^2\big(6N^3+98N^2+459N+655\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3(N,\varepsilon) = \big( &- 64\varepsilon^{12} - 8\varepsilon^{11}(113N+298) - 8\varepsilon^{10}\big(519N^2+2948N+3896\big) \\
&- 4\varepsilon^9\big(1444N^3+13839N^2+39746N+34305\big) \\
&+ 4\varepsilon^8\big(1948N^4+17868N^3+63837N^2+112966N+84655\big) \\
&+ 16\varepsilon^7\big(1456N^5+20460N^4+112365N^3+304963N^2+412258N+221769\big) \\
&- 8\varepsilon^6\big(320N^6+2050N^5+4192N^4+27408N^3+174901N^2+411759N+324872\big) \\
&- 16\varepsilon^5\big(1756N^7+33154N^6+265889N^5+1186719N^4+3218059N^3+5349388N^2 \\
&+ 5071913N+2113696\big) \\
&+ 32\varepsilon^4\big(188N^8+4802N^7+59527N^6+439922N^5+2025336N^4+5813984N^3 \\
&+ 10076450N^2+9621283N+3878602\big) \\
&+ 64\varepsilon^3\big(140N^9+2768N^8+22500N^7+99545N^6+287700N^5+723136N^4 \\
&+ 1854572N^3+3714620N^2+4272517N+2031600\big) \\
&- 128\varepsilon^2\big(24N^{10}+830N^9+14362N^8+152630N^7+1053620N^6+4834279N^5 \\
&+ 14824351N^4+29964399N^3+38244797N^2+27875896N+8824032\big) \\
&+ 256\varepsilon(N+2)(N+3)(N+4)\big(118N^7+2639N^6+24247N^5+118311N^4+329565N^3 \\
&+ 520306N^2+426076N+136854\big) \\
&- 512(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)\big(12N^3+97N^2+230N+144\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_4(N,\varepsilon) = \big(&64\varepsilon^{12} + 192\varepsilon^{11}(5N+14) + 16\varepsilon^{10}\big(297N^2+1769N+2451\big)\\
&+16\varepsilon^9\big(453N^3+4462N^2+13094N+11244\big)\\
&-8\varepsilon^8\big(1084N^4+11117N^3+47258N^2+103981N+94650\big)\\
&-8\varepsilon^7\big(3304N^5+51138N^4+311957N^3+948722N^2+1440105N+858544\big)\\
&+16\varepsilon^6\big(420N^6+5507N^5+36275N^4+169650N^3+536911N^2+952507N+694370\big)\\
&+16\varepsilon^5\big(1828N^7+38868N^6+353301N^5+1801014N^4+5604391N^3+10664390N^2\\
&+11433064N+5260048\big)\\
&-32\varepsilon^4\big(316N^8+8356N^7+105800N^6+802421N^5+3836854N^4+11588223N^3\\
&+21401558N^2+22066744N+9745752\big)\\
&-64\varepsilon^3\big(116N^9+2424N^8+19923N^7+82966N^6+208191N^5+530980N^4+1847484N^3\\
&+4687014N^2+6120858N+3111104\big)\\
&+128\varepsilon^2\big(24N^{10}+826N^9+14897N^8+172000N^7+1314686N^6+6710299N^5\\
&+22873183N^4+51298261N^3+72551278N^2+58573022N+20544948\big)\\
&-256\varepsilon(N+2)(N+3)\big(106N^8+3278N^7+42903N^6+310942N^5+1366350N^4\\
&+3729418N^3+6173159N^2+5657732N+2191212\big)\\
&+512(N+1)(N+2)^2(N+3)^2(N+4)(N+5)(N+6)\big(12N^3+121N^2+396N+431\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_5(N,\varepsilon) = (N+5)\big( &-128\varepsilon^{11} - 128\varepsilon^{10}(11N+26) - 32\varepsilon^{9}\big(115N^2+592N+647\big) \\
&+ 32\varepsilon^{8}\big(63N^3+430N^2+1665N+2384\big) \\
&+ 16\varepsilon^{7}\big(714N^4+7881N^3+33802N^2+66225N+47654\big) \\
&- 16\varepsilon^{6}\big(234N^5+2444N^4+13989N^3+50862N^2+104083N+87848\big) \\
&- 16\varepsilon^{5}\big(580N^6+10181N^5+76586N^4+319207N^3+772120N^2+1012046N+547832\big) \\
&+ 16\varepsilon^{4}\big(244N^7+5456N^6+61605N^5+401216N^4+1536277N^3+3408574N^2 \\
&+ 4066436N+2026928\big) \\
&+ 64\varepsilon^{3}\big(26N^8+357N^7+583N^6-11139N^5-65193N^4-120264N^3+11864N^2 \\
&+ 272830N+222624\big) \\
&- 64\varepsilon^{2}(N+3)\big(12N^8+298N^7+4684N^6+49024N^5+306907N^4+1122441N^3 \\
&+ 2350650N^2+2607576N+1185072\big) \\
&+ 256\varepsilon(N+2)(N+3)\big(25N^7+743N^6+8856N^5+55358N^4+197497N^3+404131N^2 \\
&+ 439902N+196128\big) \\
&- 256(N+1)(N+2)^2(N+3)^2(N+4)(N+6)(N+7)\big(6N^2+35N+54\big)\big).
\end{aligned}
$$

# Example: A master integral from Ladder and $V$-topologies

[arXiv:1509.08324]

$$F(\varepsilon,N)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+N+1}$$
$$(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^N$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,N)F(\varepsilon,N)+a_1(\varepsilon,N)F(\varepsilon,N+1)+\cdots+a_5(\varepsilon,N)F(\varepsilon,N+5)=0$$

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$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

recurrence solver $\downarrow$

$F(\varepsilon, N) =$ expression in terms of special functions

# Recurrence solving

# A recurrence solver (Sigma.m)

**GIVEN** a recurrence

$a_0(N), \dots, a_d(N)$: polynomials in N
$h(N)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(N)F(N) + \dots + a_d(N)F(N+d) = h(N);$$

together with initial values $F(0), \dots, F(d-1) \in \mathbb{K}$

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**DECIDE** constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

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Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(N) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j} \qquad \text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$ (generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{N} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i} \qquad \text{(cyclotomic harmonic sums)}$$

J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

# A recurrence solver (Sigma.m)

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \qquad \text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and C. Schneider, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2\binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümmlein, A. De Freitas, A. Goedicke, C. Schneider, K. Schönwald. Nucl. Phys. B 932, 2018. [arXiv:1804.02226]

# Sigma.m is based on difference ring/field theory


1. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
2. P. Paule. Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)
3. M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
4. P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
5. M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, 2000.
6. CS. Symbolic summation in difference fields. J. Kepler University, May 2001. PhD Thesis.
7. CS. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. *An. Univ. Timişoara Ser. Mat.-Inform.*, 42(2):163–179, 2004.
8. CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
9. CS. Degree bounds to find polynomial solutions of parameterized linear difference equations in $\Pi\Sigma$-fields. *Appl. Algebra Engrg. Comm. Comput.*, 16(1):1–32, 2005.
10. CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
11. CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
12. CS. Finding telescopers with minimal depth for indefinite nested sums and product expressions. In *Proc. ISSAC'05*, pages 285–292. ACM Press, 2005.
13. CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
14. CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
15. CS. A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, pages 285–308. 2010.
16. CS. Parameterized Telescoping Proves Algebraic Independence of Sums. *Ann. Comb.*, 14(4):533–552, 2010. [arXiv:0808.2596].
17. CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
18. C. Schneider. Simplifying Multiple Sums in Difference Fields. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
19. CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. In *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].
20. CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
21. CS. Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80(3), pp. 616-664. 2017.
22. E.D. Ocansey, CS. Representing (q-)hypergeometric products and mixed versions in difference rings. In: Advances in Computer Algebra,, C. Schneider, E. Zima (ed.), Springer Proceedings in Mathematics & Statistics 226. 2018.
23. S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

# Example: A master integral from Ladder and $V$-topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's MultiIntegrate.m ↓ (9 hours)

$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

recurrence solver ↓

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$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

Sigma ↓

$$F(\varepsilon, N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \ldots$$

# A refined recurrence solver (Sigma.m)

**GIVEN** a recurrence

$a_0(\varepsilon,N),\dots,a_d(\varepsilon,N)$: polynomials in $\varepsilon,N$
$h_l(N),h_{l+1}(N),\dots,h_r(N)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon,N)F(\varepsilon,N)+\dots+a_d(\varepsilon,N)F(\varepsilon,N+d)$$
$$=h_l(N)\varepsilon^l+h_{l+1}(N)\varepsilon^{l+1}+\dots h_r(N)\varepsilon^r+O(\varepsilon^{r+1});$$

together with $\varepsilon$-expansions of $F(0),\dots,F(d-1)$ up to a certain order.

# A refined recurrence solver (Sigma.m)

**GIVEN** a recurrence

$a_0(\varepsilon,N),\dots,a_d(\varepsilon,N)$: polynomials in $\varepsilon, N$
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$$=h_l(N)\varepsilon^l+h_{l+1}(N)\varepsilon^{l+1}+\dots h_r(N)\varepsilon^r+O(\varepsilon^{r+1});$$

together with $\varepsilon$-expansions of $F(0),\dots,F(d-1)$ up to a certain order.

**DECIDE** constructively if the coefficients $F_i(N)$ of

$$F(N)=F_l(N)\,\varepsilon^l+F_{l+1}(N)\,\varepsilon^{l+1}+\dots+F_r(N)\,\varepsilon^r+O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F(N)\Big] \\
+&a_1(\varepsilon, N)\Big[F(N+1)\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F(N+d)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

# Ansatz (for power series)

$$a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]$$

$$+a_1(\varepsilon,N)\Big[F(N+1)\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon,N)\Big[F(N+d)\Big]$$

$$=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots$$

given (in terms of indefinite nested sums and products)

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,N)\Big[F(N+d)\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+&a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big] \\
&\qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N) + a_1(0,N)F_0(N+1) + \dots + a_d(0,N)F_0(N+d) = h_0(N)}
$$

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}
$$

REC solver: Given the initial values $F_0(1),F_0(2),\dots,F_0(d)$, **decide** if $F_0(N)$ can be written in terms of indefinite nested sums and products.

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

# Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+&a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon, N)\Big[F_0(N+d) + F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big] \\
&\qquad = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N) + a_1(0,N)F_0(N+1) + \dots + a_d(0,N)F_0(N+d) = h_0(N)}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N{+}1)\varepsilon+F_2(N{+}1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_1(N{+}d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h'_0(N)+h'_1(N)\varepsilon+h'_2(N)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon, N)\Big[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+ & a_1(\varepsilon, N)\Big[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+ & \\
\vdots & \\
+ & a_d(\varepsilon, N)\Big[F_1(N+d)\varepsilon + F_2(N+d)\varepsilon^2 + \dots\Big] \\
& \qquad = \underbrace{h_0'(N)}_{=\mathbf{0}} + h_1'(N)\varepsilon + h_2'(N)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by $\varepsilon$

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_1(N) + F_2(N)\varepsilon + \dots\Big] \\
+&a_1(\varepsilon, N)\Big[F_1(N+1) + F_2(N+1)\varepsilon + \dots\Big] \\
+& \\
&\vdots \\
+&a_d(\varepsilon,N)\Big[F_1(N+d) + F_2(N+d)\varepsilon + \dots\Big] = h'_1(N) + h'_2(N)\varepsilon + \dots
\end{aligned}
$$

**Now repeat for** $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+N+1}(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^N$$

↓ (package `MultiIntegrate.m`)

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+\cdots+a_5(\varepsilon,N)F(N+5)=0$$

$$F(2)=\frac{20}{27\varepsilon^3}-\frac{40}{27\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1393}{486}+\frac{5\zeta_2}{18}\right)+\dots$$

$$\vdots$$

$$F(6)=\frac{22}{147\varepsilon^3}-\frac{535}{2058\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{630043}{1234800}+\frac{11\zeta_2}{196}\right)+\dots$$

$$F(N)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+N+1}(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^N$$

↓ (package MultiIntegrate.m)

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+\cdots+a_5(\varepsilon,N)F(N+5)=0$$

$$F(2)=\frac{20}{27\varepsilon^3}-\frac{40}{27\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1393}{486}+\frac{5\zeta_2}{18}\right)+\dots$$

$$\vdots$$

$$F(6)=\frac{22}{147\varepsilon^3}-\frac{535}{2058\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{630043}{1234800}+\frac{11\zeta_2}{196}\right)+\dots$$

↓ (summation package Sigma.m)

$$\boxed{F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots}$$

We get

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

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$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N\left(3N^3+18N^2+31N+18\right)}{3(N+1)^3(N+2)^2} - \frac{4\left(6N^3+32N^2+51N+26\right)}{3(N+1)^3(N+2)^2}$$

We get

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N\left(3N^3+18N^2+31N+18\right)}{3(N+1)^3(N+2)^2} - \frac{4\left(6N^3+32N^2+51N+26\right)}{3(N+1)^3(N+2)^2}$$

$$\begin{aligned}F_{-1}(N) = (-1)^N \Bigg( & \frac{2\left(9N^5+81N^4+295N^3+533N^2+500N+204\right)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)} \\ & + \frac{2\left(18N^5+150N^4+490N^3+755N^2+536N+132\right)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)} \\ & + \left(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\right) S_2 \\ & + \left(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\right) S_{-2}\end{aligned}$$

# Example: A master integral from Ladder and $V$-topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1}$$
$$(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

Sigma $\downarrow$ (2 hours)

$$F(\varepsilon, N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \cdots + F_4(N)\varepsilon^4 + O(\varepsilon^5)$$

# Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

# Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

**$\varepsilon$-recurrence solver**

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

# Symbolic summation

# Find a recurrence for the integral/sum

$$F(N)=\int_0^1\cdots\int_0^1 \Phi(\varepsilon,N,x_1,x_2,\ldots,x_7)dx_1dx_2\ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\ldots$$

**$\varepsilon$-recurrence solver**

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon,N)F(N)+\ldots+a_d(\varepsilon,N)F(N+d)=h(\varepsilon,N)$$

# Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

**$\varepsilon$-recurrence solver**

Mellin-Barnes-
and ${}_pF_q$-technologies

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

# Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\overset{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \ldots$$

**$\varepsilon$-recurrence solver**

Mellin-Barnes- and ${}_pF_q$-technologies

multivariate Almkvist/Zeilberger (J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \ldots, i_7)$$

Wegschaider's MultiSum Package

$$a_0(\varepsilon, N)F(N) + \ldots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

# Find a recurrence for the integral/sum

$$F(N)=\int_0^1\cdots\int_0^1\Phi(\varepsilon,N,x_1,x_2,\dots,x_7)dx_1dx_2\dots dx_7$$
$$\stackrel{?}{=}F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots$$

**$\varepsilon$-recurrence solver**

Mellin-Barnes-
and ${}_pF_q$-technologies

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$\sum_{i_1}\cdots\sum_{i_7}f(\varepsilon,N,i_1,i_2,\dots,i_7)$$

Wegschaider's
Package

MultiSum

experimental code of a new
holonomic-difference ring approach

$$a_0(\varepsilon,N)F(N)+\dots+a_d(\varepsilon,N)F(N+d)=h(\varepsilon,N)$$

# Symbolic summation:

## the Sigma-approach

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad (=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

$\uparrow$ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1)-g(0)$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1)-g(0)$$
$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{1}{n!}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$ :

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

**no solution** ☹

## Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

**no solution** ☹

# Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

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Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a}\Big[c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)\Big]}$$

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$$\| \qquad\qquad \|$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

## Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

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# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

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$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

# Sigma's full summation machinery

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in $k$;
$N$: extra parameter

FIND a recurrence for $F(N)$

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GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

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3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

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$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

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$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

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$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\right.\right.$$

$$\left.\left.\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

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$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{N\}, \{1\}, \{\infty\}]}$

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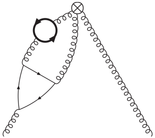
Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}[n]}{n+1}+\frac{S_1[n]}{(n+1)^2}+\frac{S_2[n]}{-n-1}$$

# Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



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The diagram is produced with axodraw (J. Vermaseren).

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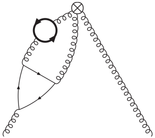

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Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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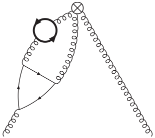

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$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

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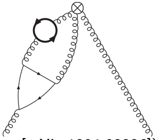

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$\varepsilon^{-1}$-contribution: 30 minutes for this sum

$\sim 1$ year for full expression

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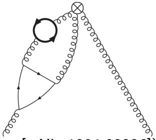

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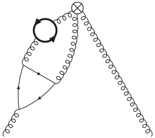
$\varepsilon^{-1}$-contribution: 30 minutes for this sum
$\sim 1$ year for full expression

$\varepsilon^{0}$-contribution: sums are not expressible
in terms of indefinite nested sums

# Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



(see [arXiv:1804.02226])

Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

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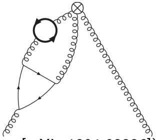

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| sum | size of sum (with $\varepsilon$) | summand size of constant term | time of calculation | | number of indef. sums |
|---|---|---|---|---|---|
| $\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$ | 17.7 MB | 266.3 MB | 177529 s | (2.1 days) | 1188 |
| $\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$ | 232 MB | 1646.4 MB | 980756 s | (11.4 days) | 747 |
| $\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$ | 67.7 MB | 458 MB | 524485 s | (6.1 days) | 557 |
| $\sum_{i_1=0}^{\infty}$ | 38.2 MB | 90.5 MB | 689100 s | (8.0 days) | 44 |
| $\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$ | 1.3 MB | 6.5 MB | 305718 s | (3.5 days) | 1933 |
| $\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$ | 11.6 MB | 32.4 MB | 710576 s | (8.2 days) | 621 |
| $\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$ | 4.5 MB | 5.5 MB | 435640 s | (5.0 days) | 536 |
| $\sum_{i_1=3}^{N-4}$ | 0.7 MB | 1.3 MB | 9017s | (2.5 hours) | 68 |

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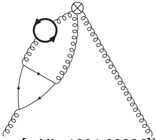

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expression (154 MB) consisting of 4110 indefinite sums

# Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left( \sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left( \sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left( \sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

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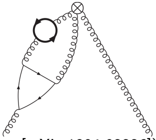

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Mellin-Barnes- and ${}_pF_q$-technologies →

expression (95 MB) with

- 150 single sums
- 1000 double sums
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- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

← Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

# Solving coupled systems

## The DE-REC approach

DE system

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

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**DE system**

$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

$\xrightarrow[\text{uncoupling algorithm}]{\text{OreSys package (S. Gerhold)}}$

**uncoupled DE system**

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$

$$\hat{I}_k(x) = \mathsf{expr}_k(D^i \hat{I}_1(x)), k > 1$$

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↓ coeff. comparison w.r.t. $x^N$ ↓

**linear recurrence**

$$\sum_i a_i'(N) I_1(N+i) = r'(N)$$

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$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

**linear recurrence**
$\sum_i a'_i(N) I_1(N+i) = r'(N)$

← `Sigma`'s REC-solver ←

**closed form solutions of $I_1(N)$ in the class of nested sums and products – if this is possible**

## The DE-REC approach (SolveCoupledSystem package)

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$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

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**uncoupled DE system**
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
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↓ coeff. comparison w.r.t. $x^N$ ↓

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

**extract coefficient**
SumProduction package

**linear recurrence**
$$\sum_i a_i'(N) I_1(N+i) = r'(N)$$

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**closed form solutions of**
$I_1(N), \mathbf{I_2(N)}, \ldots, \mathbf{I_n(N)}$
in the class of nested
sums and products
– if this is possible

J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

# A coupled differential system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

(produced by IBP [extension of REDUZE_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\hat{R}_2(x) = \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x}\hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)\left(19\varepsilon^2+36\varepsilon+16\right)}{16\varepsilon(5\varepsilon+6)(x-1)x}\hat{B}_2(x)$$
$$+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x}\hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x}\hat{B}_4(x)$$

$\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

# Solving a coupled differential system

In[6]:= $<<$ **OreSys.m**

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[7]:= $<<$ **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[8]:= $\mathbf{coupledDESys = D[\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\}, x] - A.\{\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)\};}$

In[9]:= $\mathbf{rhs = \{\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)\}}$ in power series representation;

In[10]:= $\mathbf{SolveCoupledDESystem[coupledDESys, rhs, \{I_1[x], I_2[x], I_3[x]\}, \varepsilon, -3, \{-2, -2, -2\}, rhs, \ldots]}$

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Out[10]=
$$\Big\{\frac{1}{\varepsilon^3}\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1[N]}{3(N+1)}\Big)+\frac{1}{\varepsilon^2}\Big(-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big),$$
$$\frac{4}{3\varepsilon^3}-\frac{2}{\varepsilon^2}, \frac{8}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\Big(-\frac{4\big(4N^2+7N+2\big)}{3(N+1)^2}+\frac{4(N+2)S_1[N]}{3(N+1)}\Big)\Big\}$$

# Concrete calculations:


- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x,Q^2)$ and Transversity. Nuclear Physics B 886, 2014. arXiv:1406.4654 [hep-ph].
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, 2014. arXiv:1405.4259 [hep-ph].
- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, 2014. arXiv:1402.0359 [hep-ph].
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x,Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), 2015. arXiv:1508.01449 [hep-ph].
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x,Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, 2015. arXiv:1504.08217 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x,Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, 2015. arXiv:1409.1135.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x,Q^2)$ and $F_2^{W^+-W^-}(x,Q^2)$. Physical Review D 94(11), 2016. arXiv:1609.06255 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, 2016. arXiv:1509.08324 [hep-ph].
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit, Physics Letters B 782, 2018. arXiv:1804.07313 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D, 2018. arXiv:1712.09889.

# Computing large moments and guessing recurrences

# The DE-REC approach (SolveCoupledSystem package)

**DE system**
$$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

**uncoupled DE system**
$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
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**extract coefficient**
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coeff. comparison w.r.t. $x^N$

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← `Sigma's` REC-solver ←

**closed form solutions of**
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in the class of nested
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# The method of large moments (SolveCoupledSystem)

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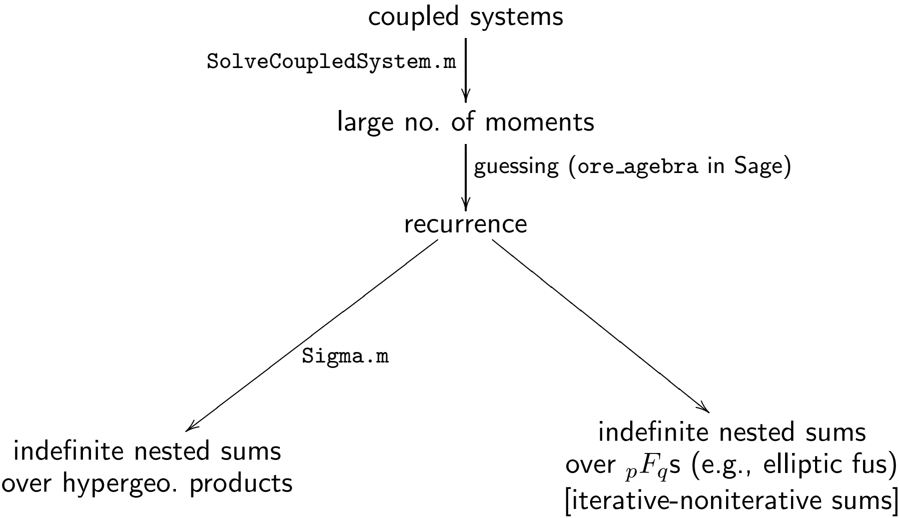
← Sigma's REC-solver ←

**closed form solutions of $I_1(N)$**
in the class of nested sums and products
– if th

Given $r'(0), r'(1), \dots, r'(5114)$,
compute $I_1(0), I_1(1), \dots, I_1(5114)$

J. Blümlein, CS, Physics Letters B 771, pp. 31-36., 2017; arXiv:1701.04614 [hep-ph].


coupled systems
SolveCoupledSystem.m
large no. of moments
guessing (ore_agebra in Sage)
recurrence
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over $_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

# Concrete calculations of large moments:

- J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, C. Schneider The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213. arXiv:1705.01508 [hep-ph].
  1. computed $\sim 2400$ moments
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- massive Wilson coefficient $A_{Qg}$:
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- special case: $A_{Qg}^{T_F^2,(3)}$ (as case study)
  1. computed 8000 moments
  2. guessed all recurrences
  3. solved some recurrences

# Special functions algorithms

(implemented in `HarmonicSums`)

# Computer algebra and special functions:

**Harmonic sums** (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . . )

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

(J. Ablinger, J. Blümlein, CS; J. Math. Phys. 2011 [arXiv:1105.6063 [math-ph]])

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$$\boxed{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}}$$

**Integral representation:**

$$=\int_0^1 \frac{x^n-1}{1-x}\left(\int_0^x \frac{\int_0^y \frac{1}{1-z}dz}{y}dy-\zeta_2\right)dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

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# Computer algebra and special functions:

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**Integral representation:**

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left( \int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

**Asymptotic expansion:**

$$= \left( \frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$
$$- \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta_3 + O\left(\frac{\ln(n)}{n^6}\right).$$

**limit computations** **numerical evaluation**

(J. Ablinger, J. Blümlein, CS; J. Math. Phys. 2011 [arXiv:1105.6063 [math-ph]])

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\,\to\,\infty}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$\overset{a\to\infty}{=} \frac{1}{n!}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# Computer algebra and special functions:

**Harmonic sums** (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . . )

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

**Integral representation:**

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left( \int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

**Asymptotic expansion:**

$$= \left( \frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$

$$- \tfrac{1}{100n^5} - \tfrac{1}{6n^4} + \tfrac{13}{36n^3} - \tfrac{1}{4n^2} - \tfrac{1}{n} + 2\zeta_3 + O(\tfrac{\ln(n)}{n^6}).$$

**limit computations** **numerical evaluation**

(J. Ablinger, J. Blümlein, CS; J. Math. Phys. 2011 [arXiv:1105.6063 [math-ph]])

- Generalized algorithms for generalized harmonic sums

$$
\sum_{k=1}^{n} \frac{2^k \sum_{i=1}^{k} \frac{2^{-i} \sum_{j=1}^{i} \frac{S_1(j)}{j}}{i}}{k} = -\frac{21\zeta_2^2}{20}\frac{1}{n} + \frac{1}{8n^2} + \frac{295}{216n^3} - \frac{1115}{96n^4} + O(n^{-5})
$$
$$
+ \Big(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\Big)\zeta_2
$$
$$
+ 2^n\Big(\frac{3}{2n} + \frac{3}{2n^2} + \frac{9}{2n^3} + \frac{39}{2n^4} + O(n^{-5})\Big)\zeta_3
$$
$$
+ \Big(\frac{1}{n} + \frac{3}{4n^2} - \frac{157}{36n^3} + \frac{19}{n^4} + O(n^{-5})\Big)(\log(n) + \gamma)
$$
$$
+ \Big(\frac{1}{2n} - \frac{3}{4n^2} + \frac{19}{12n^3} - \frac{5}{n^4} + O(n^{-5})\Big)(\log(n) + \gamma)^2)
$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta_3}{16}\Big)\zeta_2 - \frac{31\zeta_5}{8}$$
$$+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})$$
$$+\log(2)\Big(6\zeta_2-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta_3$$
$$+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{n}\Bigg\{ \Bigg[ -\frac{2}{n} + \frac{5}{12n^2} - \frac{21}{320n^3} - \frac{223}{10752n^4} + \frac{671}{49152n^5} + \frac{11635}{1441792n^6} - \frac{1196757}{136314880n^7} - \frac{376193}{50331648n^8} + \frac{201980317}{18253611008n^9} + O(n^{-10}) \Bigg] \ln(\bar{n}) - \frac{4}{n} + \frac{5}{18n^2} - \frac{263}{2400n^3} + \frac{579}{12544n^4} + \frac{10123}{1105920n^5} - \frac{1705445}{71368704n^6} - \frac{27135463}{11164188672n^7} + \frac{197432563}{7927234560n^8} + \frac{405757489}{775778467840n^9} + O(n^{-10}) \Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

# Conclusion (used techniques)

1. symbolic integration
   (method of hyperlogarithms, Almkvist-Zeilberger algorithm)
2. generalized hypergeometric functions (and extensions)
3. Mellin-Barnes techniques
4. symbolic summation (WZ-, holonomic, difference ring methods)
5. recurrence solving
6. integration by parts technique
7. differential equation solving
8. method of large moments and guessing
9. special function algorithms