

Amplitudes on plane waves

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Joint work with Adamo, Casali & Nekovar 2017-8,
arxiv:1706.08925, 1708.09249, 1809.????.

Motivation

Goal: calculate scattering amplitudes on plane-wave space-time backgrounds.

Our motivation:

- ▶ Simplicity of CHY formulae for tree amplitudes suggests simplification for fully nonlinear expression of theories.
- ▶ Ambitwistor strings are valid on curved backgrounds [Adamo, Casali, Skinner; AC & Nekovar] and provide route to nonlinear version.
- ▶ We have ambitwistor string formulae for up to three points for YM and gravity on plane waves background [Adamo, Casali, M & Nekovar 1708.09249] but need data to check against.

Conventional motivation:

- ▶ A first step towards interacting perturbative QFT on curved backgrounds with external particles.
- ▶ Plane waves give universal features of gravitational waves via Penrose limit.
- ▶ Computable: Plane waves satisfy Huygens for scalar wave equation, with separable Hamilton-Jacobi and linear fields.
- ▶ Test Gravity scattering = Double copy of Yang-Mills.

Yang-Mills amplitudes & colour-kinematic duality

Scatter n particles, momentum k_μ , polarization ϵ_μ

$$A_\mu(x) = \epsilon_\mu e^{ik \cdot x} t^a, \quad k^2 = 0, \quad k \cdot \epsilon = 0, \quad t^a \in \text{Lie } G.$$

- ▶ Suppose that YM amplitude $\mathcal{A}(k_i, \epsilon_i, t_i)$, $i = 1, \dots, n$ arises from trivalent Feynman diagrams

$$\mathcal{A} = \sum_{\Gamma} \frac{N_{\Gamma}(k_i, \epsilon_i) C_{\Gamma}(t_i)}{D_{\Gamma}}, \quad \Gamma \in \{ \text{trivalent diagrams, } n \text{ legs} \}.$$

- ▶ N_{Γ} = *kinematic factors*: polynomials in k_i , linear in each ϵ_i .
- ▶ $D_{\Gamma} = \prod_{\text{propagators } e \in \Gamma} (\sum_{i \in e} k_i)^2$ = *denominators*.
- ▶ $C_{\Gamma}(t_i)$ = *colour factor* = contract structure constants at each vertex together along propagators and with t_i at i th leg.

Definition

The N_{Γ} are said to BCJ numerators if N_{Γ} satisfy identities when C_{Γ} does via Jacobi identities: $C_{\bar{\Gamma}} = C_{\Gamma} + C_{\Gamma'} \Rightarrow N_{\bar{\Gamma}} = N_{\Gamma} + N_{\Gamma'}$.

Possible at tree-level and up to 4-loops, but not canonical.

Gravity as double copy of Yang-Mills

Zvi Bern, J J Carrasco, H Johansson, 2008

Scatter n gravity plane waves

$$h_{\mu\nu} = \epsilon_{(\mu}\epsilon_{\nu)}e^{ik\cdot x}$$

Given BCJ numerators N_Γ , the gravity tree-amplitude/loop integrand can be obtained as a *double copy* of YM amplitude

$$\mathcal{M}(k_i, \epsilon_i, \epsilon_i) = \sum_{\Gamma} \frac{N_\Gamma(k_i, \epsilon_i) N_\Gamma(k_i, \epsilon_i)}{D_\Gamma}$$

- ▶ First conjectured from string amplitude formulae and KLT relations between YM and gravity amplitudes.
- ▶ Proved up to 4-loops.
- ▶ There are many extensions to supersymmetric theories.
- ▶ Genuine tool for constructing gravity amplitudes.
- ▶ No nonperturbative or space-time explanation.

The three point amplitude

At three points, there is just one trivalent diagram

$$\mathcal{A} = (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot (k_1 - k_2) + \text{cyclic}) f_{abc} t_1^a t_2^b t_3^c = N_\lambda(\epsilon_i, k_i) C_\lambda(t_j)$$

For gravity

$$\mathcal{M} = N_\lambda(\epsilon_i, k_i) N_\lambda(\epsilon_i, k_i),$$

but very nontrivial: graviton 3-vertex is much more complicated.

Can we extend to curved backgrounds?

- ▶ How do we define momentum eigenstates?
- ▶ What are momenta and polarization vectors?
- ▶ How can we relate Yang-Mills and Gravity?

Sandwich plane waves

The Brinkman form in d -dimensions of the metric is

$$ds^2 = dudv - Hdu^2 - dx_a dx^a, \quad a = 1, \dots, d-2.$$

with $H = H(u)_{ab} x^a x^b$, $H_a^a := H_{ab} \delta^{ab} = 0$ for vacuum.

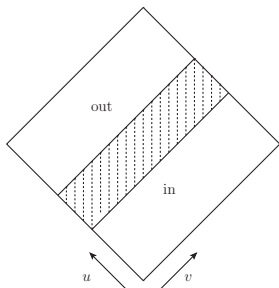


Figure: The sandwich plane wave with x^a -directions suppressed, $H_{ab}(u) \neq 0$ only in the shaded region with flat in- and out-regions.

- ▶ $H_{ab} = \text{curvature}$, supported for $u \in [0, 1]$ (shaded).
- ▶ These coordinates are global, but:
- ▶ Space-time *not* globally hyperbolic! (Penrose).

Plane wave symmetries

$$ds^2 = dudv - H_{ab}(u)x^ax^bdu^2 - dx_adx^a$$

- ▶ Symmetry group = $2d - 3$ -dimensional Heisenberg group transitive on $u = \text{const.}$, centre ∂_v .
- ▶ $e^a(u)$ with

$$\ddot{e}_a = H_{ab}e^b, \quad \cdot = \frac{d}{du}$$

gives Killing vectors $e^a\partial_{x^a} - \dot{e}^ax^a\partial_v$.

- ▶ Choose $d - 2$ -dimensional abelian subgroup

$$D_i = e_i^a\partial_{x^a} - \dot{e}_i^ax_a\partial_v, \quad i = 1 \dots d - 2,$$

commuting $\Leftrightarrow \dot{e}_{[i}^ae_{j]a} = 0$.

- ▶ Let e_a^i be inverse matrix, $e_a^ie_{ib} = \delta_{ab}$.

Momentum eigenstates on plane waves: I. Gravity

- ▶ Choose $d - 1$ commuting symmetries $(\partial_v, D_i) \rightsquigarrow$
Separable Hamilton-Jacobi soln, momenta (k_+, k_i)

$$\phi_k = k_+(v + \frac{1}{2}\sigma_{ab}x^ax^b) + k_ie_a^jx^a + \frac{k_ik_jF^{ij}(u)}{2k_+},$$

where $F^{ij}(u) = \int^u e_a^ie^jadu'$ and $\sigma_{ab} = \dot{e}_a^ie_{bi}$ 'shear'.

- ▶ Then

$$\Phi_k = |e|^{-\frac{1}{2}}e^{i\phi_k}, \quad |e| = \det(e_i^a) \quad \text{solves} \quad \square\Phi_k = 0.$$

- ▶ Such a field has a 'curved' momentum

$$K_\mu dX^\mu := d\phi_k = k_+dv + (\sigma_{ab}x^b + k_ie_a^j)dx^a + (\dots)du$$

Memory:

As $u \rightarrow -\infty$, set $e_i^a = \delta_i^a$ so $K_\mu = (k_+, k_a, k_ak^a/2k_+)$ const.

As $u \rightarrow +\infty$, $e_i^a(u) = b_i^a + uc_i^a$, $b, c = \text{const.}$, and $\sigma_{ab} \neq 0$;
so wave fronts $\phi_k = \text{const.}$ become curved.

Higher spins

- ▶ We have $d - 2$ covariantly constant spin raising operators

$$R^a = du \delta^{ab} \partial_{x^b} + dx^a \partial_v, \quad \nabla_\mu R^a = 0.$$

- ▶ Gives linear gauge field on background

$$A = \frac{\epsilon_a R^a}{k_+} \Phi_k = \epsilon_\mu dX^\mu \Phi_k,$$

with curved polarization ϵ_μ , $K^\mu \epsilon_\mu = 0$,

$$\epsilon_\mu dX^\mu = \epsilon_a dx^a + \epsilon^a \left(\frac{k_i e_a^i}{k_+} + \sigma_{ab} x^b \right) du$$

- ▶ Linear gravity on background

$$h_{\mu\nu} dX^\mu dX^\nu = \frac{\epsilon_a R^a (\epsilon_b R^b \Phi_k)}{k_+^2} = \left((\epsilon \cdot dX)^2 - \frac{i}{k_+} \epsilon_a \epsilon_b \sigma^{ab} du^2 \right) \Phi_k$$

Note potential obstruction to double copy.

Tails and Huygens

Theorem (Friedlander 1970s)

The only space-times that admit clean cut solutions to the wave equation are conformal to plane waves (or flat space).

- ▶ $\Phi = |e|^{-1/2} \delta(\phi_k)$ is clean cut solution to wave equation.
- ▶ Analogous spin-1 solution is

$$a = |e|^{-1/2} \epsilon_a R^a(\phi_k \Theta(\phi_k))$$

so

$$F = da = \delta(\phi_k) |e|^{-1/2} \epsilon_a R^a \phi_k \wedge d\phi_k + \Theta(\phi_k) |e|^{-1/2} \epsilon^a \sigma_{ab}^0 dx^a \wedge du$$

i.e., there is backscattering with a tail.

- ▶ Similar spin-2 solution has longer tail.

Momentum eigenstates on plane waves: II. Yang-Mills

Use same coordinates on flat space-time with gauge potential

$$A = \dot{A}_a(u)x^a du, \quad F = \dot{A}_a(u)dx^a \wedge du.$$

Again, take sandwich wave with $Supp(\dot{A}_a) \subset u \in [0, 1]$.

Momentum eigenstate charge e :

$$\Phi_k = e^{i(k_+ v + (k_a + eA_a)x^a + \frac{f(u)}{2k_+})}, \quad \square_{eA}\Phi_k = 0,$$

momentum $K_\mu(u) = \left(k_+, k_a + eA_a(u), \frac{f(u)}{2k_+}\right)$ with

$$K \cdot K = 0 \quad \rightsquigarrow \quad f(u) = \int_{-\infty}^u (k_a + eA_a)(k^a + eA^a) du'.$$

Memory:

Choose $A_a = 0$ for $u < 0$, then for $u > 1$, $A_a(u) = \text{const.} \neq 0$.

$$K_\mu(u) = \begin{cases} \left(k_+, k_a, \frac{k_a k^a}{2k_+}\right), & u < 0, \\ \left(k_+, k_a + eA_a(1), \frac{f(1)}{2k_+}\right), & u > 1. \end{cases}$$

Linear YM fields on the background

Let \mathfrak{h} be Cartan subalgebra of Lie algebra \mathfrak{g} of gauge group G .

- ▶ Let $\dot{A}_a(u)x^a du$ take values in \mathfrak{h} .
- ▶ Encode colour t^a in charge $e = \text{eigenvalue of } \mathfrak{h} \times \text{coupling}$.
- ▶ Charged linear YM field a_μ satisfies

$$D^\mu D_{[\mu} a_{\nu]} + a^\mu \partial_{[\mu} e A_{\nu]} = 0, \quad D_\mu = \partial_\mu + e A_\mu.$$

- ▶ Solution $a = \tilde{\epsilon}_a R^a \Phi_k = \tilde{\epsilon}_\mu dX^\mu \Phi_k$, transverse polarization

$$\tilde{\epsilon}_\mu(u) dX^\mu = \tilde{\epsilon}_a \left(dx^a + \frac{1}{k_+} (k^a + e A^a(u)) du \right), \quad \epsilon_a = \text{const..}$$

- ▶ Convention: YM background polarization vectors are tilded.

No particle creation or leakage

As $u \rightarrow -\infty$ take linear fields to become flat space-time momentum eigenstates, i.e., $e_i^a = \delta_i^a$, and $A_a = 0$;

- ▶ \pm frequency determined by sign of k_+ , doesn't change with u so no particle creation.
- ▶ Inner products are u -independent on both backgrounds:

$$\langle \Phi_k | \Phi_{k'} \rangle = \int_{u=\text{const.}} \bar{\Phi}_k^* d\Phi_l - \Phi_l^* d\bar{\Phi}_k = 2k_+ \delta(k_+ - l_+) \delta^{d-2}(k_i - l_i).$$

Similarly for spin-1

$$\langle a_1 | a_2 \rangle = \int_{u=\text{const.}} \bar{a}_1 \wedge^* da_2 - a_2 \wedge^* d\bar{a}_1 = 2\epsilon_1 \cdot \epsilon_2 k_+ \delta(k_+ - l_+) \delta^{d-2}(k_i - l_i)$$

and spin-2

$$\langle h_1 | h_2 \rangle = 2(\epsilon_1 \cdot \epsilon_2)^2 k_+ \delta(k_+ - l_+) \delta^{d-2}(k_i - l_i).$$

- ▶ Failure of global hyperbolicity does not lead to leakage: failure is in too high co-dimension in space of null geodesics (i.e., those parallel to ∂_v).

Three particle gravity amplitude

Insert linear fields into 3-vertex taken from action. In our gauge this reduces to

$$\mathcal{M}_3 = \frac{\kappa}{4} \int d^d X (h_1^{\mu\nu} \partial_\mu h_{2\rho\sigma} \partial_\nu h_3^{\rho\sigma} - 2h_1^{\rho\nu} \partial_\mu h_{2\rho\sigma} \partial_\nu h_3^{\mu\sigma}) + \text{perms}$$

This gives with our states, after some manipulation

$$\frac{\kappa}{2} \delta^{d-1} \left(\sum_{r=1}^3 k_r \right) \int \frac{du}{\sqrt{\det e_i^a}} \exp \left(\sum_{r=1}^s \frac{F^{ij} k_{ri} k_{rj}}{2k_{r0}} \right) \\ [(\varepsilon_1 \cdot \varepsilon_2 (K_1 - K - 2) \cdot \varepsilon_3 + \odot)^2 - ik_{1+} k_{2+} k_{3+} \sigma^{ab} C_a C_b]$$

where

$$C_a := \varepsilon_1 \cdot \varepsilon_2 \frac{\varepsilon_{3a}}{k_{3+}} + \odot$$

The first term is square of YM 3-pt amplitude on gravity plane wave, so tail term $\sigma^{ab} C_a C_b$ seems to obstruct double copy.

Three-point YM amplitude

Three point vertex on background now obtained from cubic part of action $\int_M a_{[\mu} a_{\nu]} D^\mu a^\nu d^4 X$ which gives

$$\mathcal{A}_3 = \int du (\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2 \tilde{\epsilon}_3 \cdot (K_1 - K_2) + \odot) \exp\left(i \sum_{r=1}^3 \frac{f_r(u)}{2k_{r0}}\right) C_\lambda(t_i)$$

in terms of flat quantities, the integrand is

$$\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2 \tilde{\epsilon}_3^a \left(\left(\frac{k_{1+}}{k_{2+}} k_{2a} - k_{1a} \right) + A_a \left(e_1 - e_2 \frac{k_{1+}}{k_{2+}} \right) \right) + \odot =: F + C$$

with the second term providing background 'tail' dependence on A_a .

Double copy replacement principle

We can establish correspondence between integrands with following rules

1. Flip charges $e_r \rightarrow -e_r$ in $\mathcal{A}_3 = F + C$ to obtain conjugate $\tilde{\mathcal{A}} = F - C$ so

$$|\mathcal{A}_3|^2 := \mathcal{A}_3 \tilde{\mathcal{A}}_3 = F^2 - C^2$$

with $F = F(k_r, \tilde{\epsilon})$ and $C = C(k_r, \tilde{\epsilon}_r, A)$.

2. In F and C replace k_{ra} by $k_i e_a^i$ and drop \sim 's on ϵ 's yielding $F(k_{ri} e_a^i, \epsilon_r)$ and $C(k_{ri} e_a^i, \epsilon_r, A)$.
- 3.

$$\text{Replace } e_r e_s A^a A^b \rightarrow \begin{cases} i k_{r0} \sigma^{ab} & r = s, \\ i(k_{r0} + k_{s0}) \sigma^{ab} & r \neq s. \end{cases}$$

This now yields a double copy relationship between the integrands that incorporates tail terms.

Four point amplitudes: YM

Work in progress w/Adamo, Casali, Nekovar

- ▶ Construct scalar Feynman propagator as

$$G^F(X, X') = \int \frac{d^d k}{k^2 + i\epsilon} \exp i(\tilde{\phi}_k(X) - \tilde{\phi}_k(X'))$$

where $\tilde{\phi}_k$ is now solution to massive Hamilton-Jacobi

$$\tilde{\phi}_k = k_+ v + (\mathbf{k}_a + eA_a)x^a + \frac{1}{2k_+} \int^u ds [k^2 + (\mathbf{k} + eA(s))^2]$$

with $k^2 = k_+ k_- - k_a k^a \neq 0$.

- ▶ For spin-1 solve $(\square_{eA} + k^2)a_\mu + 2ieF_\mu^\nu a_\nu = 0$ with

$$G_{\mu\nu}^F = \int \frac{d^d k}{k^2 + i\epsilon} P_{\mu\nu}(u, u', k_+) e^{i(\phi_k(X) - \phi_k(X'))}$$

$$P_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\delta_{ac} & \alpha \Delta A_a \\ 1 & -\alpha \Delta A_c & \frac{\alpha^2}{2} \Delta A^2 \end{pmatrix}, \quad \Delta A = A(u) - A(u'), \quad \alpha = \frac{ie}{k_+}$$

Gravity similar, but *much* more complicated.

BCJ at four points

YM diagrams are 3 exchanges in s , t and u channels and contact 4-vertex.

- ▶ BCJ form requires opening up 4-vertex into 3-exchange diagrams using $C_s \rightarrow sC_s/s$ etc..
- ▶ Then 3-channels, s , t and u in flat space give

$$\mathcal{A}_4 = \frac{N_s C_s}{s} + \frac{N_t C_t}{t} + \frac{N_u C_u}{u}$$

with $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, $u = (k_1 + k_3)^2$.

- ▶ Jacobi-identity is $C_s - C_t + C_u = 0$; for BCJ property need

$$N_s - N_t + N_u = 0,$$

but this is naively *obstructed* on a plane wave!

- ▶ Do propagators double copy? Perhaps need convolution.

Conclusions and further developments

- ▶ Explicit Feynman rules on plane waves have been found for both gravity and YM.
- ▶ The original double copy is multiplication in momentum space, and so is convolution on space-time in general.
- ▶ Satisfies nontrivial double copy rules at 3pts but work is in progress at four points: does obstruction to BCJ find natural home in curved background double copy principle?
- ▶ For Feynman propagators on backgrounds for different spins convolution must play bigger role.
- ▶ Full geometric understanding of double copy in nonlinear regime is a long way off, but framework gives first steps.
- ▶ Provides independent calculation to verify ambitwistor string computation on plane wave background in arxiv:1708.09249.
- ▶ Ambitwistor strings manifest double copy. Need to extend this framework to find curved background scattering equations that express double copy beyond four points.

The Oxford Group

Faculty

- ▶ Fernando Alday: Integrability, string dualities, conformal bootstraps.
- ▶ Francis Brown: polylogs, zeta values and number theory.
- ▶ Andrew Hodges: twistor diagrams etc..
- ▶ Also Beem, Candelas, Conlon, De la Ossa, Lukas, Schafer-Nameki, Sparks, Starinetz.

Nearby friends and collaborators:

- ▶ David Skinner, Ron Reid Edwards, twistors/ambitwistor strings (Cambridge).
- ▶ Tomek Lukowski (Hertfordshire).

Postdocs

- ▶ Erik Panzer, Feynman integrals, polylogs and string integrals.
- ▶ Carlo Meneghelli: CFTs, conformal bootstraps.
- ▶ Eduardo Casali (Davis): ambitwistor strings etc..
- ▶ Tim Adamo (Imperial): twistor actions, ambitwistor strings, etc..