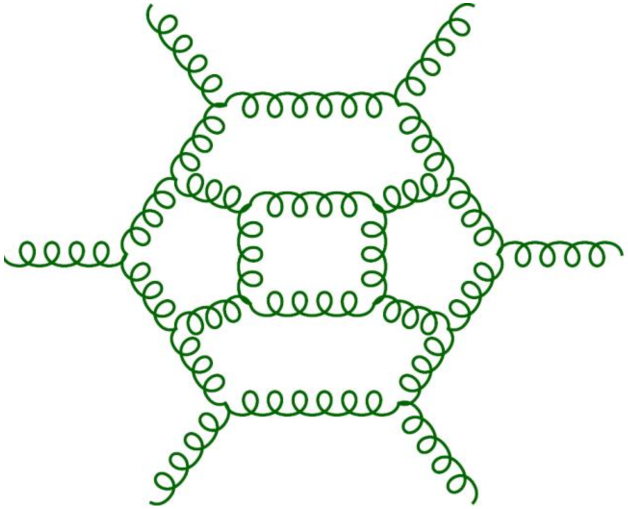
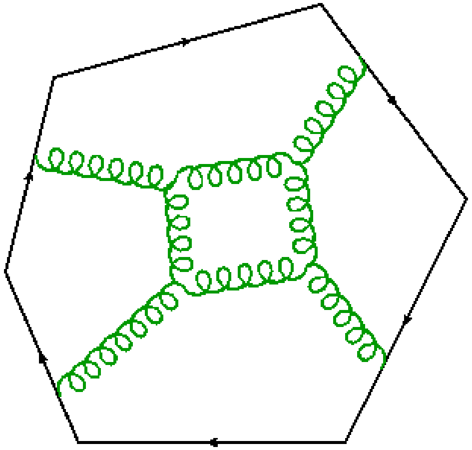
N=4 Super-Yang-Mills Amplitudes and Cosmic Galois Theory





Lance Dixon

1609.00669 with S. Caron-Huot, M. von Hippel, A. McLeod,

18mm.nnnnn also with F. Dulat and G. Papathanasiou

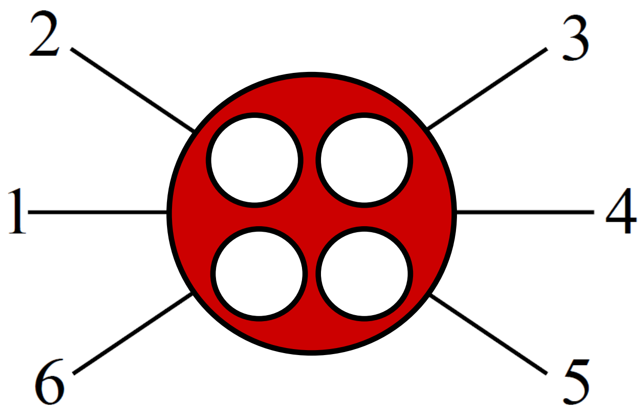
SAGEX Kickoff

September 5, 2018

Hexagon function bootstrap

LD, Drummond, Henn, 1108.4461, 1111.1704;
Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
Heptagon: Drummond, Papathanasiou, Spradlin, 1412.3763;
LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976

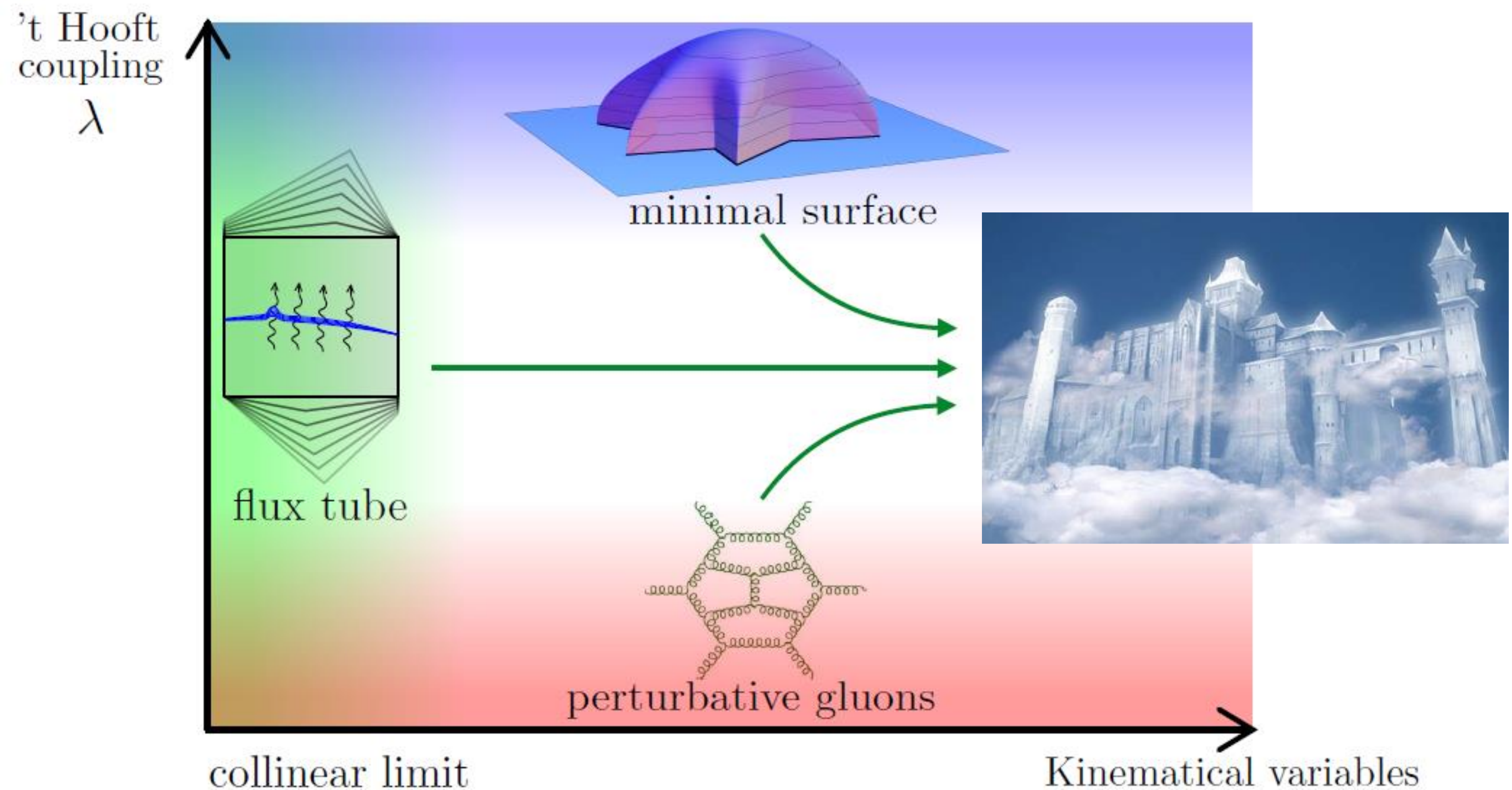
Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops



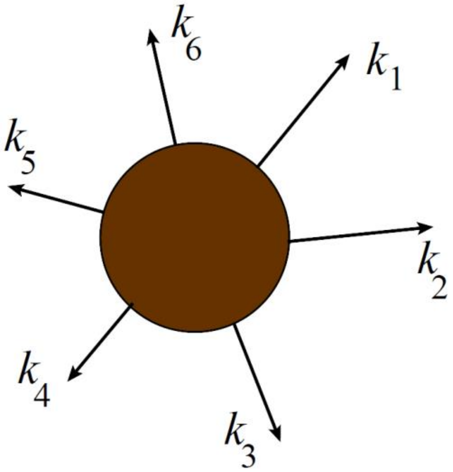
Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

Solving Planar N=4 SYM

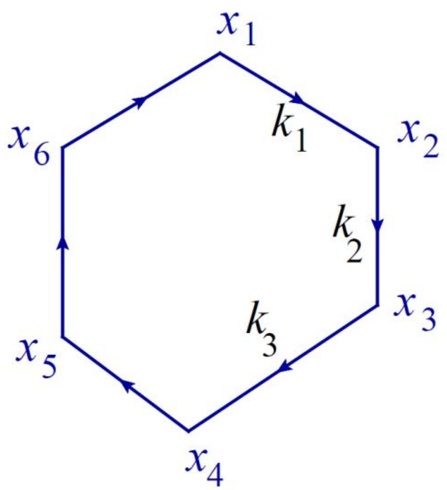
Images: A. Sever, N. Arkani-Hamed



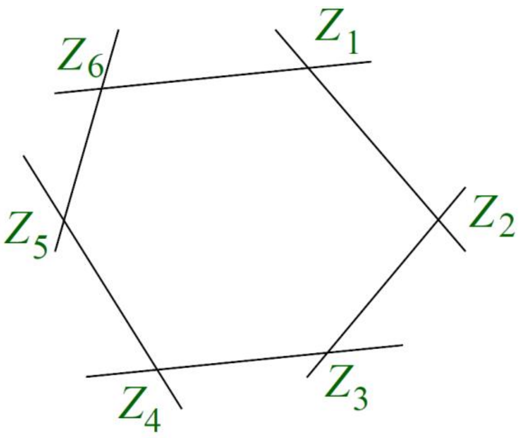
Amplitudes = Wilson loops



Spacetime



Dual Spacetime



Momentum Twistor Space

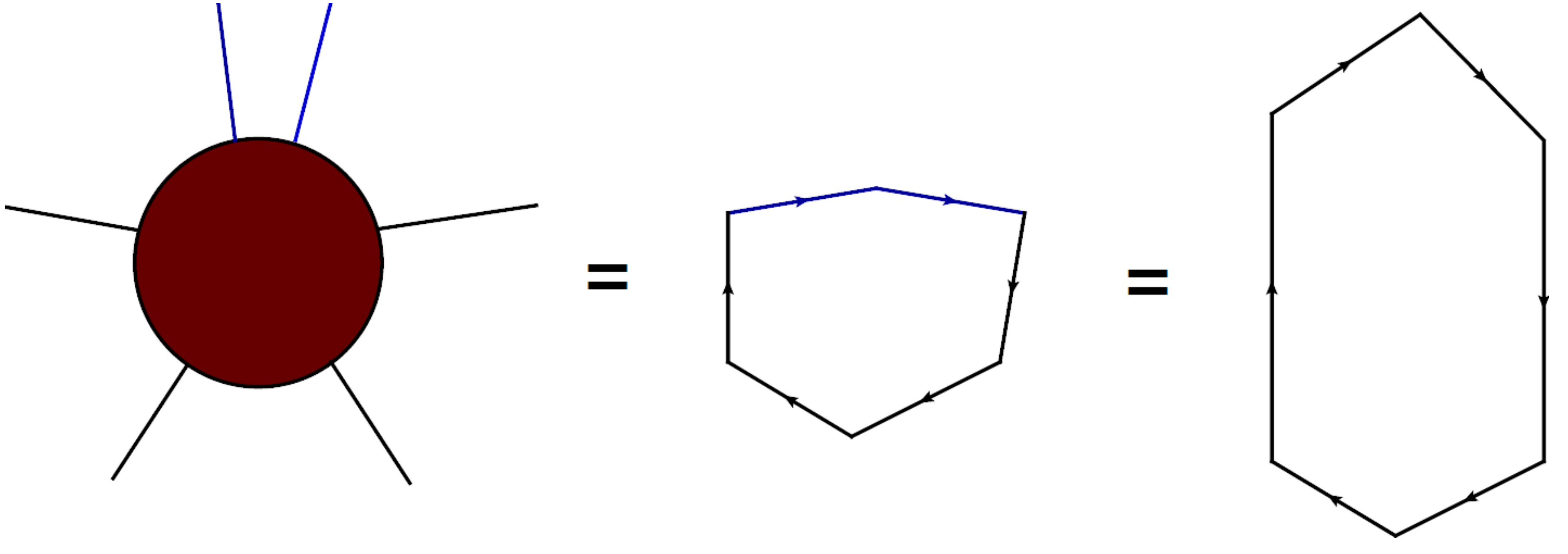
Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Hodges, 0905.1473
Arkani-Hamed et al, 0907.5418, 1008.2958, 1212.5605
Adamo, Bullimore, Mason, Skinner, 1104.2890

Rich theoretical “data” mine

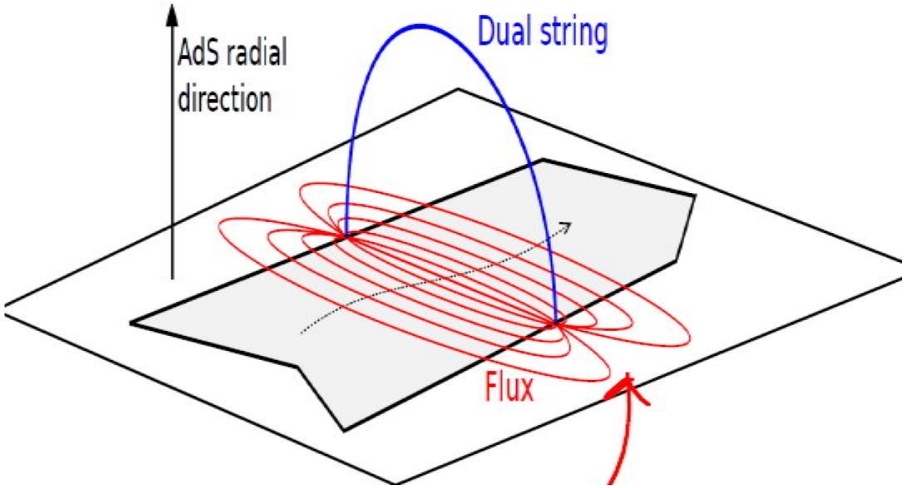
- Rare to have perturbative results to 6 loops
- Usually high loop order → single numbers
- Here we have analytic functions of 3 variables (6 variables in 7-point case)
- Rich Hopf algebraic structure
- Many limits to study (and exploit)

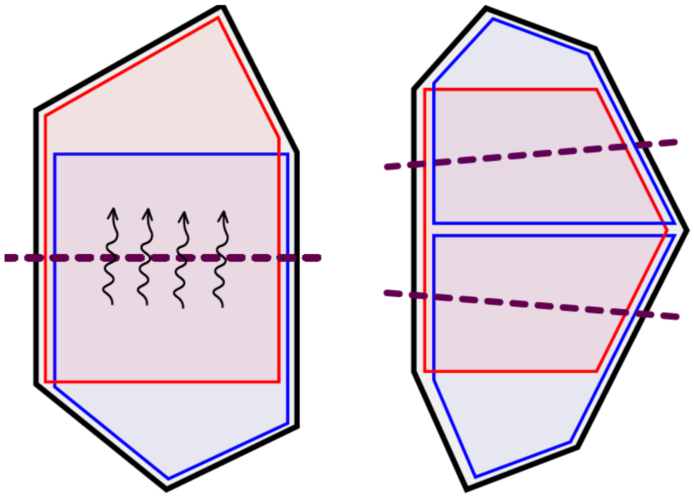
(Near) collinear (OPE) limit



Flux tubes at finite coupling

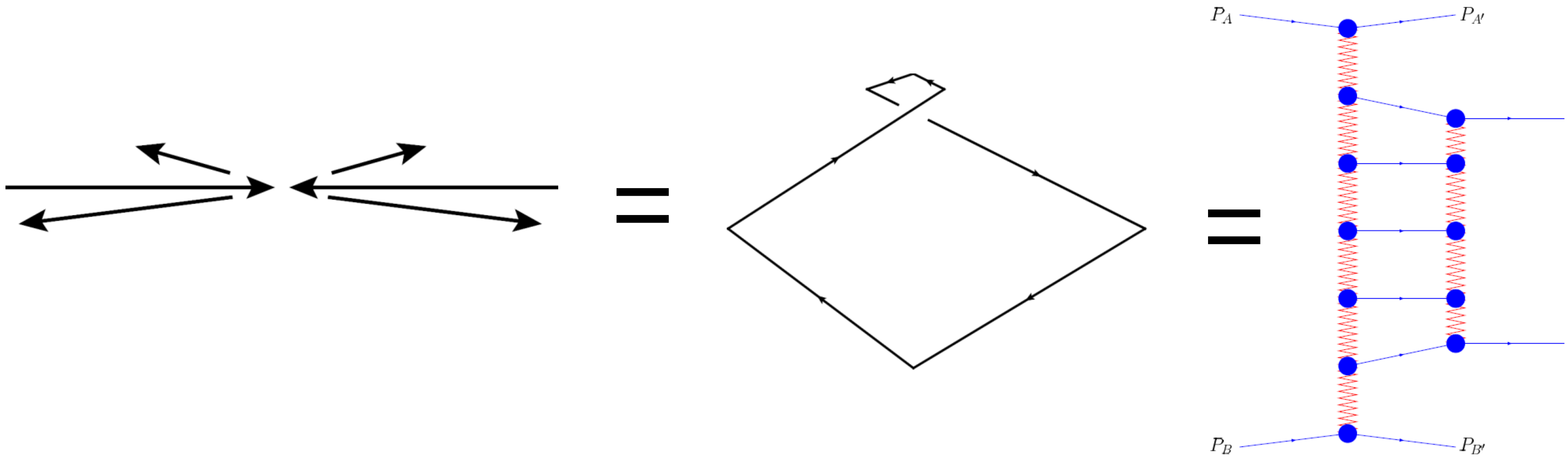
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

Multi-Regge limit



- Amplitude factorizes in Fourier-Mellin space

Bartels, Lipatov, Sabio Vera, 0802.2065, Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357;
Basso, Caron-Huot, Sever, 1407.3766 (analytic continuation from OPE limit);
Broedel, Sprenger, 1512.04963; Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1606.08807

Dual conformal invariance

- Wilson n-gon invariant under

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \qquad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

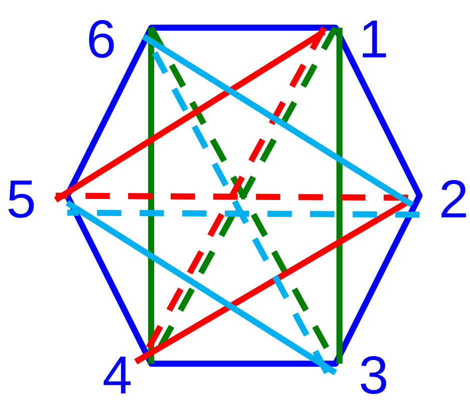
n = 6 → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$v = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

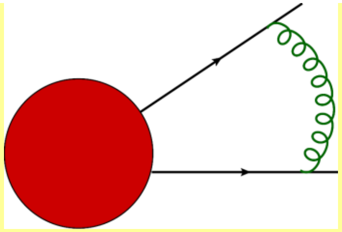
$$w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$

In general, $3n$-15 ratios

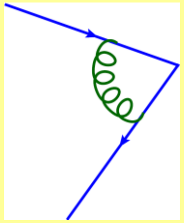


Removing Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension γ_K



- Both removed by dividing by a known function, the BDS-like ansatz Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized amplitude is finite, (dual) conformally invariant.
- BDS-like → also maintain important relation due to causality (Steinmann)

BDS-like ansatz

$$\frac{\mathcal{A}_6^{\rm BDS-like}}{\mathcal{A}_6^{\rm MHV\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon) \frac{1}{2} \hat{M}_6(L\epsilon) + C^{(L)} \right) \right]$$

where $f^{(L)}(\epsilon) = \frac{1}{4}\gamma_K^{(L)} + \epsilon\, \frac{L}{2}\mathcal{G}_0^{(L)} + \epsilon^2\, f_2^{(L)}$ are constants, and

$$\begin{aligned} \hat{M}_6(\epsilon) &= M_6^{\rm 1-loop}(\epsilon) + Y(u,v,w) \\ &= \sum_{i=1}^{6} \left[-\frac{1}{\epsilon^2}(1 - \epsilon \ln s_{i,i+1}) - \ln s_{i,i+1} \ln s_{i+1,i+2} + \frac{1}{2} \ln s_{i,i+1} \ln s_{i+3,i+4} \right] + 6\zeta_2 \end{aligned}$$

$$Y(u,v,w) = \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) + \tfrac{1}{2}(\ln^2 u + \ln^2 v + \ln^2 w)$$

- BDS-like ansatz contains all IR poles, but **no 3-particle invariants**.
- BDS-like removes Y from BDS
- Y is dual conformally invariant part of one-loop amplitude $M_6^{\rm 1-loop}$ containing all 3-particle invariants

6-point BDS-like normalized amplitude

Define

$$\frac{\mathcal{A}_6^{\text{MHV}}}{\mathcal{A}_6^{\text{BDS-like}}} \equiv \mathcal{E}(u,v,w)$$

No 3-particle invariants in denominator of $\mathcal{E}$
→ Necessary for Steinmann constraints to hold
→ A unique choice (up to constant)

Basic bootstrap assumption

- **MHV**: L loop coefficient $\mathcal{E}^{(L)}(u,v,w)$ is a linear combination of weight **2*L*** hexagon functions at any loop order ***L***
- **NMHV**: BDS-like normalized super-amplitude

$$\hat{\mathcal{P}}_{\rm NMHV} \equiv \frac{\mathcal{A}_{\rm NMHV}}{\mathcal{A}_{\rm MHV}^{\rm BDS-like}}$$

Drummond, Henn, Korchemsky, Sokatchev, 0807.1095; LD, von Hippel, McLeod, 1509.08127

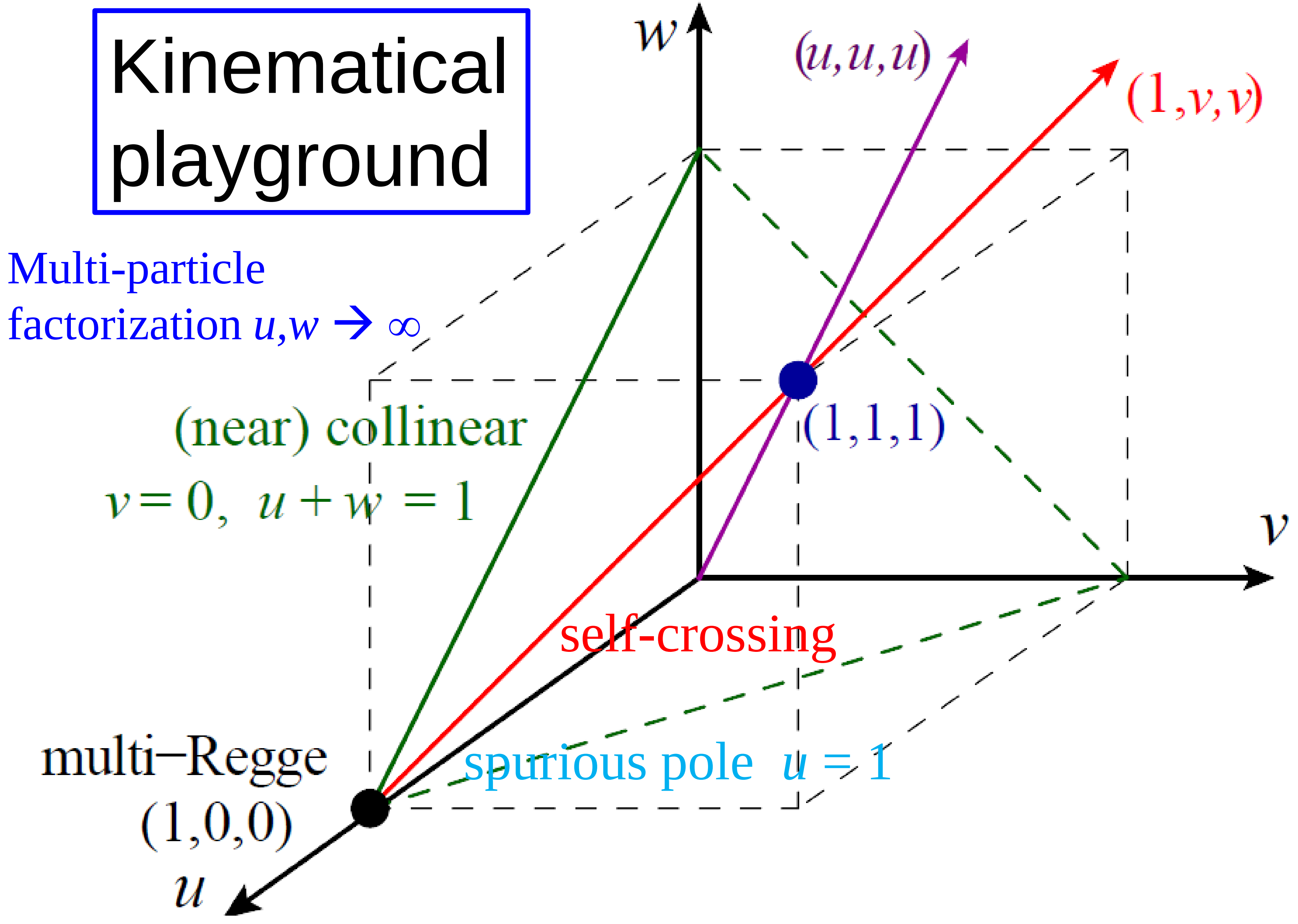
has expansion

$$\hat{\mathcal{P}}_{\rm NMHV} = \frac{1}{2}\Big[[(1)+(4)]E(u,v,w) + [(2)+(5)]E(v,w,u) + [(3)+(6)]E(w,u,v)$$
$$+[(1)-(4)]\tilde{E}(u,v,w) - [(2)-(5)]\tilde{E}(v,w,u) + [(3)-(6)]\tilde{E}(w,u,v)\Big]$$

Grassmann-containing dual superconformal invariants, (a) = [bcdef]

$E^{(L)}, \widetilde{E}^{(L)}$ = hexagon functions

Kinematical playground



Iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives:

$$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1,1\} \text{ component of a "coproduct" } \Delta$$

f^{s_k} are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \;\Rightarrow\; f^{s_j,s_k} \equiv \{n-2,1,1\}$ component
- Symbol = {1,1,…,1} component (maximally iterated)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example 1: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Gen'lize classical polylogs: $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$: 2^{2L}

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.

  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Example 2: Single-valued harmonic polylogarithms of one complex variable

Brown, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 527

- Also a subsector of hexagon functions, in the "multi-Regge limit"
- Symbol letters: $\mathcal{S} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$
- Also require function to be real analytic in $(z, \bar{z}) \in \mathbb{C} - \{0, 1\}$
- Constrains the first entry of the symbol to be $z\bar{z} \leftrightarrow \ln|z|^2$ or $(1-z)(1-\bar{z}) \leftrightarrow \ln|1-z|^2$
- Brown: One SVHPL for each HPL
- Powerful constraint: $4^{2L} \to 2^{2L}$ functions

Hexagon symbol letters

- Momentum twistors $Z_i^A,\ i=1,2,\ldots,6$ transform simply under dual conformal transformations. Hodges, 0905.1473
- Construct 4-brackets $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \langle ijkl \rangle$
- 15 projectively invariant combinations of 4-brackets can be factored into 9 basic ones:

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad 1-u = \frac{\langle 6135\rangle\langle 2346\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$

+ cyclic

Hexagon function symbol letters (cont.)

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-} \;, \;\ldots \quad \text{where} \qquad \begin{aligned} z_\pm &= \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right] \\ \Delta &= (1 - u - v - w)^2 - 4uvw \end{aligned}$$

- Function space graded by **parity**:

$$\begin{array}{rcl} i\sqrt{\Delta} & \leftrightarrow & -i\sqrt{\Delta} \\ z_+ & \leftrightarrow & z_- \\ y_i & \leftrightarrow & 1/y_i \\ u_i & \leftrightarrow & u_i \end{array}$$

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$ or v or w

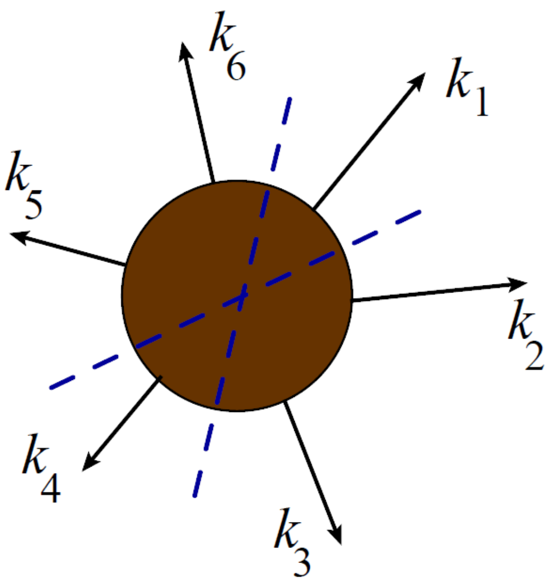
→ First symbol entry $\in \{u, v, w\}$ GMSV, 1102.0062

- Powerful constraint: At weight 8 (four loops) we would have 1,675,553 functions without it; exactly 6,916 with it.
- But most of the 6,916 functions are still unphysical.

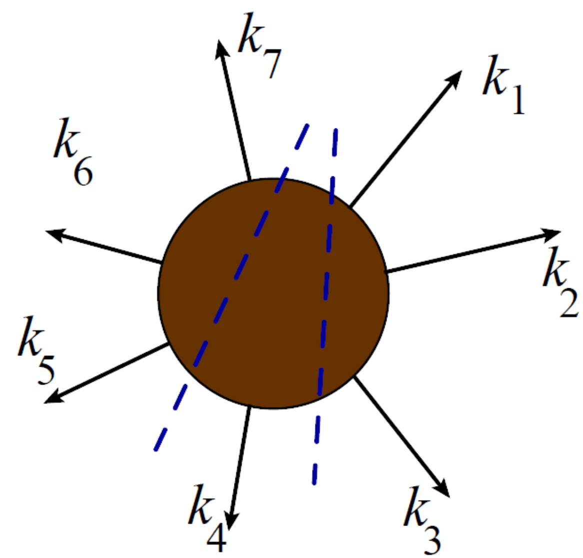
Steinmann relations

Steinmann, Helv. Phys. Acta (1960)   Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:



Not Allowed



Allowed

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

Steinmann relations (cont.)

S. Caron-Huot, LD, M. von Hippel, A. McLeod, 1609.00669

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

+ cyclic conditions

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$\ln^2 u$	$\ln^2 \frac{uv}{w}$
NO	**OK**

$$\frac{uv}{w} = \frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

First two entries restricted to 6 out of 9:

$$\mathrm{Li}_2(1-1/u) \quad \mathrm{Li}_2(1-1/v) \quad \mathrm{Li}_2(1-1/w)$$

$$\ln^2\frac{uv}{w} \quad \ln^2\frac{vw}{u} \quad \ln^2\frac{wu}{v}$$

Analogous constraints for $n=7$

LD, J. Drummond, T. Harrington, A. McLeod, G. Papathanasiou, M. Spradlin, 1612.08976

Iterative Construction of Steinmann hexagon functions

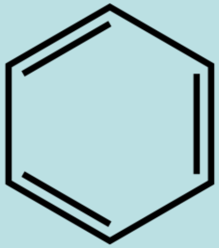
$\{n\text{-}1,1\}$ coproduct F^x characterizes first derivatives, defines F up to additive constant (a multiple zeta value).

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

$$\frac{\partial \ln y_u}{\partial u}$$

1. Insert general linear combinations for F^x
2. Apply "integrability" constraint that mixed-partial derivatives are equal (largest linear algebra computation)
3. Stay in space of functions with good branch cuts and obeying Steinmann by imposing a few more "zeta-valued" conditions in each iteration.

Simple all-loop constraints on $\mathcal{E}$



- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**".
- "Remainder function" $\boldsymbol{R_6}$ vanishes in **collinear** limit ($\boldsymbol{R_6 \rightarrow R_5 = 0}$)

$$v \to 0 \qquad\qquad u + w \to 1$$

$$\frac{\mathcal{A}_6^{\text{MHV}}}{\mathcal{A}_6^{\text{BDS-like}}} \equiv \mathcal{E}(u,v,w) = \exp\left[R_6 - \frac{\gamma_K(a)}{8} Y\right] \qquad \gamma_K(a) = \text{cusp anom. dim.}$$

$$Y(u,v,w) \equiv \text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w) + \frac{1}{2}\left(\ln^2 u + \ln^2 v + \ln^2 w\right)$$

$\bar{Q}$ equation for MHV

Bullimore, Skinner, 1112.1056; Caron-Huot, He, 1112.1060

- First derivative of $\mathcal{E}$ constrained by dual superconformal invariance.
- In terms of final entry of symbol, restricts to 6 of 9 possible letters:

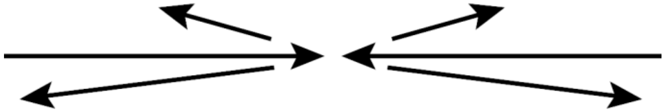
$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- In terms of $\{n\text{-}1,1\}$ coproducts, equivalent to:

$$\mathcal{E}^u + \mathcal{E}^{1-u} = \mathcal{E}^v + \mathcal{E}^{1-v} = \mathcal{E}^w + \mathcal{E}^{1-w} = 0$$

- Similar (but more intricate) constraints for NMHV
  [Caron-Huot], LD, von Hippel McLeod, 1509.08127

Multi-Regge limit



- Euclidean MRK limit vanishes
- To get nonzero result for physical region, first let

$u \to u\, e^{-2\pi i}$, then $u \to 1,\quad v, w \to 0$

$$\frac{v}{1-u} \to \frac{1}{|1-z|^2} \qquad \frac{w}{1-u} \to \frac{|z|^2}{|1-z|^2}$$

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u)\, [g_r^{(L)}(z,\bar{z}) + 2\pi i\, h_r^{(L)}(z,\bar{z})]$$

$g_r^{(L)}$ and $h_r^{(L)}$
all well understood by now;
all SVHPLs (Brown, 2004);
also NMHV behavior

$$weight = 2L - r - 1$$

Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186;
Pennington, 1209.5357;
Basso, Caron-Huot, Sever, 1407.3766;
Broedel, Sprenger, 1512.04963

Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505

Master Table

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)}, E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
0. Steinmann OLD	(7,7)	(37,39)	(174,190)	(758,839)	(3105,3434)	?????
1. Steinmann NEW	(6,6)	(25,27)	(92,105)	(313,372)	(991,1214)	(2951,742?)
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(???,???)
3. Final entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(262,102)
4. Collinear limit	(0,0)	(0,0)	(0*,0*)	(0*,2*)	(1*,5*)	(6*,17*)
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,2)
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. all MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. $T^2F^2\ln^4 T$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
12. all T^2F^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

“Steinmann NEW” = minimal function space

- Want to describe, not only $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$ to a given loop order, but also derivatives ($\{n\text{-}k,1,1,\ldots,1\}$ coproducts) of even higher loop answers.
- But nothing more.
- How many functions do we need?
- We take multiple derivatives/coproducts of amplitudes we know, and ask how much more of the Steinmann space we can remove.

Minimal function space (cont.)

- First surprise already at weight 2
- The many, many {2,1,1,…,1} coproducts of the weight 12 functions $(\mathcal{E}^{(6)}, E^{(6)} \& \tilde{E}^{(6)})$ span only a 6 dimensional subspace of the 7 dimensional Steinmann space, with basis:

$\mathrm{Li}_2(1-1/u)$ $\ln^2\frac{uv}{w} + 4\zeta_2$ plus cyclic

ζ_2 is not an independent element!

Minimal Steinmann space

- At higher weights, we find that all zeta values are not independent elements of the basis, except $\zeta_4, \zeta_6, \zeta_8, \zeta_{10}, \cdots$
- That is,

$$\zeta_2, \zeta_3, \zeta_5, \zeta_2\,\zeta_3, \zeta_3^2, \zeta_7, \zeta_2\zeta_5, \zeta_3\zeta_4, \zeta_{5,3}, \zeta_3\zeta_5, \zeta_2\zeta_3^2, \cdots$$

- are absorbable into other functions

- There are also additional Steinmann constraints, restricting pairs of adjacent entries, but deeper into the symbol than the first two entries.

Cosmic Galois Theory ←→ Co-action principle

[Cartier (2001); Connes, Marcolli (2004)]; Brown, Panzer, Schnetz (~2013+)

- Classical Galois group: Discrete group permuting points = roots of a polynomial
- Is there a kind of continuous group acting on the space of numbers/functions generated by QFT?
- Look for stability (closure) under co-action Δ:

$$\Delta H \subset H \otimes F$$

where Δ acting on weight n has components $\{n\text{-}r,r\}$, $r = 0,1,\ldots,n$. Δ can be computed from iterated integral representations of functions or MZVs.

- Closure *trivial* if H = *all* polylogs or *all* MZVs.
- *Nontrivial* if H is restricted in some way.

Motivic decomposition of MZV's

Brown, 1102.1310 [math.NT]

- Can use Hopf co-algebra to represent any MZV in terms of words composed of an "f-basis" of non-commuting letters f_3 , f_5 , f_7 , … while π^{2k} commutes.
- Generating function for independent MZVs at weight w:

$$\sum_{n=1}^{\infty} d(n) t^n = \frac{1}{1-t^2} \frac{1}{1-t^3-t^5-\cdots} = \frac{1}{1-t^2-t^3}$$
$$= 1 + t + t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + \cdots$$

- f-basis makes manifest a set of odd weight derivations, ∂_3, ∂_5 , ∂_7 , … which clip f_3 , f_5 , f_7 , … off back of a word
- This structure let's us probe co-action beyond {n-1,1}

High loop ϕ^4 primitive UV divergences (periods)

period	$\sum_m f_m^N \delta_m(P_\bullet)$
P_1	0
P_3	$6 f_3 P_1$
P_4	$20 f_5 P_1$
P_5	$\frac{441}{8} f_7 P_1$
$P_{6,1}$	$168 f_9 P_1$
$P_{6,2}$	$\frac{2}{3} f_3 P_3^2 + \frac{1\,063}{9} f_9 P_1$
$P_{6,3}$	$\frac{63}{5} f_3 P_4 - 30 f_5 P_3$
$P_{6,4}$	$-\frac{648}{5} f_3 P_4 + 720 f_5 P_3$
$P_{7,1}$	$\frac{33\,759}{64} f_{11} P_1$
$P_{7,2}$	$\frac{7}{12} f_3 P_3 P_4 - \frac{5}{18} f_5 P_3^2 - \frac{195\,379}{192} f_{11} P_1$
$P_{7,3}$	$\frac{1}{3} f_3 P_3 P_4 - \frac{31}{9} f_5 P_3^2 - \frac{960\,211}{240} f_{11} P_1$
$P_{7,4}, P_{7,7}$	$\frac{160}{21} f_3 P_5 - 20 f_5 P_4 + 70 f_7 P_3$
$P_{7,5}, P_{7,10}$	$-\frac{24}{7} f_3 P_5 + 45 f_5 P_4 - \frac{63}{2} f_7 P_3$
$P_{7,6}$	$\frac{7}{12} f_3 P_3 P_4 + \frac{145}{18} f_5 P_3^2 + \frac{502\,247}{64} f_{11} P_1$
$P_{7,8}$	$f_3(7 P_{6,3} - \frac{161}{30} P_3 P_4) + \frac{527}{9} f_5 P_3^2 + \frac{2\,756\,439}{20} f_{11} P_1$
$P_{7,9}$	$f_3(\frac{7}{2} P_{6,3} - \frac{133}{80} P_3 P_4) - \frac{217}{24} f_5 P_3^2 + \frac{4\,136\,619}{160} f_{11} P_1$
$P_{7,11}$	$f_2^6(-\frac{2\,755}{864} P_{6,1} + \frac{35}{27} P_3^3) + \frac{14}{9} f_4^6 P_5 + \frac{1\,017}{22} f_6^6 P_4 - \frac{36\,918}{43} f_8^6 P_3$
$P_{8,1}$	$1716 f_{13} P_1$
$P_{8,2}$	$f_3(\frac{145}{147} P_3 P_5 - \frac{27}{80} P_4^2) + \frac{29}{40} f_5 P_3 P_4 + \frac{47}{16} f_7 P_3^2 + \frac{94\,871\,691}{22\,400} f_{13} P_1$

…

Panzer, Schnetz, 1603.04289

At weight 11, new non-MZV numbers appear

The result for the QED contribution a_e to the electron $g-2$ is

4 loop Electron g-2

Laporta, 1704.06996;
Schnetz, 1711.05118

$$
\begin{aligned}
a_e \cong\ & \frac{1}{2}\left(\frac{\alpha}{\pi}\right) \\
&+ \left(\frac{197}{144} + \frac{1}{12}\pi^2 + \frac{27}{32} f_3^6 - \frac{1}{4} g_1^6 \pi^2\right)\left(\frac{\alpha}{\pi}\right)^2 \\
&+ \left(\frac{28259}{5184} + \frac{17101}{810}\pi^2 + \frac{139}{16} f_3^6 - \frac{149}{9} g_1^6 \pi^2 - \frac{525}{32} g_1^6 f_3^6 + \frac{1969}{8640}\pi^4 - \frac{1161}{128} f_5^6 \right. \\
&\quad \left. + \frac{83}{64} f_3^6 \pi^2\right)\left(\frac{\alpha}{\pi}\right)^3 \\
&+ \left(\frac{1243127611}{130636800} + \frac{30180451}{155520}\pi^2 - \frac{255842141}{2419200} f_3^6 - \frac{8873}{36} g_1^6 \pi^2 + \frac{126909}{2560} \frac{f_4^6}{\mathrm{i}\sqrt{3}} \right. \\
&\quad - \frac{84679}{1280} g_1^6 f_3^6 + \frac{169703}{3840} \frac{f_2^6 \pi^2}{\mathrm{i}\sqrt{3}} + \frac{779}{108} g_1^6 g_1^6 \pi^2 + \frac{112537679}{3110400}\pi^4 - \frac{2284263}{25600} f_5^6 \\
&\quad + \frac{8449}{96} g_1^6 g_1^6 f_3^6 - \frac{12720907}{345600} f_3^6 \pi^2 - \frac{231919}{97200} g_1^6 \pi^4 + \frac{150371}{256} \frac{f_6^6}{\mathrm{i}\sqrt{3}} + \frac{313131}{1280} g_1^6 f_5^6 \\
&\quad - \frac{121383}{1280} f_2^6 f_4^6 - \frac{14662107}{51200} f_3^6 f_3^6 + \frac{8645}{128} \frac{f_2^6 g_1^6 f_3^6}{\mathrm{i}\sqrt{3}} - \frac{231}{4} g_1^6 g_1^6 g_1^6 f_3^6 - \frac{16025}{48} \frac{f_4^6 \pi^2}{\mathrm{i}\sqrt{3}} \\
&\quad + \frac{4403}{384} g_1^6 f_3^6 \pi^2 - \frac{136781}{1920} f_2^6 f_2^6 \pi^2 + \frac{7069}{75} f_2^4 f_2^4 \pi^2 - \frac{1061123}{14400} f_3^6 g_1^6 \pi^2 \\
&\quad + \frac{1115}{72} \frac{f_2^6 g_1^6 g_1^6 \pi^2}{\mathrm{i}\sqrt{3}} + \frac{781181}{20736} \frac{f_2^6 \pi^4}{\mathrm{i}\sqrt{3}} - \frac{4049}{1080} g_1^6 g_1^6 \pi^4 + \frac{90514741}{54432000}\pi^6 \\
&\quad - \frac{95624828289}{2050048} f_7^6 - \frac{29295}{512} g_1^6 f_2^6 f_4^6 + \frac{107919}{512} g_1^6 f_3^6 f_3^6 + \frac{337365}{256} f_3^6 g_1^6 f_3^6 \\
&\quad - \frac{55618247}{409600} f_5^6 \pi^2 - \frac{1055}{256} g_1^6 f_2^6 f_2^6 \pi^2 + \frac{26}{3} f_1^4 f_2^4 f_2^4 \pi^2 + \frac{553}{4} g_1^6 f_3^6 g_1^6 \pi^2 \\
&\quad - \frac{35189}{1024} f_3^6 g_1^6 g_1^6 \pi^2 + \frac{79147091}{2211840} f_3^6 \pi^4 - \frac{3678803}{4354560} g_1^6 \pi^6 \\
&\quad \left. + \sqrt{3}(E_{4a} + E_{5a} + E_{6a} + E_{7a}) + E_{6b} + E_{7b} + U\right)\left(\frac{\alpha}{\pi}\right)^4 .
\end{aligned}
$$

f_k^4 = weight k primitives for 4th roots of unity

g_1^6 , f_k^6 = weight k primitives for 6th roots of unity

E_i = elliptic

U = unknown

At $(u,v,w) = (1,1,1)$, amplitude → MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear
Brown, Panzer, Schnetz

MHV

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

→ Evidence that the hexagon space is closed under elements of the co-action beyond the {n-1,1} component

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = \frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

MZV's found in full hexagon function space at (1,1,1), in *f*-basis

$\zeta_{12}\,,\quad 7f_{3,9} - 6\zeta_4 f_{3,5}\,,\quad 5f_{3,9} - 3\zeta_6 f_{3,3}\,,\quad \zeta_2 f_{3,7} - \zeta_6 f_{3,3}\,,\quad 7f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\,,\quad 5f_{7,5} - 2\zeta_2 f_{7,3}$

∂_3

$f_{11}\,,\quad \zeta_2 f_9\,,\quad \zeta_4 f_7\,,\quad \zeta_6 f_5\,,\quad \zeta_8 f_3\,,\quad 5f_{3,3,5} - 2\zeta_2 f_{3,3,3}$

$\zeta_{10}\,,\quad 7f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\,,\quad 5f_{5,5} - 2\zeta_2 f_{5,3}$

$7f_9 - 6\zeta_4 f_5\,,\quad 5f_9 - 3\zeta_6 f_3,\quad \zeta_2 f_7 - \zeta_6 f_3$

$\zeta_8\,,\quad \zeta_{5,3} + 5\zeta_3\zeta_5 - \zeta_2(\zeta_3)^2 = 5f_{3,5} - 2\zeta_2 f_{3,3}$

$7\zeta_7 - \zeta_2\zeta_5 - 3\zeta_4\zeta_3 = 7f_7 - \zeta_2 f_5 - \zeta_4 f_3$

∂_5

ζ_6

$5\zeta_5 - 2\zeta_2\zeta_3 = 5f_5 - 2\zeta_2 f_3$

ζ_4

—

ζ_2

—

1

∂_3 ∂_5

- Co-action principle manifest
- Far fewer MZVs than "expected"

$$1 + t + t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + \cdots$$

$$\rightarrow\quad 1 + t^2 + t^4 + t^5 + t^6 + t^7 + 2t^8 + 3t^9 + \ldots$$

Amplitudes at (1,1,1) in f-basis

MHV

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[5f_{3,5} - 2\zeta_2 f_{3,3}\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 384\left[7f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\right] - 312\left[5f_{5,5} - 2\zeta_2 f_{5,3}\right]$$
$$\begin{aligned}\mathcal{E}^{(6)}(1,1,1) = &-\frac{2273108143}{6219}\zeta_{12} + 2264\left[7f_{3,9} - 6\zeta_4 f_{3,5}\right] + 6536\left[5f_{3,9} - 3\zeta_6 f_{3,3}\right]\\ &- 3072\left[\zeta_2 f_{3,7} - \zeta_6 f_{3,3}\right] + 5328\left[7f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\right]\\ &+ 4224\left[5f_{7,5} - 2\zeta_2 f_{7,3}\right].\end{aligned}$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = \frac{36271}{9}\,\zeta_8 - 24\left[5f_{3,5} - 2\zeta_2 f_{3,3}\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 528\left[7f_{3,7} - \zeta_2 f_{3,5} - 3\zeta_4 f_{3,3}\right] + 384\left[5f_{5,5} - 2\zeta_2 f_{5,3}\right]$$
$$\begin{aligned}E^{(6)}(1,1,1) = &\frac{5066300219}{6219}\zeta_{12} - 4664\left[7f_{3,9} - 6\zeta_4 f_{3,5}\right] - 11384\left[5f_{3,9} - 3\zeta_6 f_{3,3}\right]\\ &+ 5664\left[\zeta_2 f_{3,7} - \zeta_6 f_{3,3}\right] - 8928\left[7f_{5,7} - \zeta_2 f_{5,5} - 3\zeta_4 f_{5,3}\right]\\ &- 6528\left[5f_{7,5} - 2\zeta_2 f_{7,3}\right].\end{aligned}$$

Confession: need BDS-like-like

- To get the amplitudes into the minimal space requires, starting at 3 loops, one more redefinition of the BDS ansatz, by a multi-loop constant ρ :

$$\mathcal{A}_6^{\mathrm{BDS-like}'} = \mathcal{A}_6^{\mathrm{BDS-like}} \times \rho$$

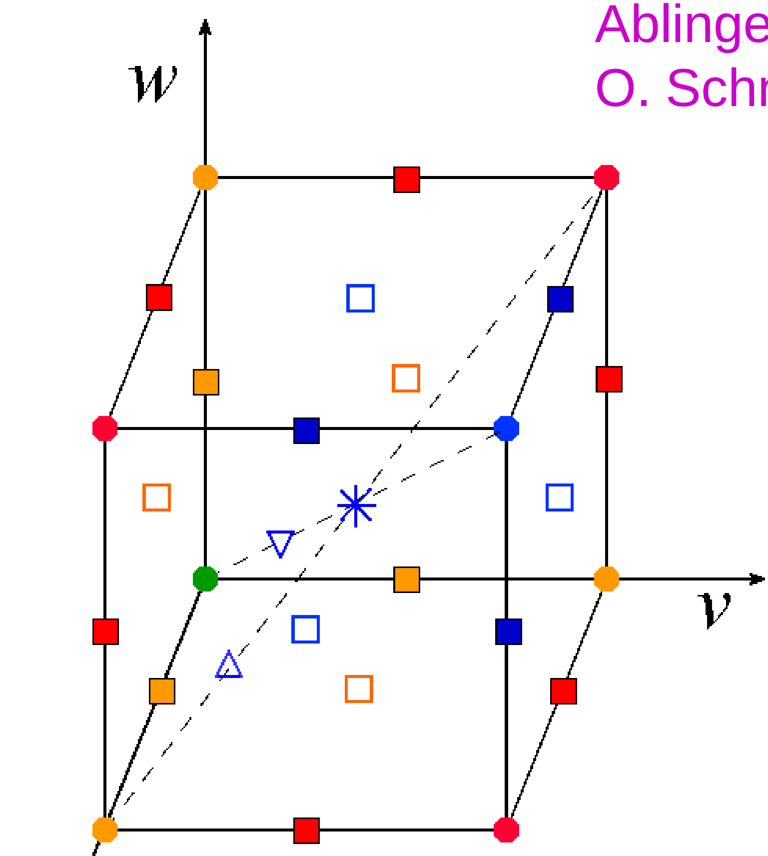
$$\begin{aligned}\rho \;=\; & 1 + (\zeta_3)^2 a^3 - 10\zeta_3\zeta_5\, a^4 \\ & + \Big[-\zeta_4(\zeta_3)^2 + \frac{105}{2}\zeta_3\zeta_7 + \frac{57}{2}(\zeta_5)^2\Big] a^5 \\ & + \Big[\frac{25}{4}\zeta_6(\zeta_3)^2 + 7\zeta_4\zeta_3\zeta_5 - 294\zeta_3\zeta_9 - \frac{651}{2}\zeta_5\zeta_7\Big] a^6 + \cdots\end{aligned}$$

$$a = \frac{N_c g^2}{8\pi^2} = \frac{\lambda}{8\pi^2}$$

- **What is the meaning of ρ ?**

Menagerie of “cyclotomic” polylogs at unity

Ablinger, Blumlein, Schneider, 1105.6063, 1310.5645;
O. Schnetz, **HyperlogProcedures**



- ● MZVs
- ■, □ Alternating sums
- ✳ 4th roots of unity
- ▽, △ 6th roots of unity

finite
1 variable singular
2 variables singular
3 variables singular

Galois co-action principle applies to entire function space at every point at which we have checked it!!

many more such points

e.g.

$$u = v = w, \qquad y_u = y_v = y_w = y,$$
$$u = \frac{y}{(1+y)^2} \qquad 1 - u = \frac{1+y+y^2}{(1+y)^2}$$

On the line $(u,u,1)$, everything collapses to HPLs of u.
In a linear representation, and a very compressed notation,

$$H_1^u H_{2,1}^u = H_1^u H_{0,1,1}^u = 3H_{0,1,1,1}^u + H_{1,0,1,1}^u \to 3h_7^{[4]} + h_{11}^{[4]}$$

2 and 3 loop answers:

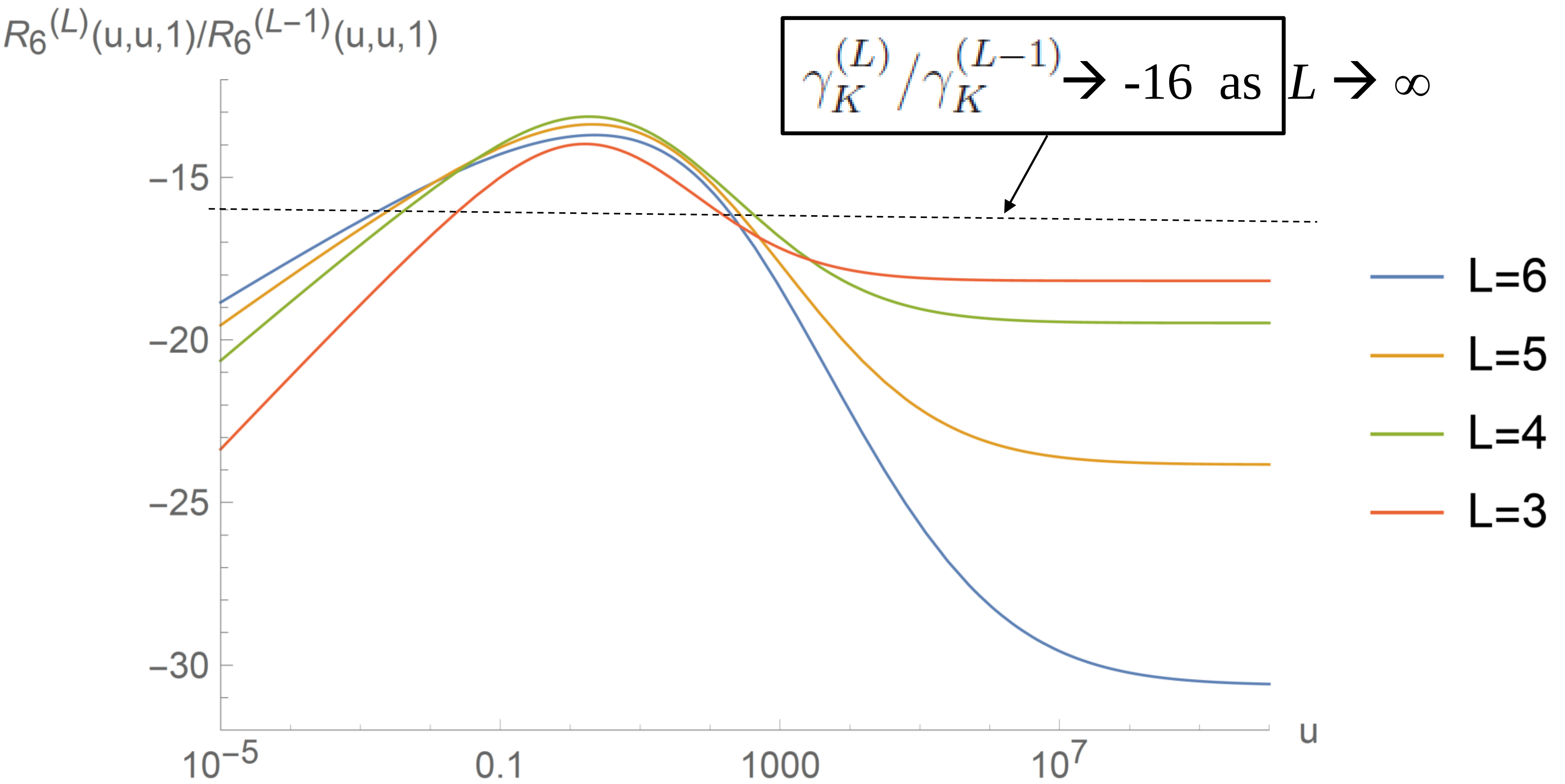
$$\begin{aligned}
R_6^{(2)}(u,u,1) &= h_1^{[4]} - h_3^{[4]} + h_9^{[4]} - h_{11}^{[4]} - \frac{5}{2}\zeta_4\,, \\
R_6^{(3)}(u,u,1) &= -3h_1^{[6]} + 5h_3^{[6]} + \frac{3}{2}h_5^{[6]} - \frac{9}{2}h_7^{[6]} - \frac{1}{2}h_9^{[6]} - \frac{3}{2}h_{11}^{[6]} - h_{13}^{[6]} - \frac{3}{2}h_{17}^{[6]} \\
&\quad + \frac{3}{2}h_{19}^{[6]} - \frac{1}{2}h_{21}^{[6]} - \frac{3}{2}h_{23}^{[6]} - 3h_{33}^{[6]} + 5h_{35}^{[6]} + \frac{3}{2}h_{37}^{[6]} - \frac{9}{2}h_{39}^{[6]} \\
&\quad - \frac{1}{2}h_{41}^{[6]} - \frac{3}{2}h_{43}^{[6]} - h_{45}^{[6]} - \frac{3}{2}h_{49}^{[6]} + \frac{3}{2}h_{51}^{[6]} - \frac{1}{2}h_{53}^{[6]} - \frac{3}{2}h_{55}^{[6]} \\
&\quad + \zeta_2\left[-h_1^{[4]} + 3h_3^{[4]} + 2h_5^{[4]} - h_9^{[4]} + 3h_{11}^{[4]} + 2h_{13}^{[4]}\right] \\
&\quad + \zeta_4\left[-2h_1^{[2]} - 2h_3^{[2]}\right] + \zeta_3^2 + \frac{413}{24}\zeta_6\,,
\end{aligned}$$

4 loop answer →

$$\begin{aligned}
R_6^{(4)}(u,u,1) = {} & 15h_1^{[8]} - 41h_3^{[8]} - \frac{31}{2}h_5^{[8]} + \frac{105}{2}h_7^{[8]} - \frac{7}{2}h_9^{[8]} + \frac{53}{2}h_{11}^{[8]} + 12h_{13}^{[8]} - 42h_{15}^{[8]} \\
& + \frac{5}{2}h_{17}^{[8]} + \frac{11}{2}h_{19}^{[8]} + \frac{9}{2}h_{21}^{[8]} - \frac{41}{2}h_{23}^{[8]} + h_{25}^{[8]} - 13h_{27}^{[8]} - 7h_{29}^{[8]} - 5h_{31}^{[8]} \\
& + 6h_{33}^{[8]} - 11h_{35}^{[8]} - 3h_{37}^{[8]} + 3h_{39}^{[8]} - 4h_{43}^{[8]} - 4h_{45}^{[8]} - 11h_{47}^{[8]} + \frac{3}{2}h_{49}^{[8]} - \frac{3}{2}h_{51}^{[8]} \\
& - 3h_{53}^{[8]} - 5h_{55}^{[8]} + \frac{3}{2}h_{57}^{[8]} - \frac{3}{2}h_{59}^{[8]} + 9h_{65}^{[8]} - 25h_{67}^{[8]} - 9h_{69}^{[8]} + 27h_{71}^{[8]} - 2h_{73}^{[8]} \\
& + 9h_{75}^{[8]} + 2h_{77}^{[8]} - 23h_{79}^{[8]} + 2h_{81}^{[8]} - h_{85}^{[8]} - 8h_{87}^{[8]} + 2h_{89}^{[8]} - 3h_{91}^{[8]} + \frac{5}{2}h_{97}^{[8]} \\
& - \frac{7}{2}h_{99}^{[8]} - \frac{1}{2}h_{101}^{[8]} + \frac{5}{2}h_{103}^{[8]} + \frac{1}{2}h_{105}^{[8]} + \frac{1}{2}h_{107}^{[8]} + \frac{1}{2}h_{109}^{[8]} - \frac{5}{2}h_{111}^{[8]} + 15h_{129}^{[8]} \\
& - 41h_{131}^{[8]} - \frac{31}{2}h_{133}^{[8]} + \frac{105}{2}h_{135}^{[8]} - \frac{7}{2}h_{137}^{[8]} + \frac{53}{2}h_{139}^{[8]} + 12h_{141}^{[8]} - 42h_{143}^{[8]} \\
& + \frac{5}{2}h_{145}^{[8]} + \frac{11}{2}h_{147}^{[8]} + \frac{9}{2}h_{149}^{[8]} - \frac{41}{2}h_{151}^{[8]} + h_{153}^{[8]} - 13h_{155}^{[8]} - 7h_{157}^{[8]} \\
& - 5h_{159}^{[8]} + 6h_{161}^{[8]} - 11h_{163}^{[8]} - 3h_{165}^{[8]} + 3h_{167}^{[8]} - 4h_{171}^{[8]} - 4h_{173}^{[8]} \\
& - 11h_{175}^{[8]} + \frac{3}{2}h_{177}^{[8]} - \frac{3}{2}h_{179}^{[8]} - 3h_{181}^{[8]} - 5h_{183}^{[8]} + \frac{3}{2}h_{185}^{[8]} - \frac{3}{2}h_{187}^{[8]} \\
& + 9h_{193}^{[8]} - 25h_{195}^{[8]} - 9h_{197}^{[8]} + 27h_{199}^{[8]} - 2h_{201}^{[8]} + 9h_{203}^{[8]} + 2h_{205}^{[8]} - 23h_{207}^{[8]} \\
& + 2h_{209}^{[8]} - h_{213}^{[8]} - 8h_{215}^{[8]} + 2h_{217}^{[8]} - 3h_{219}^{[8]} + \frac{5}{2}h_{225}^{[8]} - \frac{7}{2}h_{227}^{[8]} - \frac{1}{2}h_{229}^{[8]} \\
& + \frac{5}{2}h_{231}^{[8]} + \frac{1}{2}h_{233}^{[8]} + \frac{1}{2}h_{235}^{[8]} + \frac{1}{2}h_{237}^{[8]} - \frac{5}{2}h_{239}^{[8]} \\
& + \zeta_2\Big[2h_1^{[6]} - 14h_3^{[6]} - \frac{15}{2}h_5^{[6]} + \frac{37}{2}h_7^{[6]} - \frac{5}{2}h_9^{[6]} + \frac{25}{2}h_{11}^{[6]} + 7h_{13}^{[6]} - \frac{1}{2}h_{17}^{[6]} \\
& \qquad + \frac{5}{2}h_{19}^{[6]} + \frac{7}{2}h_{21}^{[6]} + \frac{9}{2}h_{23}^{[6]} - 3h_{25}^{[6]} + 3h_{27}^{[6]} + 2h_{33}^{[6]} - 14h_{35}^{[6]} - \frac{15}{2}h_{37}^{[6]} \\
& \qquad + \frac{37}{2}h_{39}^{[6]} - \frac{5}{2}h_{41}^{[6]} + \frac{25}{2}h_{43}^{[6]} + 7h_{45}^{[6]} - \frac{1}{2}h_{49}^{[6]} + \frac{5}{2}h_{51}^{[6]} + \frac{7}{2}h_{53}^{[6]} \\
& \qquad + \frac{9}{2}h_{55}^{[6]} - 3h_{57}^{[6]} + 3h_{59}^{[6]}\Big] \\
& + \zeta_4\left[\frac{15}{2}h_1^{[4]} - \frac{55}{2}h_3^{[4]} - \frac{41}{2}h_5^{[4]} + \frac{15}{2}h_9^{[4]} - \frac{55}{2}h_{11}^{[4]} - \frac{41}{2}h_{13}^{[4]}\right] \\
& + \left(\zeta_2\zeta_3 - \frac{5}{2}\zeta_5\right)\left[h_3^{[3]} + h_7^{[3]}\right] - \left(\zeta_3^2 - \frac{73}{4}\zeta_6\right)\left[h_1^{[2]} + h_3^{[2]}\right] \\
& - \frac{3}{2}\zeta_2\zeta_3^2 - \frac{5}{2}\zeta_3\zeta_5 - \frac{471}{4}\zeta_8 + \frac{3}{2}\zeta_{5,3}\,.
\end{aligned}$$

5 loop answer is several pages
6 loop answer is a novel!

Numerical values on $(u,u,1)$ – and finite radius of convergence of perturbation theory



$\gamma_K^{(L)}/\gamma_K^{(L-1)} \rightarrow$ -16 as $L \rightarrow \infty$

Beyond 6 gluons

- Cluster algebras provide strong clues to “the right functions”

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Spradlin talk at Amplitudes 2016; Drummond, Foster, Gurdogan, 1710.10953

- Power seen particularly in symbol of 3-loop MHV 7-point amplitude. 6 variables, 42 letters.

Drummond, Papathanasiou, Spradlin 1412.3763

- With Steinmann relations, can go to 4-loop MHV and 3-loop non-MHV LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976, and in progress

Summary & Outlook

- Steinmann hexagon (heptagon) functions provide solution space for planar N=4 SYM amplitudes/WLs over full kinematical phase space, for 6 (7) gluons, both MHV and NMHV, to high loop orders.
- 6 point: used only multi-Regge limits, OPE at 6 loops
- 7 point (symbol): only basic collinear limits needed.
- Rich algebraic structure: Lots of evidence for closure of hexagon functions under co-action principle [Schnetz], as also seen in g-2, ϕ^4
- Can we go to finite coupling for generic kinematics? What are the right finite-coupling functions? Clues from OPE/integrability?

Extra Slides

Cosmic Galois Theory

Studies the symmetries of 'periods' (integrals of rational functions over domains given by rational inequalities)

- The space of functions appearing in the six-point amplitude is (conjecturally) stable under the coaction
- This property can be formulated as a 'coaction principle'

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H}^{\pi}$$

  which incorporates the branch cut condition, but also constrains the constants that can show up

- This can be alternately formulated in terms of the action of the 'cosmic Galois group' C which is dual to this coaction

$$C \times \mathcal{H}^{\text{hex}} \to \mathcal{H}^{\text{hex}}$$

Cosmic Galois Theory

- The Lie algebra of C includes a set of elements ∂_{2m+1} that act on the zeta values as

$$\partial_{2m+1}\zeta_{2n+1} = \delta_{m,n}$$

  and that satisfy the Leibniz rule. So, for example,

$$\partial_3(\zeta_7\zeta_3^2) = 2\zeta_7\zeta_3$$

- There is no ∂_2, because including even zeta values on both sides of the coaction leads to contradictions
- These operators also act nontrivially on multiple zeta values

  Brown, arXiv:1102.1310 [math.NT]

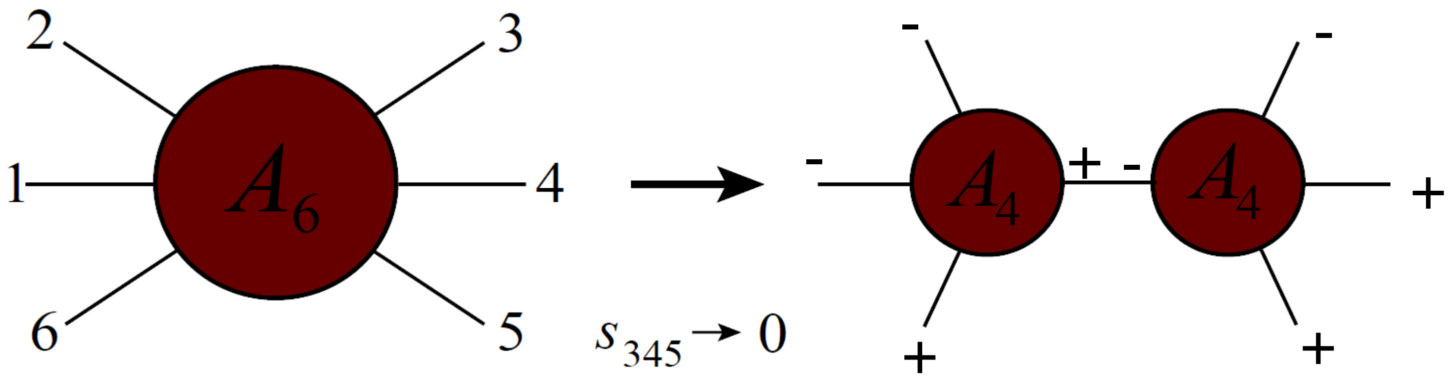
MZV restrictions

Weight	All MZVs	$\mathcal{H}^{\text{hex}}(1,1,1)$	$\mathcal{H}^{\text{hex}}$ indep.
0	1	1	1
1	–	–	–
2	ζ_2	ζ_2	–
3	ζ_3	–	–
4	ζ_4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_2\zeta_3$	$5\zeta_5 - 2\zeta_2\zeta_3$	–
6	$\zeta_6,\ (\zeta_3)^2$	ζ_6	ζ_6
7	$\zeta_7,\ \zeta_2\zeta_5,\ \zeta_3\zeta_4$	$7\zeta_7 - \zeta_2\zeta_5 - 3\zeta_3\zeta_4$	–
8	$\zeta_8,\ \zeta_{5,3},\ \zeta_3\zeta_5,\ \zeta_2(\zeta_3)^2$	$\zeta_8,\ \zeta_{5,3} + 5\zeta_3\zeta_5 - \zeta_2(\zeta_3)^2$	ζ_8

∂_3 ∂_5 ∂_3

NMHV Multi-Particle Factorization

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505



$$A_6^{\rm NMHV}(k_i) \xrightarrow{s_{345}\to 0} A_4(k_6,k_1,k_2,K)\,\frac{F_6(K^2,s_{i,i+1})}{K^2}\,A_4(-K,k_3,k_4,k_5)$$

Look at for NMHV: MHV tree has no pole

$$\mathcal{A}_{\rm MHV}^{(0)} = i\,\frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \to \infty \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}} \to \infty$$

$$u/w \text{ and } v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \text{ fixed}$$

Multi-Particle Factorization (cont.)

$(1) = (4) \to \infty, \text{ rest finite}$

→ look at $E(u,v,w)$

Or rather at $U(u,v,w) = \ln E(u,v,w)$

$$\frac{\mathcal{A}_{\mathrm{NMHV}}}{\mathcal{A}_{\mathrm{BDS-like}}} \approx e^{U} [(1) + (4)]$$

Factorization limit of U

$$U^{(1)}(u,v,w) = -\frac{1}{4}\ln^2(uw/v) - \zeta_2$$

$$U^{(2)}(u,v,w)|_{u,w\to\infty} = \frac{3}{4}\zeta_2 \ln^2(uw/v) - \frac{1}{2}\zeta_3 \ln(uw/v) + \frac{71}{8}\zeta_4$$

$$U^{(3)}(u,v,w)|_{u,w\to\infty} = \frac{1}{3}\zeta_3 \ln^3(uw/v) - \frac{75}{8}\zeta_4 \ln^2(uw/v) + (7\,\zeta_5 + 8\,\zeta_2\,\zeta_3)\ln(uw/v) - \frac{721}{8}\zeta_6 - 3\,(\zeta_3)^2$$

…

- Simple polynomial in $\ln(uw/v)$, form dictated by Steinmann relations

$$\frac{uw}{v} = \frac{s_{12}s_{34}}{s_{56}} \cdot \frac{s_{45}s_{61}}{s_{23}} \cdot \frac{1}{s_{345}^2}$$

- Sudakov logs due to on-shell intermediate state

- All orders form available via analytic continuation from the near-collinear (OPE) limit.

Basso, Sever, Vieira
(Sever talk at Amplitudes 2015)

All hexagon letter are rational in terms of y_i

$$u = \frac{y_u(1-y_v)(1-y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad v = \frac{y_v(1-y_w)(1-y_u)}{(1-y_v y_w)(1-y_v y_u)}, \quad w = \frac{y_w(1-y_u)(1-y_v)}{(1-y_w y_u)(1-y_w y_v)}$$

$$1-u = \frac{(1-y_u)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad \text{etc.}, \quad \sqrt{\Delta} = \frac{(1-y_u)(1-y_v)(1-y_w)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_v y_w)(1-y_w y_u)}$$

$$\mathcal{S} = \{y_i,\ 1-y_i,\ 1-y_i y_j,\ 1-y_u y_v y_w\}$$

"extra" 10th letter

MRK Master formulae

$$w = -z, \quad w^* = -\bar{z}$$

- MHV:

$$e^{R+i\pi\delta}|_{\rm MRK} = \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\Phi_{\rm Reg}(\nu,n)\times\left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

NLL: Fadin, Lipatov, 1111.0782;
Caron-Huot, 1309.6521

- NMHV:

$$\exp(R^{\rm NMHV}+i\pi\delta)|_{\rm MRK} = \mathcal{P}\exp(R^{\rm MHV}+i\pi\delta)$$
$$= \cos\pi\omega_{ab} - i\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{(i\nu+\frac{n}{2})^2}|w|^{2i\nu}$$
$$\times\,\Phi_{\rm Reg}^{\rm NMHV}(\nu,n)\left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

LL: Lipatov, Prygarin, Schnitzer, 1205.0186

NMHV MRK limit

Like g, h for R_6:
Extract p, q from V, $\tilde{V}$
→ linear combinations of SVHPLs [Brown, 2004]

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \, [g_r^{(L)}(w,w^*) + 2\pi i \, h_r^{(L)}(w,w^*)]$$

$$\mathcal{P}_{\text{MRK}}^{(L)} = (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \Big[\frac{1}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i \, q_r^{(L)}(w,w^*) \big) + \frac{w^*}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i \, q_r^{(L)}(w,w^*) \big) \big|_{(w,w^*)\to(\frac{1}{w},\frac{1}{w^*})} \Big] + \mathcal{O}(1-u)$$

- Then match p, q to master formula for factorization in Fourier-Mellin space

MRK limits agree with all-orders predictions

Basso, Caron-Huot, Sever 1407.3766

- BFKL eigenvalue:

$$E^{(1)}(\nu,n),\ E^{(2)}(\nu,n),\ E^{(3)}(\nu,n)$$

LL, NLL, NNLL, NNNLL

- Impact factors:

$$\Phi_{\rm Reg}^{\rm (N)MHV,(1)}(\nu,n),\ \Phi_{\rm Reg}^{\rm (N)MHV,(2)}(\nu,n),\ \Phi_{\rm Reg}^{\rm (N)MHV,(3)}(\nu,n),\ \Phi_{\rm Reg}^{\rm (N)MHV,(4)}(\nu,n)$$

- All zeta-valued linear combinations of:

derivatives of $\ln\Gamma(1 \pm i\nu + \frac{n}{2})$ $\frac{i\nu}{\nu^2+\frac{n^2}{4}},\ \frac{n}{\nu^2+\frac{n^2}{4}}$

$\bar{\text{Q}}$ equation for NMHV

Caron-Huot, He, 1112.1060; S. Caron-Huot (2015);
LD, von Hippel, McLeod, 1509.08127

$$\bar{Q}\hat{\mathcal{R}}_{6,1} = \frac{\gamma_K}{8}\int d^{2|3}\mathcal{Z}_7[\mathcal{R}_{7,2} - \hat{\mathcal{R}}_{6,1}\mathcal{R}_{7,1}^{\text{tree}}] + \text{ cyclic}$$

$$\bar{Q}_a^A = \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial Z_i^a} \qquad \hat{\mathcal{R}}_{6,1} \equiv \frac{\mathcal{A}_{\text{NMHV}}}{\mathcal{A}_{\text{BDS-like}}}$$

prevents second (simpler) term from generating new "final entries"

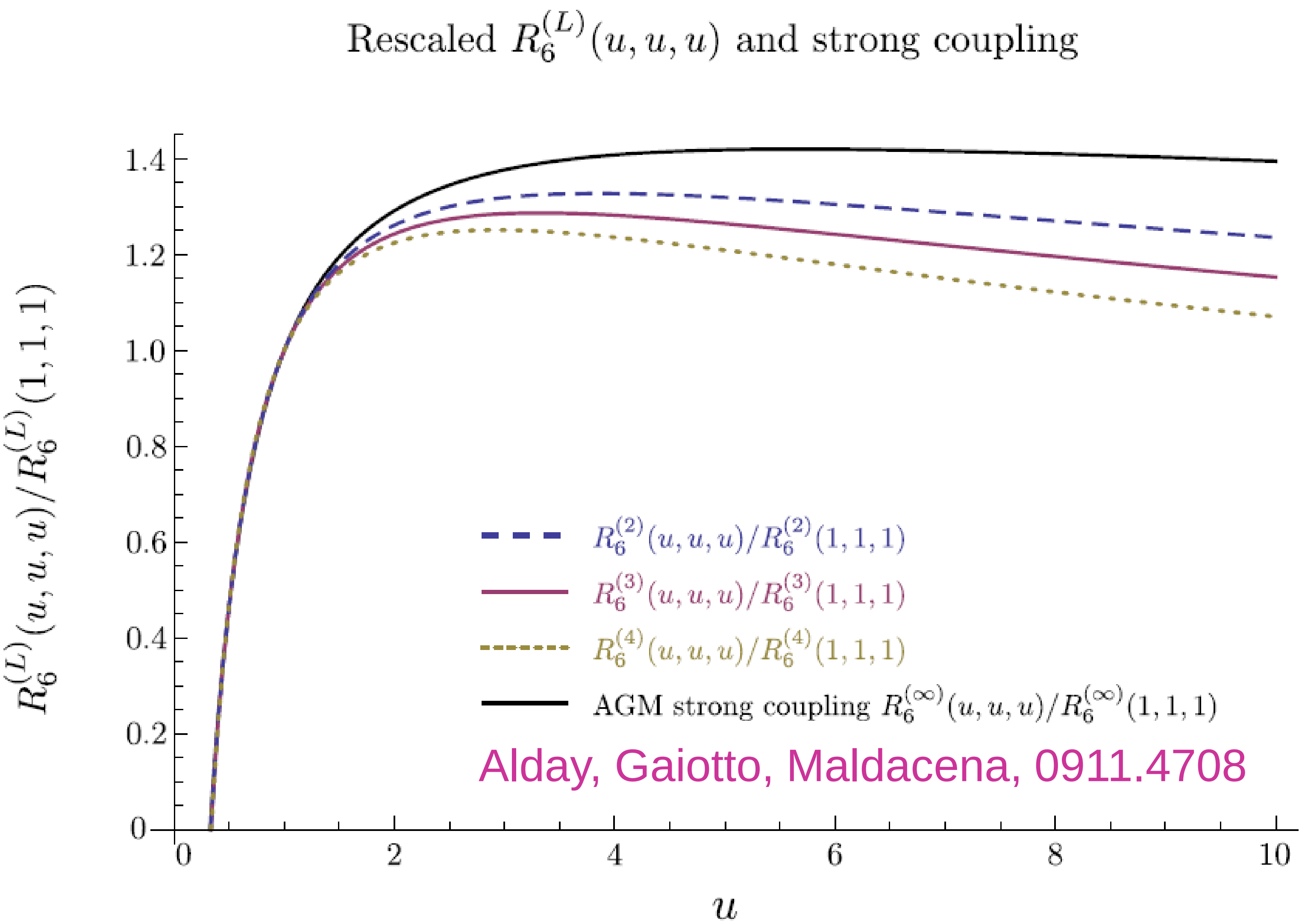
→ Only 18 out of 5 x 9 = 45 possible R-invariants x final entries:

$$(1)\, d\ln(uw/v), \quad (1)\, d\ln\Big(\frac{(1-w)u}{w(1-u)y_v}\Big),$$

$$[(2)+(5)+(3)+(6)]\, d\ln\Big(\frac{v}{1-v}\Big) + (1)\, d\ln\Big(\frac{w}{y_u(1-w)}\Big) + (4)\, d\ln\Big(\frac{u}{y_w(1-u)}\Big)$$

+ cyclic

L	$\gamma_K^{(L)}/\gamma_K^{(L-1)}$	$\bar{R}_6^{(L)}(1,1,1)$	$\overline{\ln \mathcal{W}}_{\rm hex}^{(L)}(\frac{3}{4},\frac{3}{4},\frac{3}{4})$
2	-1.6449340	∞	-2.7697175
3	-3.6188549	-7.0040885	-5.0036164
4	-4.9211827	-6.5880519	-5.8860842
5	-5.6547494	-6.7092373	-6.3453695
6	-6.0801089	-6.8736364	??
7	-6.3589220	—	—
8	-6.5608621	—	—
9	-6.7164600	—	—
10	-6.8410049	—	—
11	-6.9432839	—	—
12	-7.0288902	—	—
13	-7.1016320	—	—
14	[illegible]		



(u,u,u) → cyclotomic polylogs (weak coupling)
arccos^2(1/4/u) (strong coupling)

Iterative construction

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}}F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}}F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}}F^{y_w}$$

- F weight n, from F^x weight n-1 (already classified)
- Just need to impose: 1. mixed-partials: $\frac{\partial^2 F}{\partial u_i \partial u_j} = \frac{\partial^2 F}{\partial u_j \partial u_i}, \qquad i \neq j$

$$\begin{aligned}
F^{u,v} &= F^{v,u} - F^{y_u,y_v} + F^{y_v,y_u}\,,\\
F^{v,w} &= F^{w,v} - F^{y_v,y_w} + F^{y_w,y_v}\,,\\
F^{w,u} &= F^{u,w} - F^{y_w,y_u} + F^{y_u,y_w}\,,\\
F^{1-u,1-v} &= F^{1-v,1-u} + F^{y_u,y_v} - F^{y_u,y_w} - F^{y_v,y_u} + F^{y_v,y_w} + F^{y_w,y_u} - F^{y_w,y_v}\\
F^{1-v,1-w} &= F^{1-w,1-v} + F^{y_v,y_w} - F^{y_v,y_u} - F^{y_w,y_v} + F^{y_w,y_u} + F^{y_u,y_v} - F^{y_u,y_w}\\
F^{1-w,1-u} &= F^{1-u,1-w} + F^{y_w,y_u} - F^{y_w,y_v} - F^{y_u,y_w} + F^{y_u,y_v} + F^{y_v,y_w} - F^{y_v,y_u}\\
F^{u,1-v} &= F^{1-v,u} + F^{y_u,y_w} - F^{y_w,y_u}\,,\\
F^{v,1-w} &= F^{1-w,v} + F^{y_v,y_u} - F^{y_u,y_v}\,,\\
F^{w,1-u} &= F^{1-u,w} + F^{y_w,y_v} - F^{y_v,y_w}\,,\\
F^{u,1-w} &= F^{1-w,u} + F^{y_u,y_v} - F^{y_v,y_u}\,,\\
F^{v,1-u} &= F^{1-u,v} + F^{y_v,y_w} - F^{y_w,y_v}\,,\\
F^{w,1-v} &= F^{1-v,w} + F^{y_w,y_u} - F^{y_u,y_w}\,,
\end{aligned}$$

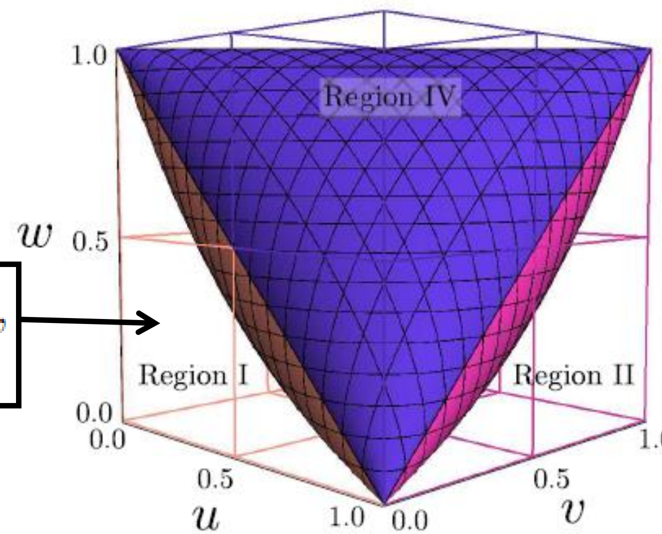
$$\begin{aligned}
F^{u,y_u} &= F^{y_u,u}\,,\\
F^{v,y_v} &= F^{y_v,v}\,,\\
F^{w,y_w} &= F^{y_w,w}\,,\\
F^{u,y_w} &= F^{w,y_u} - F^{y_u,w} + F^{y_w,u}\,,\\
F^{v,y_u} &= F^{u,y_v} - F^{y_v,u} + F^{y_u,v}\,,\\
F^{w,y_v} &= F^{v,y_w} - F^{y_w,v} + F^{y_v,w}\,,\\
F^{1-v,y_v} &= F^{y_v,1-v} - F^{y_u,1-u} + F^{1-u,y_u} + F^{y_u,w} - F^{w,y_u} - F^{y_w,v} + F^{v,y_w}\\
F^{1-w,y_w} &= F^{y_w,1-w} - F^{y_v,1-v} + F^{1-v,y_v} + F^{y_v,u} - F^{u,y_v} - F^{y_u,w} + F^{w,y_u}\\
F^{1-u,y_u} &= F^{y_u,1-u} - F^{y_w,1-w} + F^{1-w,y_w} + F^{y_w,v} - F^{v,y_w} - F^{y_v,u} + F^{u,y_v}\\
F^{1-u,y_v} &= F^{y_v,1-u} + F^{y_v,w} - F^{w,y_v}\,,\\
F^{1-v,y_w} &= F^{y_w,1-v} + F^{y_w,u} - F^{u,y_w}\,,\\
F^{1-w,y_u} &= F^{y_u,1-w} + F^{y_u,v} - F^{v,y_u}\,,\\
F^{1-u,y_w} &= F^{y_w,1-u} + F^{y_w,v} - F^{v,y_w}\,,\\
F^{1-v,y_u} &= F^{y_u,1-v} + F^{y_u,w} - F^{w,y_u}\,,\\
F^{1-w,y_v} &= F^{y_v,1-w} + F^{y_v,u} - F^{u,y_v}\,.
\end{aligned}$$

- 2. No bad branch cuts: $F^{1-u_i}(y_i = 1, y_j, y_k) = 0$

Hexagon functions as generalized polylogarithms in y_i

$$G(a_1,\ldots,a_n;z)=\int_0^z \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t)$$

$$\text{Region I}: \quad \begin{cases} \Delta>0\,, \quad 0<u_i<1\,, \quad \text{and} \quad u+v+w<1, \\ 0<y_i<1\,. \end{cases}$$

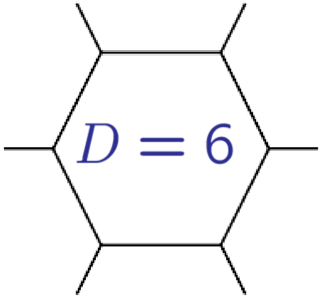


$$\mathcal{G}=\Big\{G(\vec{w};y_u)|w_i\in\{0,1\}\Big\}\cup\Big\{G(\vec{w};y_v)\Big|w_i\in\Big\{0,1,\frac{1}{y_u}\Big\}\Big\}\cup\Big\{G(\vec{w};y_w)\Big|w_i\in\Big\{0,1,\frac{1}{y_u},\frac{1}{y_v},\frac{1}{y_uy_v}\Big\}\Big\}$$

- Useful for analytics and for numerics for $\Delta > 0$

  GiNaC implementation: Vollinga, Weinzierl, hep-th/0410259

First true (*y*-containing) hexagon function



$$\Rightarrow \quad \tilde{\Phi}_6(u,v,w)$$

A real integral so it must be Steinmann

- Weight 3, totally symmetric in $\{u,v,w\}$
- First parity odd function, so:

$$\tilde{\Phi}_6^u = \tilde{\Phi}_6^v = \tilde{\Phi}_6^w = \tilde{\Phi}_6^{1-u} = \tilde{\Phi}_6^{1-v} = \tilde{\Phi}_6^{1-w} = 0$$

- Only independent {2,1} coproduct:

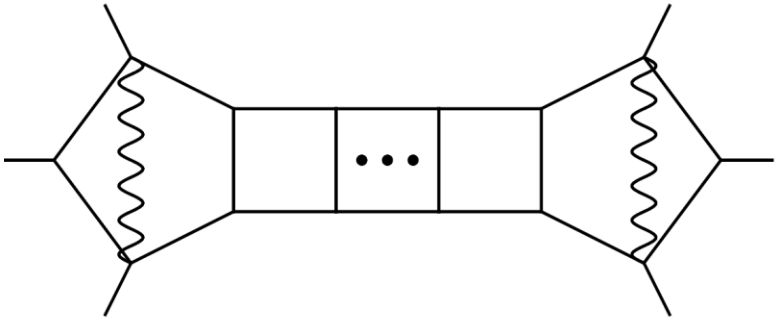
$$\tilde{\Phi}_6^{y_u} = -\Omega^{(1)}(v,w,u) = -H_2^u - H_2^v - H_2^w - \ln v \, \ln w + 2\,\zeta_2$$

$$H_2^u = \mathrm{Li}_2(1-u)$$

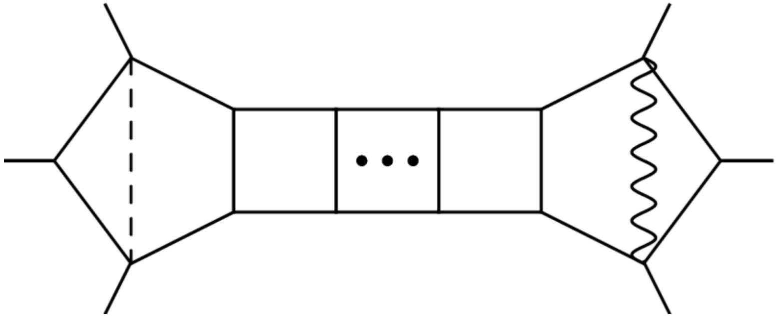
- Encapsulates first order differential equation found earlier

LD, Drummond, Henn, 1104.2787

Infinite class of integrals



$\Omega^{(L)}$



$\tilde{\Omega}^{(L)}$

- Differential equations Drummond, Henn, Trnka, 1010.3679
  easy to solve in space of Steinmann hexagon functions

Caron-Huot, LD, von Hippel, McLeod, Papathanasiou, 1806.01361

6 variables, 42 letters

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

- plus cyclic, $i \rightarrow i+1$ (mod 7), $a_{ji} \rightarrow a_{j,i+1}$ (6 x 7 = 42)

Number of (first 2 entry) Steinmann heptagon symbols

Weight $k =$	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table 1. Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7.

Enough to get symbols of 4 loop MHV & 3 loop NMHV amplitude.
Even less boundary data needed: just well-defined collinear limits.

6 loops at (1,1,1)

MHV

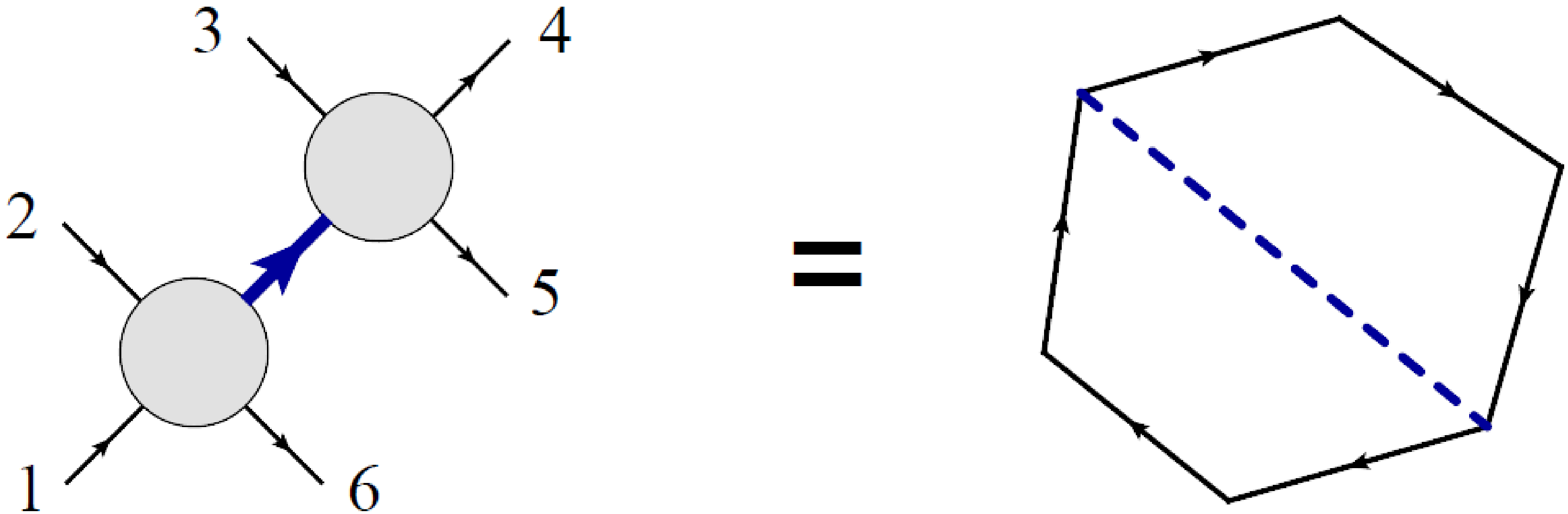
$$\mathcal{E}^{(6)}(1,1,1) = -\frac{2273108143}{6219}\zeta_{12} + \frac{260}{3}\Big[140\zeta_5\zeta_7 - 56\zeta_2\zeta_3\zeta_7 - 10\zeta_2(\zeta_5)^2 - 60\zeta_4\zeta_3\zeta_5 + 49\zeta_6(\zeta_3)^2\Big]$$
$$+ 384\Big[\zeta_2\zeta_{7,3} + 14\zeta_2\zeta_3\zeta_7 + 3\zeta_2(\zeta_5)^2 - 7\zeta_6(\zeta_3)^2\Big]$$
$$+ 120\Big[4\zeta_4\zeta_{5,3} + 20\zeta_4\zeta_3\zeta_5 - 7\zeta_6(\zeta_3)^2\Big]$$
$$+ \frac{5392}{3}\Big[\zeta_{9,3} + 27\zeta_3\zeta_9 + 20\zeta_5\zeta_7 - 2\zeta_2\zeta_3\zeta_7 - \zeta_2(\zeta_5)^2 - 6\zeta_4\zeta_3\zeta_5 - 5\zeta_6(\zeta_3)^2\Big]$$

NMHV

$$E^{(6)}(1,1,1) = \frac{5066300219}{6219}\zeta_{12} - \frac{344}{3}\Big[140\zeta_5\zeta_7 - 56\zeta_2\zeta_3\zeta_7 - 10\zeta_2(\zeta_5)^2 - 60\zeta_4\zeta_3\zeta_5 + 49\zeta_6(\zeta_3)^2\Big]$$
$$- 528\Big[\zeta_2\zeta_{7,3} + 14\zeta_2\zeta_3\zeta_7 + 3\zeta_2(\zeta_5)^2 - 7\zeta_6(\zeta_3)^2\Big]$$
$$+ 60\Big[4\zeta_4\zeta_{5,3} + 20\zeta_4\zeta_3\zeta_5 - 7\zeta_6(\zeta_3)^2\Big]$$
$$- \frac{9952}{3}\Big[\zeta_{9,3} + 27\zeta_3\zeta_9 + 20\zeta_5\zeta_7 - 2\zeta_2\zeta_3\zeta_7 - \zeta_2(\zeta_5)^2 - 6\zeta_4\zeta_3\zeta_5 - 5\zeta_6(\zeta_3)^2\Big]$$

Factorization on multi-particle pole

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505;
Basso, Sever, Vieira (Sever talk at Amplitudes 2015)



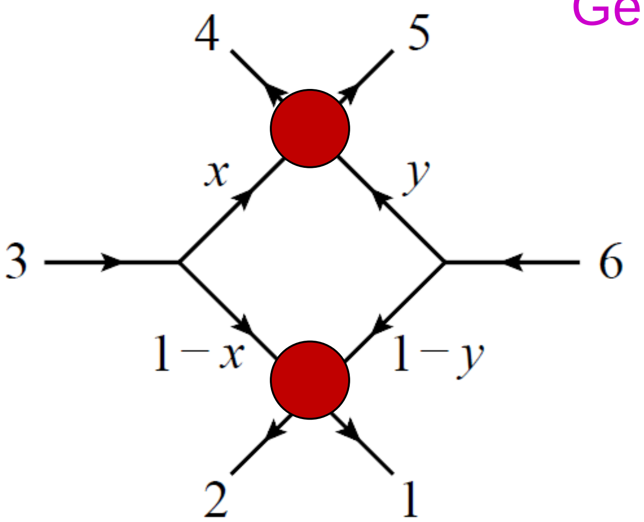
- Virtual Sudakov region, $A \sim \exp[-\ln^2 \delta]$,

$$\delta \sim s_{345}$$

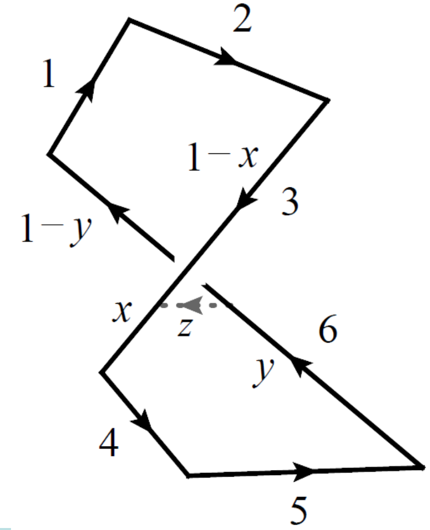
- Can study to very high accuracy in planar N=4 SYM

Double-parton-scattering-like limit

Georgiou, 0904.4675; LD, Esterlis, 1602.02107



=



- Self-crossing limit of Wilson loop
- Overlaps MRK limit
- Another Sudakov region
- Singularities ~ Wilson line RGE

  Korchemsky and Korchemskaya hep-ph/9409446