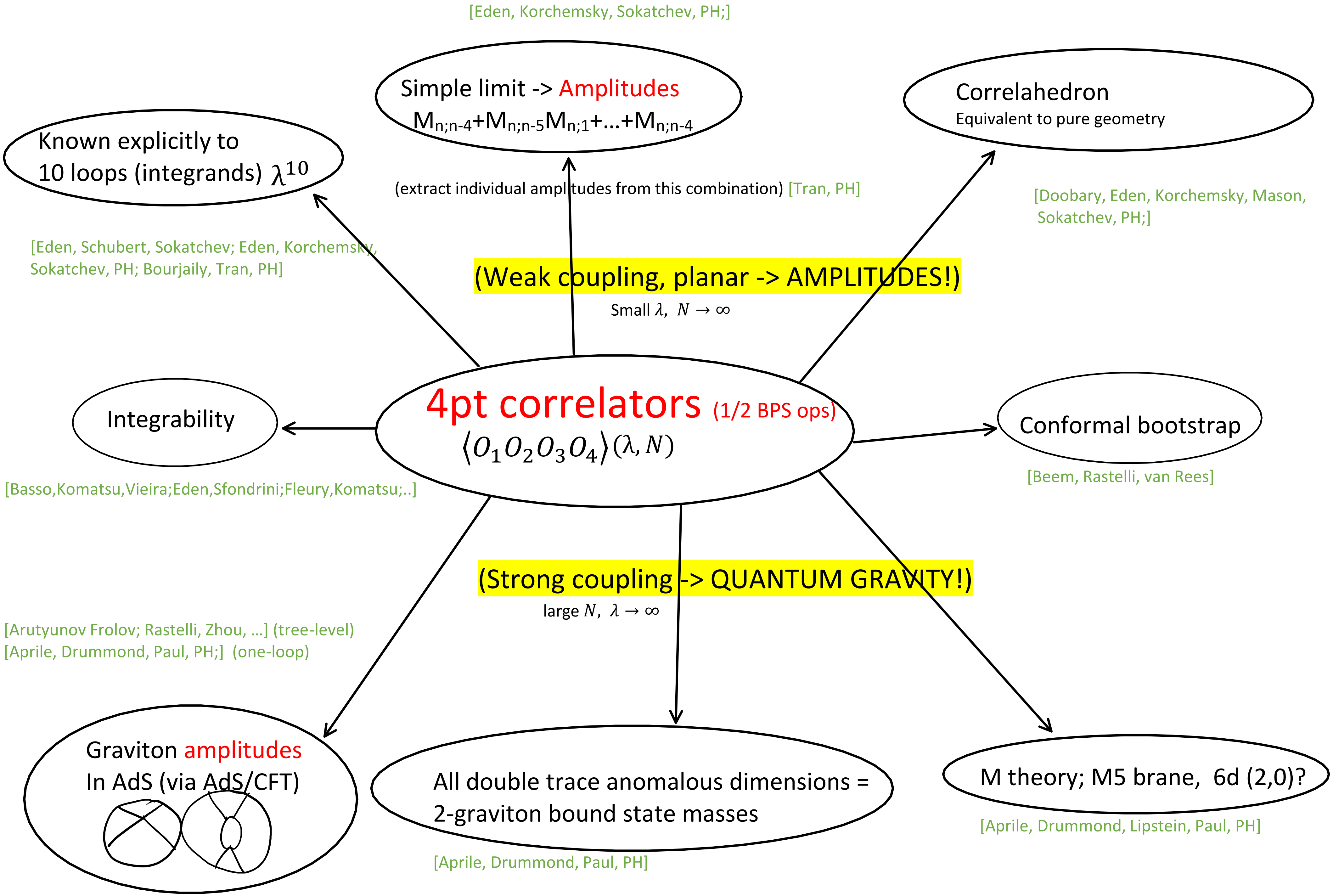
4pt Correlators in N=4 SYM at weak and strong coupling

SAGEX kickoff meeting

QMUL

Paul Heslop



[Eden, Korchemsky, Sokatchev, PH;]
Simple limit -> Amplitudes
$M_{n;n-4}+M_{n;n-5}M_{n;1}+\ldots+M_{n;n-4}$
(extract individual amplitudes from this combination) [Tran, PH]
Correlahedron
Equivalent to pure geometry
[Doobary, Eden, Korchemsky, Mason, Sokatchev, PH;]
Known explicitly to
10 loops (integrands) λ^{10}
[Eden, Schubert, Sokatchev; Eden, Korchemsky, Sokatchev, PH; Bourjaily, Tran, PH]
(Weak coupling, planar -> AMPLITUDES!)
Small λ, $N\to\infty$
4pt correlators (1/2 BPS ops)
$\langle O_1 O_2 O_3 O_4\rangle(\lambda, N)$
Integrability
[Basso,Komatsu,Vieira;Eden,Sfondrini;Fleury,Komatsu;..]
Conformal bootstrap
[Beem, Rastelli, van Rees]
(Strong coupling -> QUANTUM GRAVITY!)
large N, $\lambda\to\infty$
[Arutyunov Frolov; Rastelli, Zhou, …] (tree-level)
[Aprile, Drummond, Paul, PH;] (one-loop)
Graviton amplitudes
In AdS (via AdS/CFT)
All double trace anomalous dimensions =
2-graviton bound state masses
[Aprile, Drummond, Paul, PH]
M theory; M5 brane, 6d (2,0)?
[Aprile, Drummond, Lipstein, Paul, PH]

Half BPS correlators in N=4

- **Half BPS ops**: $O_p := \mathrm{Tr}(\Phi^p)$ $\Phi(x)$ = one of the 6 scalars in the theory, Tr over adjoint rep of gauge group SU(N)

- Plus all other ops related via internal SU(4) and SUSY. Use analytic superspace,

  [GIKOS; Howe,West]

- **4 pt Correlators:** $\langle O_p\, O_q\, O_r\, O_s \rangle$

- Focus mostly on <2222>

- Entire interacting supercorrelator <2222> given in terms of a single conformally invariant function of space-time only $f(x_1,x_2,x_3,x_4;\, \lambda, N)$ [Eden, Petkou, Schubert, Sokatchev]

- Perturbative l-loop (λ^l) planar $(N \to \infty)$ Integrand: $f(x_1,x_2,x_3,x_4;x_5,x_6,\ldots,x_{4+l})$

Weak coupling integrands

Hidden symmetry (inherited from crossing symmetry)
[Eden Korchemsky Sokatchev PH]:

$$f^{(\ell)}(x_1,\dots,x_{4+\ell}) = f^{(\ell)}(x_{\sigma_1},\dots,x_{\sigma_{4+\ell}}) \qquad \forall\, \sigma \in S_{4+\ell}$$

- NB, the symmetry mixes external variables $x_1,\dots x_4$ with integration variables $x_5 \dots x_{4+\ell}$

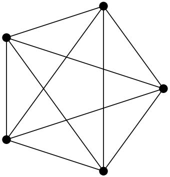
- Huge reduction in the number of integrands

- 4 point lightlike limit yields the square of the 4-point amplitude integrand

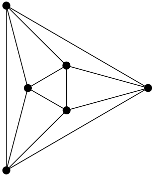
$$x_{12}^2, x_{23}^2, x_{34}^2, x_{14}^2 \to 0 \qquad \Rightarrow \qquad \left(M_4(\lambda)\right)^2$$

[Eden, Korchemsky, Sokatchev]

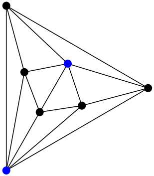
$$f^{(1)} = \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}$$



$$f^{(2)} = \frac{x_{12}^2 x_{34}^2 x_{56}^2 \; + \; S_6 \text{ perms}}{\prod_{1\le i<j\le 6} x_{ij}^2}$$

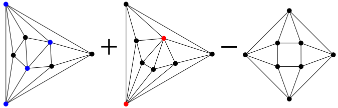


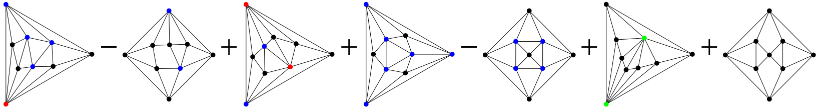
$$f^{(3)} = \frac{(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \; + \; S_7 \text{ perms}}{\prod_{1\le i<j\le 7} x_{ij}^2}$$



(Unique (planar) possibilities) $\quad f^{(3)} =$

Four- and five-loops

$f^{(4)} =$ 

$f^{(5)} =$ 

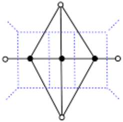
- Very compact writing (each f-graph represents a number of different integrals which you can read off from the f-graph)
- Hidden (permutation) symmetry uniquely fixes the four-point planar $\langle 2222 \rangle$ to 3 loops
- Fixes 4 loops planar to 3 constants
- 5 loops planar to 7 constants
- 6 loops planar to 36 constants etc.
- Amplitude limit can be seen graphically:

f-graphs to amplitude graphs

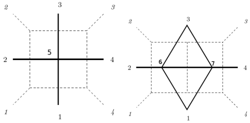
- Taking the lightlike limit is equivalent to projecting onto terms containing a 4-cycle
- 4-cycles on the surface (faces) $\rightarrow\ M_4^{(\ell)}$
- 4-cycles "inside" $\rightarrow$ product graphs $M_4^{(k)} M_4^{(\ell-k)}$

Eg at 3-loops lightlike limit gives: $2\mathcal{M}_4^{(3)} + \mathcal{M}_4^{(1)}\mathcal{M}_4^{(2)}$

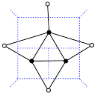
$\rightarrow$



$\rightarrow$



$\rightarrow$



Coefficient fixing

- Free coefficients fixed to 10 loops using graphical rules:
  Triangle rule, square rule, pentagon rule [Eden, Korchemsky, Sokatchev, PH;Bourjaily Tran, PH]

- These rules relate l loop coefficients to l+1 loop coefficients and only require graphical operations.

Higher point lightlike limits

$$\lim_{\substack{x_{12}^2, x_{23}^2, \dots \\ x_{n1}^2 \to 0}} f^{(l)}(x_1 \dots x_{4+l}) \Rightarrow \sum_{l'=0}^{l-(n-4)} \sum_{k=0}^{n-4} M^{(l')}_{n;k} \, \overline{M}^{(4+l-n-l')}_{n;k}$$

- 4-point correlator yields a combination of n-point amplitudes for any n
- Further, conjecture, assuming Yangian symmetry and planarity, individual amplitudes can be extracted from this combination
- Thus all amplitude integrands are contained in the 4-point correlator integrand!!
- Thus 10 point 4 loop, 9-point 5 loop, 6 point 8 loop etc. (In fact only parity even combination full amplitude requires further loop drop.) Shown at 6,7 points to 2 loops. [Heslop, Tran]

Integrals and higher charges

- Integrals are known to 3 loops; [Eden,Drummond,Duhr,Smirnov,Pennington,PH]
- Higher charge correlators are also known to 3 loops;[Chicherin,Drummond,Sokatchev,PH]

$$\boxed{\langle p_1 p_2 p_3 p_4 \rangle}$$

Correction! 5 loops….
[Chicherin, Georgoudis, Goncalves, Pereira]

- Same integrands as $\langle 2222 \rangle$ but different coefficients (no hidden symmetry),
- Thus known analytically to 3 loops from above
- 4 loops, some integrals known [Eden,Smirnov]

Correlahedron

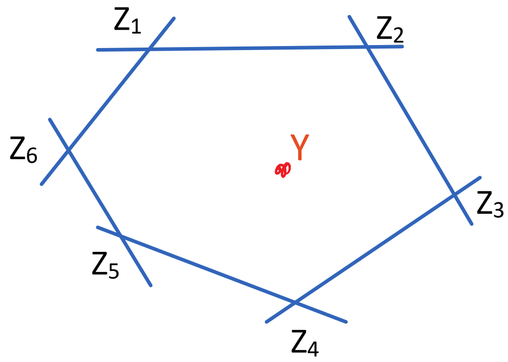
[Eden, Mason, PH; See also Arkani-Hamed, Thomas, Trnka]

Toy model:

Polygons in P^2

$Y, Z_i \in P^2 = Gr(1,3)$

[Arkani-Hamed, Trnka]



{ Y : point inside the polygon }

=

A { Y : $\langle Z_i Z_{i+1} Y \rangle > 0$ }

=

B { $Y = C_1 Z_1 + C_2 Z_2 + \ldots + C_n Z_n : C_i > 0$ }

Amplituhedron

(tree level n-point)

$Y \in Gr(k,k+4)$

$Z_i \in P^{k+3}$

[Loop level, additional Gr(2,k+4)'s representing loop integration variables]

B { $Y = C_{ai} Z_i$: ordered kxk minors of C >0 }

[Arkani-Hamed, Trnka]

Squared Amplituhedron?

(tree level n-point N^kMHV)

$Y \in Gr(k,k+4)$

$Z_i \in R^{k+4}$

[Loop-level: additional l Gr(2,k+4)s]

[B more explicit than A]

A { Y: $\langle Z_i Z_{i+1} Z_j Z_{j+1} Y \rangle > 0$ }

$\boxed{B \subset A}$

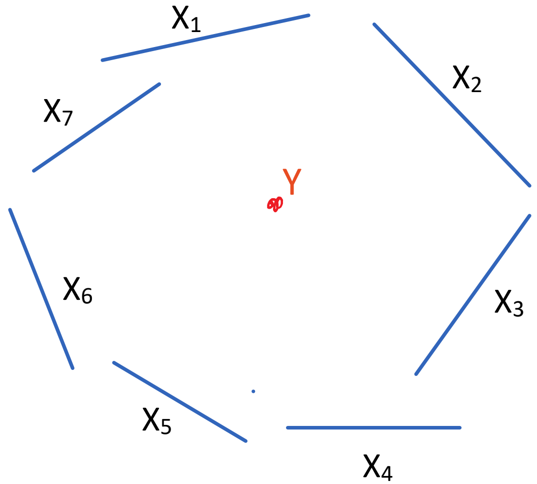
Non-trivial checks for k=n-4. Also at loop Level.

[Eden, Mason, PH; Arkani-Hamed, Thomas, Trnka]

Correlahedron

$Y \in Gr(n+k, n+k+4)$

$X_i \in Gr(2, n+k+4)$



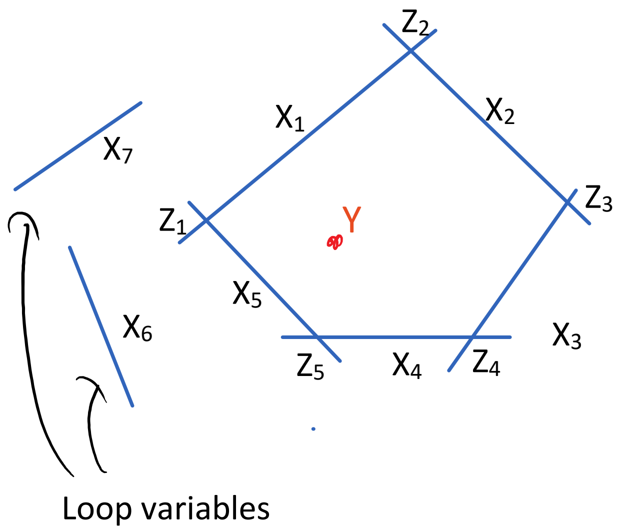
$$\{ Y : \langle X_i \, X_j \, Y \rangle > 0 \}$$

Lightlike limits

m out of n lines sequentially intersect

$\Downarrow$

m-point n-m loop
Squared amplituhedron



Freeze and Project: m dimensions of Y frozen to the intersection points: $\langle X1\ X2\ Y \rangle = 0$ etc.
Project to the space "orthogonal" to these m dimensions

Checks:

- **Freeze and project takes correlahedron -> squared amplituhedron proven**
- **Freeze and project takes correlator -> squared amplitude non-trivial checks**
- **Squared amplituhedron -> squared amplitude. 5,6,7-point $\overline{\text{MHV}}$; 4,5-point 1-loop, 4 point 2-loop.**

$$
\begin{tikzcd}
\text{correlahedron} \arrow[rr, "??"] \arrow[d, Rightarrow, "freeze/project"] & & \text{correlator} \arrow[d, Rightarrow, "freeze/project"] \\
\text{amplituhedron} \arrow[rr, "cylindrical\ decomp"] & & \text{amplitude.}
\end{tikzcd}
$$

(squared) **(squared)**

("Canonical form" with no divergences inside, log divergences on the boundary)

To Do:

- **Better understand / check the non-maximal case of the squared amplituhedron k < n-4**
- **Decompose the squared amplituhedron into different topological pieces** (Winding number?) [Arkani-Hamed, Thomas, Trnka]
- **Define the canonical form for the correlahedron**
- **Focus on maximal case, contains all amplitudes!**
- **WARNING! Momentum twistor Feynman diagrams NOT a good tessellation** [Agarwala, Marcott; Stewart, PH]

Strong coupling: AdS/CFT

AdS/CFT. canonical example:

$$\boxed{\mathcal{N}=4 \text{ SYM} = \text{IIB string theory on } AdS^5 \times S_5}$$

(CFT) (gravity)

(CFT)	(gravity)
N gauge group $SU(N)$	$l_p \sim N^{-1/4}$ Planck length (Q.G loops)
$\lambda = g^2 N$ 't Hooft coupling	$\alpha' \sim \frac{1}{\sqrt{\lambda}}$ String corrections.

'Quantum Gravity' $\boxed{\alpha' \to 0 \Leftrightarrow \lambda \to \infty}$ (no massive string states)

$\frac{1}{N}$ expansion $\Leftrightarrow$ Q.G loop expansion

$$\boxed{\text{Free gravity} + \frac{1}{N^2} \times (\text{tree level}) + \frac{1}{N^4} \times (1 \text{ loop gravity})}$$

Spectrum.

- $\alpha' \to 0 \Rightarrow$ Massive string states supressed
  ~~Only SUGRA states remain~~ [including SUGRA multi-particle bound states]

- AdS/CFT: string state in IIB $\longleftrightarrow$ gauge invariant op. in $N=4$ SYM.

AdS_5 $\boxed{\text{graviton multiplet}}$ $\longleftrightarrow$ $\boxed{O_2 := Tr(\phi^2)}$, related by $SU(4)+SUSY$

graviton S_5 modes $\boxed{(\text{Kaluza-Klein})}$ $\longleftrightarrow$ $\boxed{O_n = Tr(\phi^n) + \text{"}\tfrac{1}{N} Tr\,Tr\text{"}}$

[ϕ = one of 6 scalars in $N=4$ SYM]

Understand these corrections. New basis of HBPS ops?

4-point functions and the OPE.

- Focus on $\langle O_2 O_2 O_2 O_2 \rangle \longleftrightarrow$ AdS_5 4-point graviton amplitude

- OPE:

$$\boxed{O_2(x_1)\, O_2(x_2) = \sum_{\hat{O}} C_{22\hat{O}}\, F(x_{12}, \partial_2)\, \hat{O}(x_2)}$$

primary

all ops in theory

3-pnt / OPE coefficient

known

- Block decomposition. Insert twice into

$$\boxed{\langle O_2 O_2 O_2 O_2 \rangle = \sum_{\hat{O}} C^2_{22\hat{O}}\, u^{a\eta_{\hat{O}}} F_{\hat{O}}(x_1, x_2, x_3, x_4)} \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

half anomolous dimension of $\hat{O}$.

primary ops

Conformal partial waves or blocks. Known. Depend on quantum numbers of $\hat{O}$ only.

- Both sides depend on the parameter $\boxed{a = \frac{1}{N^2}}$ $(\lambda \to \infty)$
- (Anomalous) dimensions of operators $\hat{O}$ also depend on a: $\boxed{\eta_{\hat{O}}(a)}$
- Expanding the block decomposition in a:

① $$\langle O_2 O_2 O_2 O_2 \rangle \Big|_{a^0} = \sum_{\hat{O}} C_{22\hat{O}}^2 \Big|_{a^0} F_{\hat{O}}(x_1, x_2, x_3, x_4)$$

② $$\langle O_2 O_2 O_2 O_2 \rangle \Big|_{a^1} = \sum_{\hat{O}} C_{22\hat{O}}^2 \Big|_{a^0} \log u \; \eta_{\hat{O}}^{(0)} F_{\hat{O}}(x_1, x_2, x_3, x_4) + \text{no log}$$

Known tree SUGRA results [d'Hoker, Freedman, Mathur, Matusis, Rastelli; Arutyunov, Frolov]

lower order data

③ $$\langle O_2 O_2 O_2 O_2 \rangle \Big|_{a^2} = \sum_{\hat{O}} C_{22\hat{O}}^2 \Big|_{a^0} \frac{1}{2} \eta_{\hat{O}(0)}^{2} \log^2 u \; F_{\hat{O}}(x_1, x_2, x_3, x_4) + \log u + \text{no log}.$$

$\boxed{\text{Idea? Extract } C_{22\hat{O}},\ \eta_{\hat{O}} \text{ from ①, ② Plug into ③} \to \text{full double discontinuity of 1-loop amplitude}}$

Technical details.

Problem. Although the blocks $F_{\hat{O}}$ are independent for ops. with different quantum numbers, there are many ops with identical quantum numbers: degeneracy.

[Dolan, Osborn; Bissi, Lukowski; Doobary P.H.]

Solution part 1. Expand in superblocks. Separate contributions from each supermultiplet (super primary)

But. Still degenaracy. Multiplets with same quantum numbers.

$\therefore$ **Solution part 2** Consider more general correlators: $\boxed{\langle O_p O_p O_q O_q \rangle}$

Degeneracy of multiplets

- Taking the $\langle 2222 \rangle$ $N^0, \frac{1}{N^2}$ result (long piece) and decomposing into superblocks gives the combinations:

N^0: $\boxed{\sum_{\mathcal{O}} \left(C_{22\mathcal{O}}\right)^2}$ $\frac{1}{N^2}$: $\boxed{\sum_{\mathcal{O}} \left(C_{22\mathcal{O}}\right)^2 \eta_{\mathcal{O}}}$

Sum over all (long) multiplets with the same quantum numbers (superconf. rep.)

- Only $SU(4)$ singlets appear in $\langle 2222 \rangle$
- triple and higher trace ops $\frac{1}{N}$ suppressed.

$\Rightarrow$ Double trace singlet operators

Double trace singlet long multiplets: twist 4.

- $\boxed{\mathcal{O}_2 (\partial_\mu)^l \mathcal{O}_2}$ twist $(= \Delta - l) = \boxed{4}$, spin l.
- Only one operator for each (even) spin with

No degeneracy at twist 4.

- Superblock decomposition gives:

$$C_{220}^2 = \frac{4}{3}(l+1)(l+6) \times \frac{(l+3)!^2}{(2l+6)!} \; ; \quad C_{220}^2 \eta_0 = -64 \times \frac{(l+3)!^2}{(2l+6)!}$$

[Dolan, Osborn]

$$\Rightarrow \boxed{\eta = -\frac{48}{(l+1)(l+6)}}$$

twist 6 long singlet double trace

- $\boxed{\underset{\hat{K}_1}{O_3 \partial_\mu^l O_3}, \quad \underset{\hat{K}_2}{O_2 \Box \partial_\mu^l O_2}}$ ← not pure.

  2 operators for each spin.

- Superblock decomposition of $\langle 2222 \rangle$ gives:

$$C_{22K_1}^2 + C_{22K_2}^2 = \frac{2}{5}(l+1)(l+8) \times \frac{(l+4)!^2}{(2l+8)!}$$

$$C_{22K_1}^2 \eta_{K_1} + C_{22K_2}^2 \eta_{K_2} = -96 \times \frac{(l+4)!^2}{(2l+8)!}$$

Problem

Not enough info, 2 equations
4 unknowns. ! $C_{22K_1}, C_{22K_2}, \eta_{K_1}, \eta_{K_2}$

Now known [Rastelli, Zhou; see also Dolan, Nirschl, Osborn; Uruchurtu...]

Convert from Mellin space + fix normalization (via light-like limit ⇕ low twist cancellation)

Solution

Consider $\langle 2233 \rangle$ and $\langle 3333 \rangle$ correlators too.
⇒ 6 equations for 6 unknowns (C_{33K_1}, C_{33K_2} and)

twist 6

- Defining $C_{ppK_i}^2 = \frac{(l+4)!^2}{(2l+8)!} c_{pi}^2$ we get:

$$\boxed{\begin{aligned} c_{21}^2 + c_{22}^2 &= \frac{2}{5}(l+1)(l+8)\,, \\ c_{31}^2 + c_{32}^2 &= \frac{9}{40}(l+1)(l+2)(l+7)(l+8)\,, \\ c_{21}c_{31} + c_{22}c_{32} &= 0\,. \end{aligned}}$$

from $\langle 2222\rangle|_{N^0}$
$\langle 2233\rangle|_{N^0}$
$\langle 3333\rangle|_{N^0}$

$$\boxed{\begin{aligned} c_{21}^2\eta_1 + c_{22}^2\eta_2 &= -96\,, \\ c_{31}^2\eta_1 + c_{32}^2\eta_2 &= -54(l^2+9l+44)\,, \\ c_{21}c_{31}\eta_1 + c_{22}c_{32}\eta_2 &= 432\,, \end{aligned}}$$

from $\langle 2222\rangle|_{1/N^2}$
$\langle 2233\rangle|_{1/N^2}$
$\langle 3333\rangle|_{1/N^2}$

Twist 6 data.

$$\eta_1 = -\frac{240}{(l+1)(l+2)}, \qquad \eta_2 = -\frac{240}{(l+7)(l+8)},$$

$$c_{21} = -\sqrt{\frac{2(l+1)(l+2)(l+8)}{5(2l+9)}}, \qquad c_{22} = -\sqrt{\frac{2(l+1)(l+7)(l+8)}{5(2l+9)}},$$

$$c_{31} = \sqrt{\frac{9(l+1)(l+2)(l+7)^2(l+8)}{40(2l+9)}}, \qquad c_{32} = -\sqrt{\frac{9(l+1)(l+2)^2(l+7)(l+8)}{40(2l+9)}}.$$

- η = rational of linear factors in l.
- c's = $\sqrt{\text{rational of linear factors in } l}$.

Higher twist, $\Delta - l = 2t$

(double trace long)

- Basis of singlet operators of twist $2t$, spin l:

$$\boxed{K^{\text{free}}_{t,l,i} = \mathcal{O}_{i+1}\Box^{t-i-1}\partial^l \mathcal{O}_{i+1} + \ldots \qquad i = 1 \cdots t-1}$$

- These mix, $K_{t,l,i}$ pure operators at strong coupling
- Matrix of 3-point functions

$$C(t|l) = \begin{pmatrix} C_{22K_{t,l,1}} & C_{22K_{t,l,2}} & \cdots & C_{22K_{t,l,t-1}} \\ C_{33K_{t,l,1}} & C_{33K_{t,l,2}} & \cdots & \\ \cdots & & & \\ C_{ttK_{t,l,1}} & & & \end{pmatrix}$$

- Superblock decomposition of N^0 correlator then gives the eqn:

$$\boxed{C\, C^T = \hat{A}} \qquad \hat{A} \text{ block coeffs of } \left.\begin{pmatrix} \langle 2222\rangle & \langle 2233\rangle & \cdots & \langle 22tt\rangle \\ \langle 3322\rangle & \langle 3333\rangle & \cdots & \langle 33tt\rangle \\ \vdots & \vdots & & \vdots \\ \langle tt22\rangle & \langle tt33\rangle & \cdots & \langle tttt\rangle \end{pmatrix}\right|_{N^0}$$

- Superblock decomposition of $\frac{1}{N^2}$ correlator then gives the eq^n:

$$\boxed{C \eta C^T = \hat{M}} \qquad \hat{M} \text{ block coeffs of } \left.\begin{pmatrix} \langle 2222\rangle & \langle 2233\rangle & \cdots & \langle 22tt\rangle \\ \langle 3322\rangle & \langle 3333\rangle & \cdots & \langle 33tt\rangle \\ \vdots & \vdots & & \vdots \\ \langle tt22\rangle & \langle tt33\rangle & \cdots & \langle tttt\rangle \end{pmatrix}\right|_{\frac{1}{N^2}}$$

$$\eta = \begin{pmatrix} \eta_{K_1} & & & \\ & \eta_{K_2} & & \\ & & \ddots & \\ & & & \eta_{K_{t-1}} \end{pmatrix}$$

(from SUGRA results Rastelli, Zhou)

- <u>Unknowns</u>: $C : (t-1)\times(t-1)$
  $+\ \eta : t-1$
  $= \boxed{(t-1)t}$

<u>Equations</u>: $\hat{A} : \frac{(t-1)t}{2}$ (symmetric)
$+\ \hat{M} : \frac{(t-1)t}{2}$
$= \boxed{(t-1)t}$

unique solution.

- Very useful feature: $\boxed{\hat{A} \text{ is diagonal}}$

Why? $\langle ppqq \rangle\big|_{N^0} = \begin{cases} g_{12}^{p}\, g_{34}^{q} & p \neq q \longrightarrow \emptyset \text{ contrib to long ops.} \\ g_{12}^{p}\, g_{34}^{p} + g_{13}^{p}\, g_{24}^{p} + g_{14}^{p}\, g_{23}^{p} & p = q \end{cases}$

contribute to $\frac{1}{2}$ BPS ops $O_{|p-2|}$ only in OPE.

- Define $\tilde{c}(t|l)$.

$$\boxed{\tilde{c}\,\tilde{c}^{T} = \mathrm{Id}_{t-1},} \qquad C = \widehat{\mathcal{A}}^{\frac{1}{2}} \cdot \tilde{c}(t|l)$$

$\tilde{c}$ orthonormal

- eg twist 4: $\tilde{c} = 1$

twist 6:

$$\tilde{c}(3|l) = \begin{pmatrix} \sqrt{\frac{l+2}{2l+9}} & \sqrt{\frac{l+7}{2l+9}} \\ -\sqrt{\frac{l+7}{2l+9}} & \sqrt{\frac{l+2}{2l+9}} \end{pmatrix}$$

Higher twist examples.

Twist 8.

$$\tilde{c}(4|l) = \begin{pmatrix} \sqrt{\frac{7(l+2)(l+3)}{6(2l+9)(2l+11)}} & \sqrt{\frac{5(l+3)(l+8)}{3(2l+9)(2l+13)}} & \sqrt{\frac{7(l+8)(l+9)}{6(2l+11)(2l+13)}} \\ -\sqrt{\frac{2(l+2)(l+8)}{(2l+9)(2l+11)}} & -\sqrt{\frac{35}{(2l+9)(2l+13)}} & \sqrt{\frac{2(l+3)(l+9)}{(2l+11)(2l+13)}} \\ \sqrt{\frac{5(l+8)(l+9)}{6(2l+9)(2l+11)}} & -\sqrt{\frac{7(l+2)(l+9)}{3(2l+9)(2l+13)}} & \sqrt{\frac{5(l+2)(l+3)}{6(2l+11)(2l+13)}} \end{pmatrix},$$

$$\eta_{4,l,i} = \left\{ -\frac{720(l+7)}{(l+1)(l+2)(l+3)}, -\frac{720}{(l+3)(l+8)}, -\frac{720(l+4)}{(l+8)(l+9)(l+10)} \right\}.$$

Twist 10.

$$\tilde{c}(5|l) = \begin{pmatrix} \sqrt{\frac{3}{2}\frac{(2)(3)(4)}{[9][11][13]}} & \sqrt{\frac{5}{2}\frac{(3)(4)(9)}{[9][13][15]}} & \sqrt{\frac{5}{2}\frac{(4)(9)(10)}{[11][13][17]}} & \sqrt{\frac{3}{2}\frac{(9)(10)(11)}{[13][15][17]}} \\ -\sqrt{\frac{27}{8}\frac{(2)(3)(9)}{[9][11][13]}} & -\sqrt{\frac{5}{8}\frac{(l+18)(3)}{[9][13][15]}} & \sqrt{\frac{5}{8}\frac{(l-5)(10)}{[11][13][17]}} & \sqrt{\frac{27}{8}\frac{(4)(10)(11)}{[13][15][17]}} \\ \sqrt{\frac{5}{2}\frac{(2)(9)(10)}{[9][11][13]}} & -\sqrt{\frac{3}{2}\frac{(l-3)(10)}{[9][13][15]}} & -\sqrt{\frac{3}{2}\frac{(l+16)(3)}{[11][13][17]}} & \sqrt{\frac{5}{2}\frac{(3)(4)(11)}{[13][15][17]}} \\ -\sqrt{\frac{5}{8}\frac{(9)(10)(11)}{[9][11][13]}} & \sqrt{\frac{27}{8}\frac{(2)(10)(11)}{[9][13][15]}} & -\sqrt{\frac{27}{8}\frac{(2)(3)(11)}{[11][13][17]}} & \sqrt{\frac{5}{8}\frac{(2)(3)(4)}{[13][15][17]}} \end{pmatrix} \qquad (n) = \sqrt{l+n}\,, \qquad [n] = \sqrt{2l+n}$$

$$\eta_{5,l,i} = \left\{ -\frac{1680(l+7)(l+8)}{(l+1)(l+2)(l+3)(l+4)}, -\frac{1680}{(l+3)(l+4)}, -\frac{1680}{(l+9)(l+10)}, -\frac{1680(l+5)(l+6)}{(l+9)(l+10)(l+11)(l+12)} \right\}$$

General Formulae

- From these plus higher twist results, deduce general formulae:

- Anomalous dimensions of all double trace singlets

$$\eta_{t,l,i}^{[0,0,0]} = -\frac{2(t-1)_4(t+l)_4}{(l+2i-1)_6}$$

- OPE coeffs have the structure:

$$\tilde{C}_{pi}^{[0,0,0]} = \sqrt{\frac{2^{1-t}(2l+4i+3)\left((l+i+1)_{t-i-p+1}\right)^{\sigma_1}\left((t+l+p+2)_{i-p+1}\right)^{\sigma_2}}{\left(l+i+\frac{5}{2}\right)_{t-1}}}$$

$$\times \sum_{k=0}^{\min(i-1,p-2,t-i-1,t-p)} l^k a_{(p,i,k)}^{[0,0,0]}. \qquad \sigma_1 = \mathrm{sgn}(t-p-i+1)\,, \qquad \sigma_2 = \mathrm{sgn}(i-p+1)$$

Fixing the remaining coefficients without calculating.

- Remarkably the remaining coefficients $\boxed{a^{[000]}_{p,i,k}}$ are always uniquely fixed by orthonormality of $\tilde{C}$
- Orthonormality $\Rightarrow$ $\boxed{\text{linear equations in } a^{[000]}_{p,i,k}}$ (or a^2)
- Unique (up to signs) solution always exists!
- Allows quick solution. Complete data up to $\boxed{\text{twist } 48}$
  (without needing tree SUGRA result!)
- Analytic formula for $a^{[000]}_{p,i,k}$?

$$\boxed{a^{[0,0,0]}_{(2,i,0)} = \frac{2^{t-1}(2i+2)!(t-2)!(2t-2i+2)!}{3(i-1)!(i+1)!(t+2)!(t-i-1)!(t-i+1)!},}$$

$\langle O_2 O_2 K_{t,l,i} \rangle$ We originally wanted!

$\boxed{\text{What are those orthonormal matrices?}}$ Ideas welcome!

General operator anomalous dimensions

- We already have more data than we need for bootstrapping $\langle 2222\rangle|_{1/N^4}$.
- Before continuing, consider generalizing these results for more general long double-trace operators, other SU(4) reps
- Simplest generalization: $\boxed{\text{Operators in the } [n,0,n]\ SU(4) \text{ rep.}}$

twist $2t$ spin l basis:

$$\{\mathcal{O}_{2+n}\Box^{t-n-2}\partial^l\mathcal{O}_{2+n}, \mathcal{O}_{3+n}\Box^{t-n-3}\partial^l\mathcal{O}_{3+n}, \ldots, \mathcal{O}_t\Box^0\partial^l\mathcal{O}_t\}$$

(same input as [000])

- Similar procedure: $\langle ppqq\rangle|_{N^0}$, $\langle ppqq\rangle|_{1/N^2} \Rightarrow$ $\boxed{\text{Equations for 3-point coeffs and anomalous dimensions.}}$

General picture.

- Proceeding further a general picture of all double-trace anomalous dimensions emerges:
- Operators: Twist τ, spin l, $su(4)$ rep $[a,b,a]$ ops.

$$\boxed{\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q \,,} \qquad (p \leq q)$$

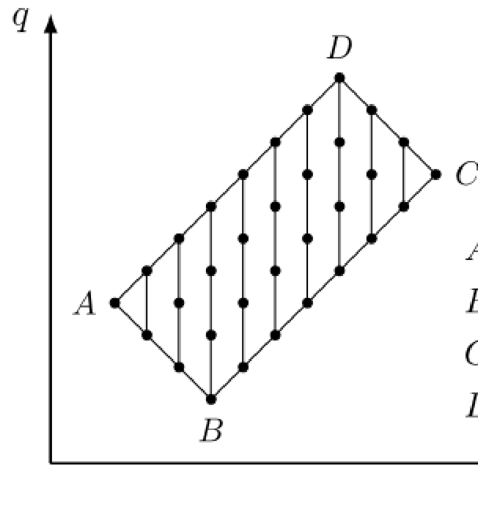
$p,q \in \mathcal{D}^{\text{long}}_{\tau,l,a,b}$

$$p = i + a + 1 + r\,, \qquad q = i + a + 1 + b - r$$

$$i = 1, \ldots, (t-1)\,, \qquad r = 0, \ldots, (\mu - 1)\,,$$

so that $d = \mu(t-1)$ with

$$t \equiv (\tau - b)/2 - a\,, \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a+l \text{ even}, \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a+l \text{ odd}. \end{cases}$$



$A = (a+2, a+b+2)$;
$B = (a+1+\mu, a+b+3-\mu)$;
$C = (a+\mu+t, a+b+2+t-\mu)$;
$D = (a+1+t, a+b+1+t)$;

Anomalous dimensions

$f(a,b,\tau,l,p,q)$

$$\boxed{\Delta_{pq} = \tau + l - \frac{2}{N^2}\frac{2M_t^{(4)}M_{t+l+1}^{(4)}}{\left(l+2p-2-a-\frac{1+(-)^{a+l}}{2}\right)_6}}$$

$$M_t^{(4)} \equiv (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$$

- Degeneracy: Independent of q!

- Obtained from $\langle pq\, p'q' \rangle$ at $O(N^0)$, $O(1/N^2)$
  For $p,q \in D^{long}_{\tau,l,a,b}$ $p',q' \in D^{long}_{\tau,l,a,b}$
- Checked for $0 \leq a \leq 3$, $0 \leq b \leq 6$, twist 24

Back to $\langle 2222 \rangle$. Next order?

- Recall $\boxed{\langle 2222 \rangle \Big|_{1/N^4} \Big|_{\log^2 u} = \sum_0 C_{220}^2 \frac{\eta_0^2}{2} \times \text{superblock}_0}$

We now have this data analytically

[All tree amplitudes given in terms of $\bar{D}$ pqrs $\bigotimes^{p\,q}_{s\,r} = \frac{1}{(x-\bar{x})^*} \times \left(\begin{array}{c}\Sigma \text{ wt.} \le 2 \text{ polylog} \\ \times \text{polynomial}\end{array}\right)$]

- Perform the sum to high order.
- Match to a plausible ansatz in terms of polylogarithms:

$$\boxed{\frac{\text{Polynomial}(x,\bar{x}) \times \text{wt.2 polylogs} + \text{lower weight}}{(x-\bar{x})^n}}$$

n some high integer.

$n=15$.

$$= \frac{1}{uv}\left[p(u,v)\frac{\text{Li}_1(x)^2 - \text{Li}_1(\bar{x})^2}{x-\bar{x}} + 2\left[p(u,v) + p\left(\frac{1}{v},\frac{u}{v}\right)\right]\frac{\text{Li}_2(x)-\text{Li}_2(\bar{x})}{x-\bar{x}} + q(u,v)(\text{Li}_1(x)+\text{Li}_1(\bar{x})) + r(u,v)\frac{\text{Li}_1(x)-\text{Li}_1(\bar{x})}{x-\bar{x}} + s(u,v)\right].$$

$$p(u,v) = 24uv\partial_x^2\partial_{\bar{x}}^2\left[\frac{u^2v^2(1-u-v)[(1-u-v)^4 + 20uv(1-u-v)^2 + 30u^2v^2]}{(x-\bar{x})^{10}}\right]$$

Uplift to full function.

- Look for a crossing symmetric (invariant under permutations of x_1, x_2, x_3, x_4) function whose $\log^2 u$ coefficient (double discontinuity) is the one given.

- Ansatz:

$$\frac{\text{Polynomial}(x,\bar{x}) \times \text{Polylog wt 4}(x,\bar{x}) + \text{Polynomial}(x,\bar{x}) \times \text{Polylog wt 3}(x,\bar{x}) + \ldots}{(x-\bar{x})^{15}}$$

degree 10

- Impose:
  1.) Crossing symmetry
  2.) Finite as $x \to \bar{x}$
  3.) $\log^2 u$ part matches

Solution with 1 free parameter

Ambiguity.

Ambiguity $= \alpha \frac{1}{uv}[(1+u\partial_u + v\partial_v)u\partial_u v\partial_v]^2\Phi^{(1)}(u,v)$ $\left(= \bar{D}_{4444} \right)$ $\frac{1}{(x-\bar{x})^{13}}$

- R^4 term in string theory effective action?
- 1-loop SUGRA counterterm and tree-level α'^3 correction?

$$\boxed{\text{Full 1-loop quantum sugra 4-pnt amplitude.}}$$

Extracting 1-loop data.

- Contains info about $\frac{1}{N^4}$ anomalous dimensions.
- Problem: operator mixing 1.) with other double trace ops (same as before)
  2.) [worse] potentially with triple trace ops (further suppressed?)
- Twist 4 avoids both problems! $\boxed{O_2 \cdot \partial^l O_2}$ only twist 4 op.
- Skipping the details: $\langle 2222 \rangle \big|_{1/N^4} \big|_{\log u} \big|_{u^2}$ contains this info
- Superblock expansion:

$$\boxed{\eta_l^{(2)} = \begin{cases} \frac{1344(l-7)(l+14)}{(l-1)(l+1)^2(l+6)^2(l+8)} - \frac{2304(2l+7)}{(l+1)^3(l+6)^3} & l = 2, 4, \ldots \\ \frac{9}{14}\alpha + \frac{1148}{3} & l = 0 \end{cases}}$$

[$l = 2, 4$ found independently by Alday, Bissi]

Future work.

- Many directions!
- $\boxed{\langle 3333\rangle|_{1/N^4} \quad \text{1-loop.}}$ [We have all the ingredients.]
- $\boxed{\langle 2222\rangle|_{1/N^{6,8,\ldots}} \quad \text{2-loops, }}$ ... Triple divergence $\leftarrow$ $\boxed{\sum_0 C^2_{220}\, \eta_0^{(1)3}}$
- $\boxed{\text{Simplifying } \langle 2222\rangle|_{1/N^4}}$. Pulling out Casimir operators. (Pull out 8th order differential ops from leading discs.) Pull out from full function?
- $\boxed{\text{3-point functions}}$: degeneracy of anomalous dims $\Rightarrow$ no unique sol^{n}.
  $\Rightarrow$ no ansatz for $\tilde{a}$

  $\Rightarrow$ Higher than $\langle 3333\rangle|_{1/N^4}$ difficult because of this $\longrightarrow$
- $\boxed{\text{M theory? M5 brane} \quad \text{6d CFT}}$ { First correction (stringy) [Lipstein,PH]
  Problem going further: superblocks tricky!
  but progress... [Aprile, Drummond, Lipstein, Paul, PH]

Durham group

Patrick Dorey: 2d amplitudes, integrability

Arthur Lipstein: ambi-twistor strings, CHY, N=4, N=8 etc

Simon Badger: new amplitude techniques -> pheno

Valya Khoze: Higgsplosion

Daniel Maitre: Blackhat

Paul Mansfield: Faraday's Lines of Force as Strings

Herbert Gangl: Number theory, symbol

Many others with wider but related interests; SUSY field theory, Holography, more pheno IPPP etc.

Appendix: Superblocks.

[Dolan, Osborn; Bissi, Lukowski; Doobary P.H.]

[0 p 0] SU(4) rep   [Y·Y = 0]

$$\boxed{O_p(X) := \mathrm{Tr}\left(\left(\varphi_I(x)\,Y_I\right)^p\right) + \dots}$$

$X = (x, Y)$ internal co-ordinate
deal with all 6 scalars.
(in fact all ops in multiplet for free - analytic superspace)

Then the superblock decomposition is:

$$\boxed{\begin{aligned}&\langle \mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)\rangle \\ &= \sum_{\gamma,\underline{\lambda}} A^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}}\, g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac{1}{2}p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{1}{2}p_{43}} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}\gamma} F^{\alpha\beta\gamma\underline{\lambda}}(Z),\end{aligned}}$$

$$\alpha = \tfrac{1}{2}(\gamma - p_{12}) \quad \beta = \tfrac{1}{2}(\gamma + p_{34})\ ,$$

where $\boxed{g_{ij} = \dfrac{Y_i \cdot Y_j}{x_{ij}^2}}$ superpropagator

$\gamma = \Delta - l$ = twist of multiplet (LWS)

$$\boxed{A^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}} = \sum_{\mathcal{O}} C_{p_1p_2\mathcal{O}_{\gamma\underline{\lambda}}}\, C_{p_3p_4\mathcal{O}_{\gamma\underline{\lambda}}}}$$

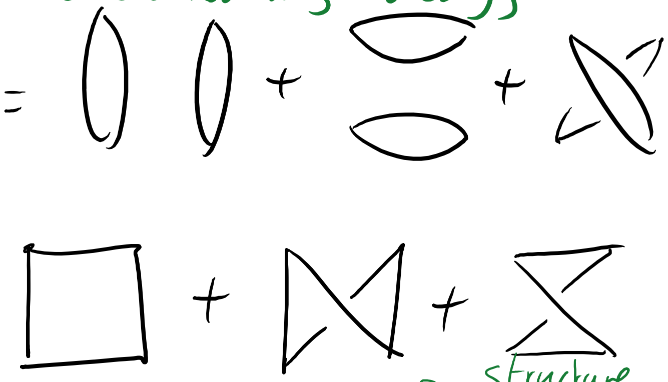
Multiplets with same quantum numbers.

$\underline{\lambda}$ other quantum numbers.
[SU(4) rep, spin l]

Supercorrelators.

$$\boxed{\begin{aligned}\langle 2222\rangle = \; & g_{12}^2 g_{34}^2 + g_{13}^2 g_{24}^2 + g_{14}^2 g_{23}^2 \\ & + \frac{1}{N^2}\left(g_{12}g_{34}g_{13}g_{24} + \text{crossing}\right) \\ & + \frac{1}{N^2} g_{13}^2 g_{24}^2 (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y}) \times \bar{D}_{2422}\end{aligned}}$$

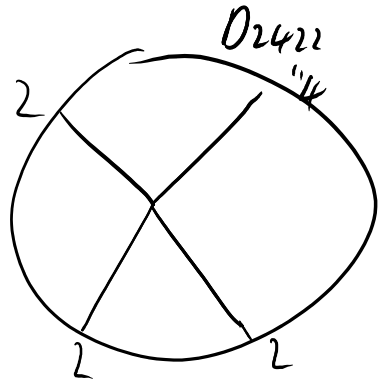
[disconnected free theory]



structure [Eden, Schubert, Sokatchev]

[d'Hoker, Freedman; Arutyunov, Frolov]

$$\left[\begin{aligned} & u = x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \\ & y\bar{y} = \frac{Y_1\cdot Y_2 \; Y_3\cdot Y_4}{Y_1\cdot Y_3 \; Y_2\cdot Y_4} \qquad (1-y)(1-\bar{y}) = \frac{Y_1\cdot Y_4 \; Y_2\cdot Y_3}{Y_1\cdot Y_3 \; Y_2\cdot Y_4}\end{aligned}\right]$$



$$\boxed{\bar{D}_{2422} = -4\,\partial_u \partial_v \left(1 + u\partial_u + v\partial_v\right) \mathrm{Box}(x,\bar{x})}$$

$$= \frac{\overbrace{\text{polynomial}(x,\bar{x})}^{\text{degree } 4} \times \text{wt } 2 \text{ polylogs in } x,\bar{x} + \text{poly}(x,\bar{x}) \times \text{wt } 1 + \text{poly}(x,\bar{x}) \text{ wt } 0}{(x-\bar{x})^7}$$

Long superblocks

- Easy! $\boxed{= (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y}) \times \text{bosonic block}(x,\bar{x}) \times \underbrace{\text{internal block}(y,\bar{y})}_{=1 \text{ for } \langle 2222\rangle}}$

- Superblock expansion of the dynamical part of the correlator is (relatively) straightforward, only contains long ops.

- Expansion of free part is more complicated. (short ops.)
- Explicit superblocks - next slide.
- Alternatively lift across to a bosonic $SU(p,p)$ block.

[v. simple determinantal blocks in p bosonic variables] [Doobary, P.H; Aprile, Drummond, Hynek, P.H.]

Superblocks (continued)

[Doobary, P.H.]

$$F^{\alpha\beta\gamma\underline{\lambda}}(x|y) = \delta_{\underline{\lambda}:0} + D^{-1}\left[\left(\frac{f(x_2,y_2)}{x_1-y_1} - y_1 \leftrightarrow y_2\right) - x_1 \leftrightarrow x_2\right] + \boxed{D^{-1} f(x_1,x_2,y_1,y_2).} \quad (7$$

where here

$$D^{-1} = \frac{(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2)}{(x_1-x_2)(y_1-y_2)}. \quad (7.$$

$$x_1 x_2 = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u$$

$$(1-x_1)(1-x_2) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v$$

$$y_1 y_2 = \frac{Y_1\cdot Y_2\ Y_3\cdot Y_4}{Y_1\cdot Y_3\ Y_2\cdot Y_4}$$

$$(1-y_1)(1-y_2) = \frac{Y_1\cdot Y_4\ Y_2\cdot Y_3}{Y_1\cdot Y_3\ Y_2\cdot Y_4}$$

The functions are given explicitly as

$\lambda_2 > 1$ **(long)** :

$f(x,y) = 0$

$$\boxed{f(x_1,x_2,y_1,y_2) = (-1)^{\lambda'_1+\lambda'_2}\left(F^{\alpha\beta\gamma}_{\lambda_1}(x_1)\,F^{\alpha\beta\gamma}_{\lambda_2-1}(x_2) - x_1 \leftrightarrow x_2\right)\left(G^{\alpha\beta\gamma}_{\lambda'_1}(y_1)\,G^{\alpha\beta\gamma}_{\lambda'_2-1}(y_2) - y_1 \leftrightarrow y_2\right)}$$

$\lambda_2 = 0,1$ **(semi-short / quarter BPS)** :

$f(x,y) = (-1)^{\lambda'_1} F^{\alpha\beta\gamma}_{\lambda_1}(x) G^{\alpha\beta\gamma}_{\lambda'_1}(y)$

$$f(x_1,x_2,y_1,y_2) = \sum_{j=\lambda'_1+1}^{p} (-1)^{\lambda'_1}\left(F^{\alpha\beta\gamma}_{1-j}(x_2)\,F^{\alpha\beta\gamma}_{\lambda_1}(x_1) - (x_1 \leftrightarrow x_2)\right)\left(G^{\alpha\beta\gamma}_{j}(y_2)\,G^{\alpha\beta\gamma}_{\lambda'_1}(y_1) - (y_1 \leftrightarrow y_2)\right)$$
$$+ \sum_{j=2}^{\lambda'_1}(-1)^{\lambda'_1}\left(F^{\alpha\beta\gamma}_{2-j}(x_2)\,F^{\alpha\beta\gamma}_{\lambda_1}(x_1) - (x_1 \leftrightarrow x_2)\right)\left(G^{\alpha\beta\gamma}_{j-1}(y_2)\,G^{\alpha\beta\gamma}_{\lambda'_1}(y_1) - (y_1 \leftrightarrow y_2)\right)$$

$\underline{\lambda} = 0$ **(half BPS)** :

$$f(x,y) = -\sum_{i=1}^{p} F^{\alpha\beta\gamma}_{1-i}(x) G^{\alpha\beta\gamma}_{i}(y)$$

$$f(x_1,x_2,y_1,y_2) = \sum_{1\le i<j\le p}\left(F^{\alpha\beta\gamma}_{1-i}(x_2)\,F^{\alpha\beta\gamma}_{1-j}(x_1) - F^{\alpha\beta\gamma}_{1-i}(x_1)\,F^{\alpha\beta\gamma}_{1-j}(x_2)\right)\left(G^{\alpha\beta\gamma}_{i}(y_1)\,G^{\alpha\beta\gamma}_{j}(y_2) - G^{\alpha\beta\gamma}_{i}(y_2)\,G^{\alpha\beta\gamma}_{j}(y_1)\right)$$

where we have defined the functions

$$F^{\alpha\beta\gamma}_{\lambda}(x) := [x^{\lambda-1}{}_2F_1(\lambda+\alpha,\lambda+\beta;2\lambda+\gamma;x)]$$
$$G^{\alpha\beta\gamma}_{\lambda'}(y) := y^{\lambda'-1}{}_2F_1(\lambda'-\alpha,\lambda'-\beta;2\lambda'-\gamma;y)$$

Free theory CPW coeffs via "bosonised superblocks" $SU(2,2|4) \to SU(m,m|2n)$

$\downarrow$

$SU(m,m)$

- Free theory:

$$\prod_{i<j} g_{ij}^{d_{ij}} = \mathcal{P}^{(\mathrm{OPE})}_{\{p_i\}} \times \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}(\gamma-p_4+p_3)} \times \left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{d_{23}}$$

$$\left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{d_{23}} = \left(\frac{(1-y_1)(1-y_2)}{(1-x_1)(1-x_2)}\right)^{d_{23}} = \mathrm{sdet}^{-d_{23}}(1-Z) \qquad Z \sim \left(\begin{array}{cc|cc} x_1 & & & \\ & x_2 & & \\ \hline & & y_1 & \\ & & & y_2 \end{array}\right)$$

$$= \sum A_{\gamma\underline{\lambda}} F^{\alpha\beta\gamma\underline{\lambda}}(Z)$$

superblocks above

$SU(2,2|4)$

$\downarrow$

$SU(m,m)$

simple bosonised blocks

$\downarrow$

$$Z \to \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{pmatrix}$$

$$F^{\alpha\beta\gamma\underline{\lambda}}(\underline{x}) = \frac{\det\left(x_i^{\lambda_j+m-j}\,{}_2F_1(\lambda_j+1-j+\alpha, \lambda_j+1-j+\beta; 2\lambda_j+2-2j+\gamma; x_i)\right)_{1\le i,j\le m}}{\det\left(x_i^{m-j}\right)_{1\le i,j\le m}}$$

Multi-particle SUGRA states

- Only other states remaining when $\alpha' \to 0$ = $\boxed{\text{bound states of these}}$

can have derivatives $\partial_\mu^{\,l} \Box^n$

2-particle states $\longrightarrow$ Double trace $\mathrm{Tr}(\varphi^n)\,\mathrm{Tr}(\varphi^m) + \frac{1}{N}\cdots$

3-particle states $\longrightarrow$ Triple trace $\mathrm{Tr}(\varphi^n)\,\mathrm{Tr}(\varphi^m)\,\mathrm{Tr}(\varphi^p) + \frac{1}{N}$

etc.

can be different scalars

More precise definition of single trace ops : Operators dual to single-particle states are defined as those orthogonal to all multi-particle ops.

eg. $\mathrm{Tr}(\varphi^2)$ is a single particle op. $\Rightarrow$ $\mathrm{Tr}(\varphi^2)\,\mathrm{Tr}(\varphi^2)$ is 2-particle op.

$$\Rightarrow \boxed{O_4 = \mathrm{Tr}(\varphi^4) - \frac{2N^2-3}{N(N^2+1)}\,\mathrm{Tr}(\varphi^2)^2}$$ is wt. 4 single particle op.

$$\boxed{\langle O_4\, \mathrm{Tr}(\varphi^2)^2 \rangle = 0}$$

[n,0,n] ops.

- $\frac{1}{N^2}$ Anomalous dimensions. 4-parameter analytic formula

$$\boxed{\eta^{[n,0,n]}_{t,l,i} = -\frac{2(t-1-n)t(t+1)(t+2+n)(t+l-n)(t+l+1)(t+l+2)(t+l+3+n)}{(l+2i+n-1)_6}}$$

- 3-pnt functions v. similar natural generalization of the orthogonal matrices

(found the generalization before we found its application!)

[0,1,0] ops

[n,1,n] similar

- Operators $\{\mathcal{K}_{t,l,1},\mathcal{K}_{t,l,2},\ldots\mathcal{K}_{t,l,t-1}\} \sim \{\mathcal{O}_2\partial^l\Box^{t-2}\mathcal{O}_3, \mathcal{O}_3\partial^l\Box^{t-3}\mathcal{O}_4,\ldots,\mathcal{O}_t\partial^l\mathcal{O}_{t+1}\}|_{[0,1,0]}$
- Input Correlators: $\langle p\, p{+}1\, q\, q{+}1\rangle$ at $O(N^0)$, $O(\frac{1}{N^2})\log u$

(previous results, l even only)

- Output:

$$\eta^{(1)}_{t,l,i} = \begin{cases} -\frac{2(t-1)_2(t+2)_2(l+t)_2(l+t+3)_2}{(l+2i-1)_6} & l=0,2,\ldots \\ -\frac{2(t-1)_2(t+2)_2(l+t)_2(l+t+3)_2}{(l+2i)_6} & l=1,3,\ldots \end{cases}$$

(Similar 3-pt structure)

Anomalous dimensions of [020] double trace ops.

- Next case $\boxed{[020]}$. $\boxed{O_p \Box^n \partial^l O_p, \; O_p \Box^n \partial^l O_{p+2}}$

$\Rightarrow$ $\boxed{\langle pp\,qq\rangle, \; \langle p\,p{+}2\;qq\rangle, \; \langle p\,p{+}2\;q\,q{+}2\rangle \;\; O(N^0), \; O\!\left(\tfrac{1}{N^2}\right)}$

- New feature: degeneracy of $\boxed{O\!\left(\tfrac{1}{N^2}\right)}$ anomalous dimensions

$\Rightarrow$ Spoils structure of 3-pnt functions:

No unique diagonal basis...

- Anomalous dimensions can still be found (eigenvalues)
- $\boxed{\text{Is this degeneracy lifted?}}$

Properties of the data.

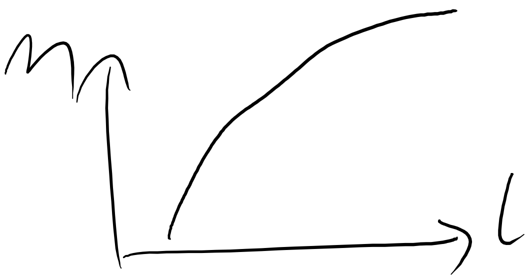
- $\boxed{\langle 2233\rangle\big|_{1/N^4} \text{ computed similarly.}}$

- All data satisfies a non-trivial symmetry as analytic functions of ℓ. Trajectories in ℓ. [swap even & odd spin trajectories and order of degenerate ops $i \to -i$]

$$\boxed{\ell \to -\ell - \underbrace{T(\ell; a)}_{\text{Full (anomalous) twist}} - 3}$$

Equivalent to reciprocity [Basso, Korchemsky; Alday, Bissi]

- Anomalous dimensions, negative, monotonic, convex. [Fitzpatrick, Kaplan, Poland, Simons-Duffin; Khomargodski, Zhiboedov; Alday, Bissi, Perlmutter]



$$\boxed{\begin{array}{l} \eta < 0 \\ \eta' > 0 \\ \eta'' < 0 \end{array}}$$

Until finite value of ℓ.

1-loop correlator.

Full result:

$$\langle 2222\rangle = \langle 2222\rangle_{\text{free}} + g_{13}^2 g_{24}^2 s(x,\bar{x};y,\bar{y})F(u,v)$$

$$F(u,v) = aF^{(1)}(u,v) + a^2F^{(2)}(u,v) + O(a^3)$$

$$F^{(1)}(u,v) = -4\partial_u\partial_v(1+u\partial_u+v\partial_v)\Phi^{(1)}(u,v)$$

$$F^{(2)}(u,v) = \frac{1}{uv}\left[f(u,v) + \frac{1}{u}f\left(\frac{1}{u},\frac{v}{u}\right) + \frac{1}{v}f\left(\frac{1}{v},\frac{u}{v}\right)\right] + \text{ambiguity}$$

$$f(u,v) = \Delta^{(4)}g(u,v)\,, \qquad \Delta^{(4)} = (x-\bar{x})^{-1}uv\partial_x^2\partial_{\bar{x}}^2(x-\bar{x})$$

$$g = (x-\bar{x})^{-10}[g^{(4)}+g^{(3)}+g^{(2)}+g^{(1)}+g^{(0)}]$$

$$\begin{aligned}
g^{(4)}(u,v) &= P_-^{(4)}(u,v)\Phi^{(2)}(u,v)\\
g^{(3)}(u,v) &= P_+^{(3)}(u,v)\Psi(u,v) + P_-^{(3)}(u,v)\log(uv)\Phi^{(1)}(u,v)\\
g^{(2)}(u,v) &= P_+^{(2)}(u,v)\log u\log v + P_-^{(2)}(u,v)\Phi^{(1)}(u,v)\\
g^{(1)}(u,v) &= P_+^{(1)}(u,v)\log(uv)\\
g^{(0)}(u,v) &= P_+^{(0)}(u,v)\,.
\end{aligned}$$

$$\Phi^{(l)}(u,v) = -\frac{1}{x-\bar{x}}\phi^{(l)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right),$$

L-Loop Ladder

$$\phi^{(l)}(x,\bar{x}) = \sum_{r=0}^{l}(-1)^r\frac{(2l-r)!}{r!(l-r)!l!}\log^r(x\bar{x})(\mathrm{Li}_{2l-r}(x)-\mathrm{Li}_{2l-r}(\bar{x}))$$

$$\begin{aligned}\Psi(u,v) &= (x-\bar{x})(u\partial_u+v\partial_v)[(x-\bar{x})\Phi^{(2)}(u,v)] \\ &= [x(1-x)\partial_x-\bar{x}(1-\bar{x})\partial_{\bar{x}}]\phi^{(2)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right)\end{aligned}$$

$$\begin{aligned}P_-^{(4)}(u,v) &= 96p^2\bar{s}[\bar{s}^4+20p\bar{s}^2+30p^2]\,,\\ P_+^{(3)}(u,v) &= \frac{8}{5}p^2[137\bar{s}^4+1214p\bar{s}^2+512p^2]\,,\\ P_-^{(3)}(u,v) &= 336p^2[\bar{s}(1-\bar{s})(6-6\bar{s}+\bar{s}^2)+2p(3-14\bar{s}+4\bar{s}^2)-16p^2]\,,\\ P_+^{(2)}(u,v) &= 2[(1-\bar{s})^2\bar{s}^6-2p\bar{s}^4(20-33\bar{s}+14\bar{s}^2)\\ &\quad+8p^2(756-1323\bar{s}+601\bar{s}^2-54\bar{s}^3+30\bar{s}^4)\\ &\quad-32p^3(583-25\bar{s}+26\bar{s}^2)+1024p^4]\,,\end{aligned}$$

$$\begin{aligned}P_-^{(2)}(u,v) &= 56p^2[-\bar{s}^2(2-\bar{s})(18-18\bar{s}+5\bar{s}^2)\\ &\quad+2p(108-144\bar{s}+128\bar{s}^2-11\bar{s}^3)-8p^2(63-\bar{s})]\,,\\ P_+^{(1)}(u,v) &= \frac{1}{3}[5\bar{s}^7(2-3\bar{s})-2p\bar{s}^5(158-193\bar{s})\\ &\quad+16p^2\bar{s}(378-567\bar{s}+233\bar{s}^2-147\bar{s}^3)\\ &\quad+32p^3(378-139\bar{s}+129\bar{s}^2)+256p^4]\,,\\ P_+^{(0)}(u,v) &= \frac{2}{15}(x-\bar{x})^2[20(1-\bar{s})\bar{s}^6-5p\bar{s}^4(102-75\bar{s}-4\bar{s}^2)\\ &\quad+8p^2(630-630\bar{s}+481\bar{s}^2-255\bar{s}^3-30\bar{s}^4)\\ &\quad-16p^3(217-215\bar{s}-60\bar{s}^2)-1280p^4]\,.\end{aligned}$$

$$\bar{s} = 1-u-v\,,\qquad p = uv$$

String expansion / M theory

- Full expansion

$$\langle 2222\rangle = \langle 2222\rangle + \frac{1}{N^2}\langle 2222\rangle^{(1,0)} + \frac{1}{N^4}\langle 2222\rangle^{(2,0)} + \cdots + \frac{1}{N^2\lambda^{3/2}}\langle 2222\rangle^{(1,1)} + \cdots$$

gravity loop corrections

string corrections

$$\alpha' = \frac{1}{\sqrt{\lambda}}$$

- $\frac{1}{N^2\lambda^{3/2}} = \frac{1}{N^{7/2}g_{YM}^3}$, for fixed g_{YM} this is lower order than the loop correction!

- $\langle 2222\rangle^{(1,1)}$ is known $= \boxed{\bar{D}_{4444}}$ (same as the ambiguity at 1-loop) (from R^4 effective action presumably)

[Alday, Bissi; Goncalves]

- One way to derive this: $\boxed{u^a v^b \bar{D}_{pqrs} + \text{crossing}}$ Require no twist 2 ops in OPE $\Rightarrow O(u^2)$ [$\mathrm{Tr}(\varphi\cdot\partial^l\varphi) \leftarrow$ string states]

$\boxed{\text{Constrains } a,b,p,q,r,s}$. $\boxed{p+q+r+s \geq 16}$ ....

$\bar{D}_{4444}$ minimal case. $\boxed{\text{Only spin 0 in OPE.}}$ [Heemskerk, Penedones, Polchinski, Sully; Alday, Bissi, Lukowski]

M theory $\leftrightarrow$ (2,0) theory (6d) (with Lipstein)

- Only $\frac{1}{N}$, no coupling.
- Structure similar though [superblocks: Arutyunov, Dolan, Gallot, Sokatchev; P.H.]

$$\boxed{\begin{aligned}\langle 2222\rangle = \langle 2222\rangle\big|_{N^0} + \frac{1}{N^3}\langle 2222\rangle^{(1,0)} + \frac{1}{N^6}\langle 2222\rangle^{(2,0)} \\ + \, ? \, \langle 2222\rangle^{(1,1)}\end{aligned}}$$

- Find analogue of the $\frac{\alpha'^3}{N^2}$ correction.
- Same technique. No twist 4 in $\log u$ part (anomalous dim)
  But can appear in non-log piece (no partial non-renormalization)

Minimal case: $\boxed{u v \bar{D}_{5755}}$

again

$\boxed{\text{Only spin } 0! \text{ in OPE}}$

twist n

$$\gamma^{\text{spin}-0}_{n,n} = -\frac{(n-1)_8\, n_6}{2240(2n+3)(2n+5)(2n+7)}.$$