# *QCD splitting functions and cusp anomalous dimensions at four loops*

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## *Based on work done in collaboration with:*

- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`
- *Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics*
  V.M. Braun, A.N. Manashov, S. M. and M. Strohmaier `arXiv:1703.09532`
- *Two-loop conformal generators for leading-twist operators in QCD*
  V.M. Braun, A.N. Manashov, S. M. and M. Strohmaier `arXiv:1601.05937`
- Many more papers of MVV and friends . . .
  `2001 - ...`

# *Motivation*

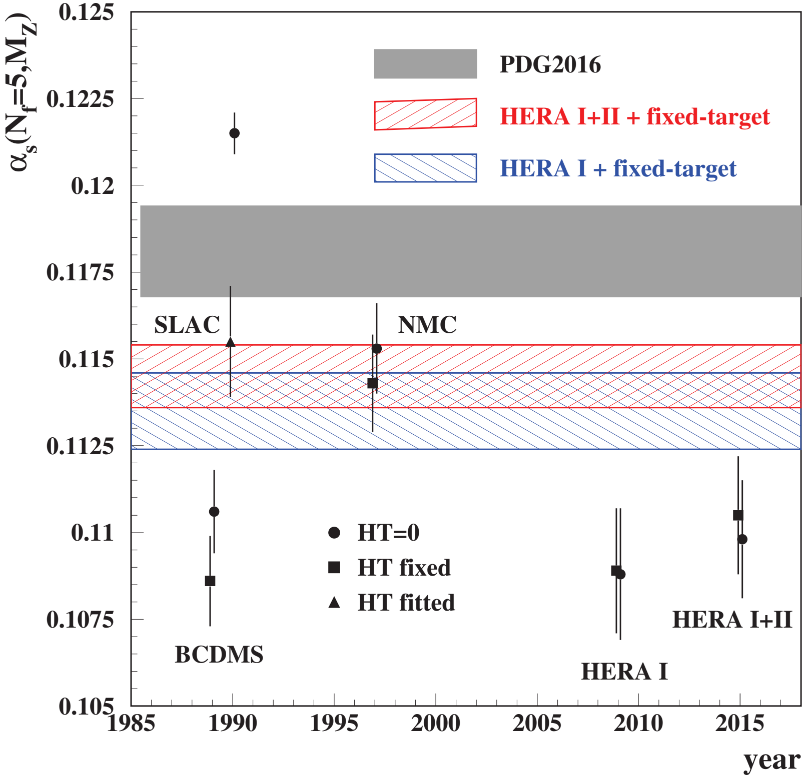
# Strong coupling constant (2018)

| | | | |
|---|---|---|---|
| BBG | $0.1134\,^{+\,0.0019}_{-\,0.0021}$ | valence analysis, NNLO | Blümlein, Böttcher, Guffanti '06 |
| JR08 | $0.1128 \pm 0.0010$ | dynamical approach | Jimenez-Delgado, Reya '08 |
| | $0.1162 \pm 0.0006$ | including NLO jets | |
| ABKM09 | $0.1135 \pm 0.0014$ | HQ: FFNS $n_f = 3$ | Alekhin, Blümlein, Klein, S.M. '09 |
| | $0.1129 \pm 0.0014$ | HQ: BSMN | |
| MSTW | $0.1171 \pm 0.0014$ | | Martin, Stirling, Thorne, Watt '09 |
| Thorne | 0.1136 | [DIS+DY, HT*] (2013) | Thorne '13 |
| $ABM11_J$ | $0.1134 \ldots 0.1149 \pm 0.0012$ | Tevatron jets (NLO) incl. | Alekhin, Blümlein, S.M. '11 |
| NN21 | $0.1173 \pm 0.0007$ | (+ heavy nucl.) | NNPDF '11 |
| ABM12 | $0.1133 \pm 0.0011$ | | Alekhin, Blümlein, S.M. '13 |
| | $0.1132 \pm 0.0011$ | (without jets) | |
| CT10 | 0.1140 | (without jets) | Gao et al. '13 |
| CT14 | $0.1150\,^{+\,0.0060}_{-\,0.0040}$ | $\Delta\chi^2 > 1$ (+ heavy nucl.) | Dulat et al. '15 |
| JR14 | $0.1136 \pm 0.0004$ | dynamical approach | Jimenez-Delgado, Reya '14 |
| | $0.1162 \pm 0.0006$ | standard approach | |
| MMHT | $0.1172 \pm 0.0013$ | (+ heavy nucl.) | Martin, Motylinski, Harland-Lang, Thorne '15 |
| ABMP16 | $0.1147 \pm 0.0008$ | | Alekhin, Blümlein, S.M., Placakyte '17 |
| NN31 | $0.1185 \pm 0.0012$ | including NLO jets | NNPDF '18 |

- Measurements at NNLO (last $\sim 10$ years) from DIS data
- Large spread of fitted values at NNLO: $\alpha_s(M_Z) = 0.1128 \ldots 0.1185$
- [yellow box] taken for 2017 PDG average: $\alpha_s(M_Z) = 0.1156 \pm 0.0021$

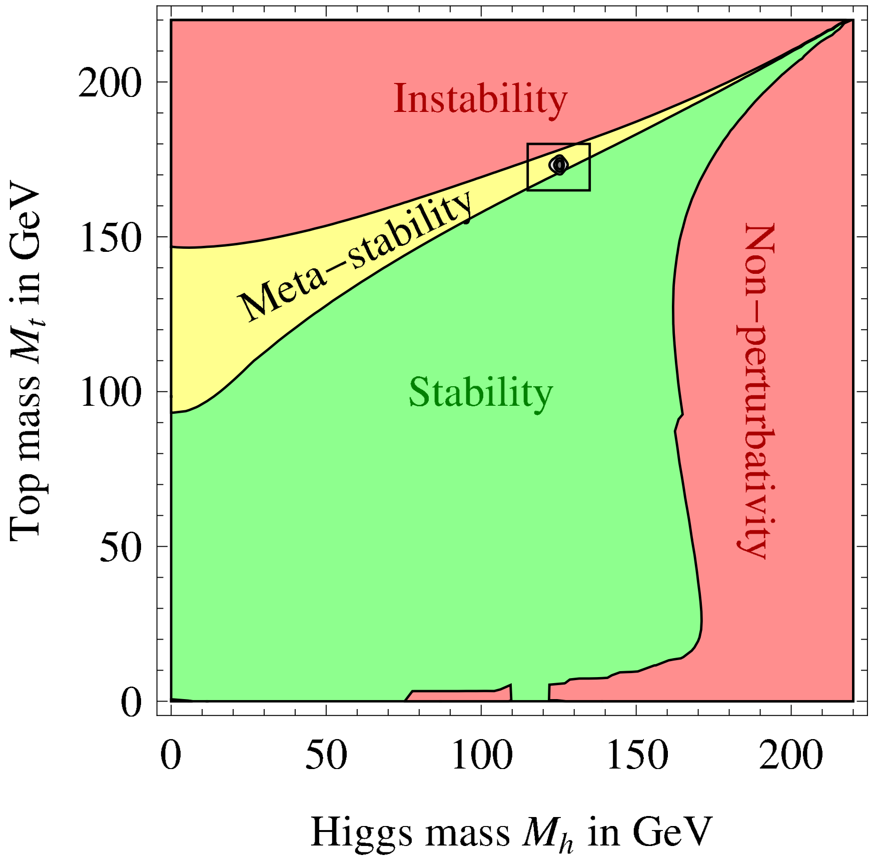
# *Theory considerations in $\alpha_s$ determinations*

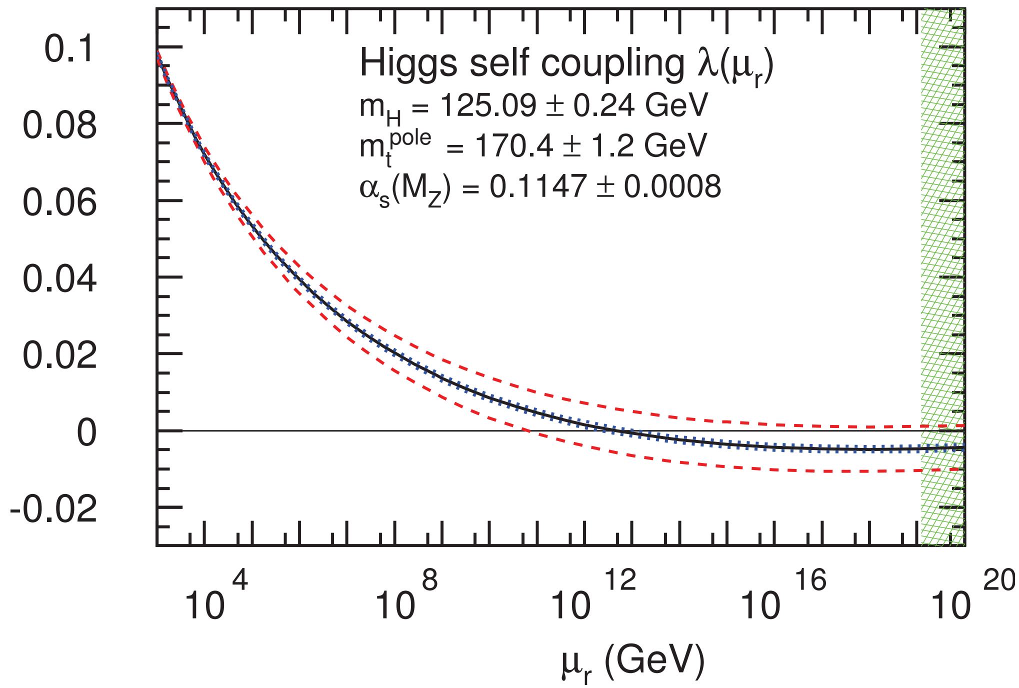
- Correlation of errors among different data DIS sets
- Target mass corrections (powers of nucleon mass $M_N^2/Q^2$)
- Higher twist $F_2^{\rm ht} = F_2 + ht^{(4)}(x)/Q^2 + \dots$
- Variants with no higher twist give larger $\alpha_s$ values Alekhin, Blümlein, S.M. '17




- Theoretical uncertainty of $\alpha_s$ at NNLO from DIS data $\gtrsim \mathcal{O}(1\dots2)\%$

# *Electroweak vacuum stability*



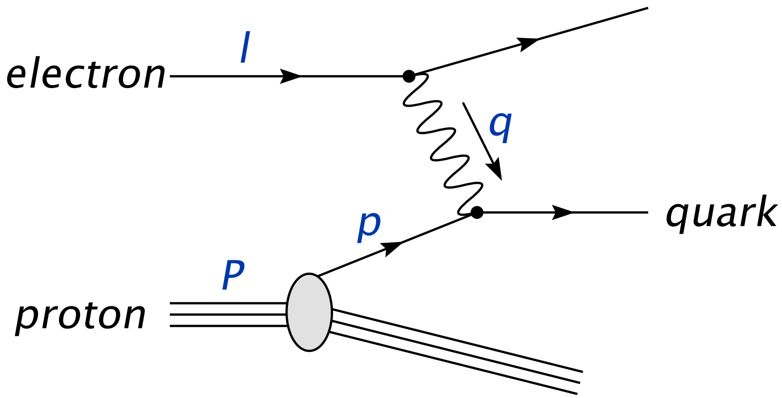

Bezrukov, Kalmykov, Kniehl, Shaposhnikov '12; Degrassi, Di Vita, Elias-Miro, Espinosa, Giudice et al. '12; Alekhin, Djouadi, S.M. '12; Masina '12; [many people]

- Condition of absolute stability of electroweak vacuum at Planck scale $M_{\rm Planck}$ requires Higgs self-coupling $\lambda(\mu_r) \geq 0$

$$m_H \geq 129.6 + 2.0 \times \left( m_t^{\rm pole} - 173.34 \text{ GeV} \right) - 0.5 \times \left( \frac{\alpha_s(M_Z) - 0.1184}{0.0007} \right) \pm 0.3 \text{ GeV}$$

# *Theoretical framework*

# *Deep-inelastic scattering*




*Kinematic variables*

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes PDF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N$^3$LO

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \ldots\right)$$

- Evolution equations up to N$^3$LO
  - non-singlet ($2n_f - 1$ scalar) and singlet ($2 \times 2$ matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$$

  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \ldots$

# Evolution equations

- Parton distribution functions $q_i(x, \mu^2)$, $\bar{q}_i(x, \mu^2)$ and $g(x, \mu^2)$ for quarks, antiquarks of flavour $i$ and gluons
- Flavor non-singlet combinations

$q^{\pm}_{\rm ns,\it ik} \;=\; (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k)$ and $q^{\rm v}_{\rm ns} \;=\; \sum_{i=1}^{n_f} (q_i - \bar{q}_i)$

  - splitting functions $P^{\pm}_{\rm ns}$ and $P^{\rm v}_{\rm ns} = P^{-}_{\rm ns} + P^{\rm s}_{\rm ns}$
- Flavor singlet evolution

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad \text{and} \quad q_{\rm s} \;=\; \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

  - quark-quark splitting function $P_{\rm qq} = P^{+}_{\rm ns} + P_{\rm ps}$
- Mellin transformation relates to anomalous dimensions $\gamma_{\rm ik}(N)$ of twist-two operators

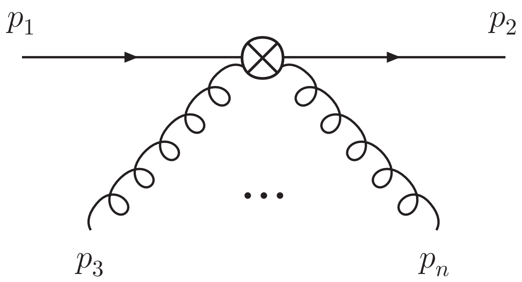
$$\gamma^{(n)}_{\rm ik}(N, \alpha_s) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P^{(n)}_{\rm ik}(x, \alpha_s)$$

## *Non-singlet*

# *Operator matrix elements*

- Non-singlet operator of spin-$N$ and twist two

$$O^{\rm ns}_{\{\mu_1,\ldots,\mu_N\}} = \overline{\psi}\,\lambda^\alpha\,\gamma_{\{\mu_1} D_{\mu_2} \ldots D_{\mu_N\}}\,\psi\ ,$$
$$\alpha = 3, 8, \ldots, (n_f^2 - 1)$$




## *Calculation*

- Anomalous dimensions $\gamma(N)$ from ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma^{\pm}_{\rm ns}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma^{\rm s}_{\rm ns}$

# Anomalous dimensions

- Anomalous dimensions $\gamma(N)$ of leading twist non-singlet local operators
  - expressible in harmonic sums up to weight $7$

$$S_{\pm m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

  - $2 \cdot 3^{w-1}$ sums at weight $w$
- Reciprocity relation $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(a)\right)$ reduces number of $2^{w-1}$ sums at weight $w$ for $\gamma_{\rm u}$
  - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects ($255$ at weight $7$ )
- Constraints at large-$x$/small-$x$ ($N \to \infty$/$N \to 0$) give additional $46$ conditions

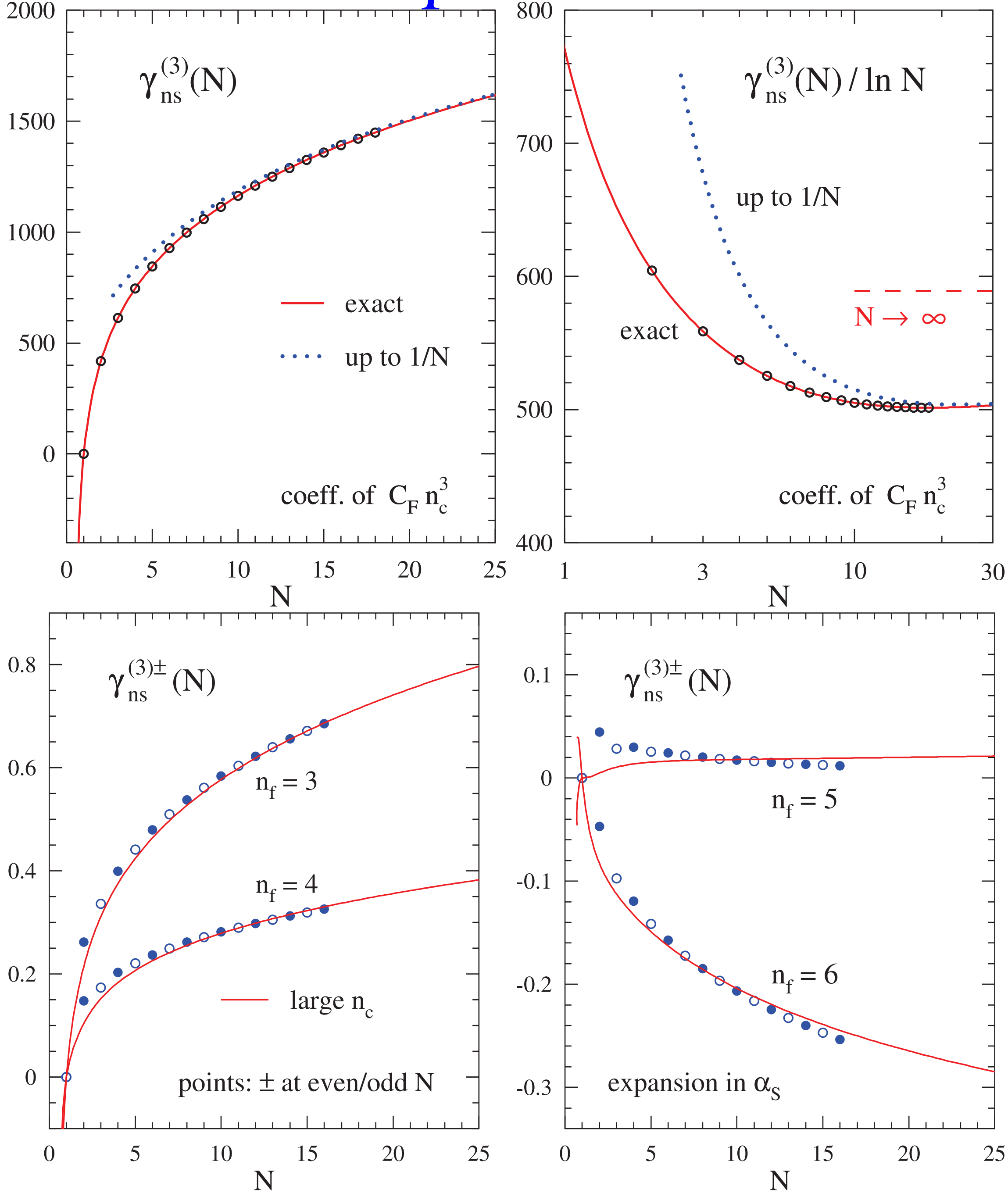
## Upshot

- Computation of Mellin moments up to $N = 18$ for anomalous dimensions feasible
- Reconstruction of analytic all-$N$ expressions in large-$n_c$ limit from solution of Diophantine equations

# Mellin moments at four loops

- Top:
  $n_f^0$ part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$
  in large-$n_c$ limit
  and large-$N$ expansion

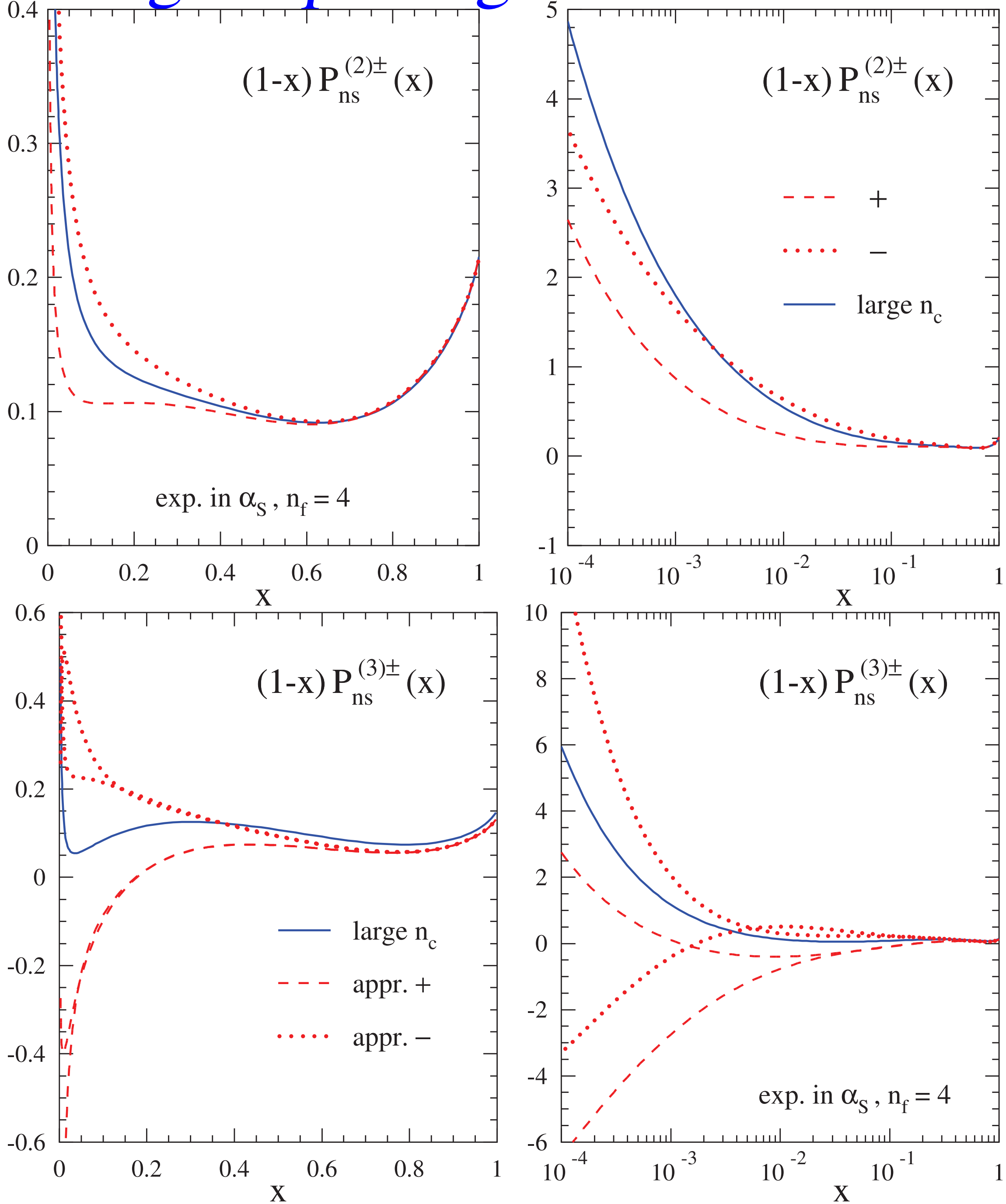
- Bottom: results for even-$N$ ( $\gamma_{\rm ns}^{(3)+}(N)$ )
  and odd-$N$ ( $\gamma_{\rm ns}^{(3)-}(N)$ )
  in large-$n_c$ limit
  for $n_f = 3, \ldots 6$



# *Four-loop non-singlet splitting functions*

- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-$n_c$ limit
  with $n_f = 4$

- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-$n_c$ limit
  with $n_f = 4$

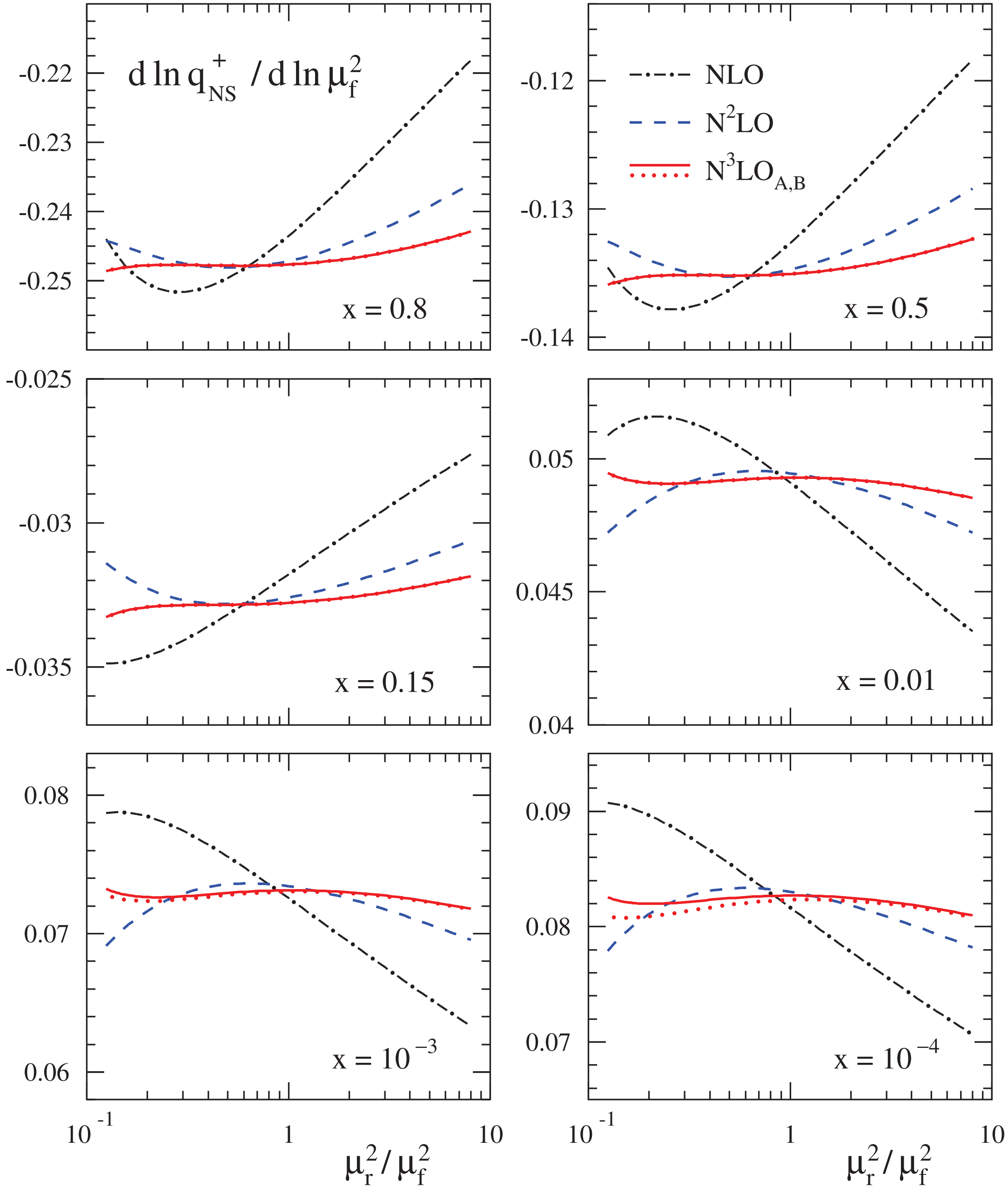


# Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

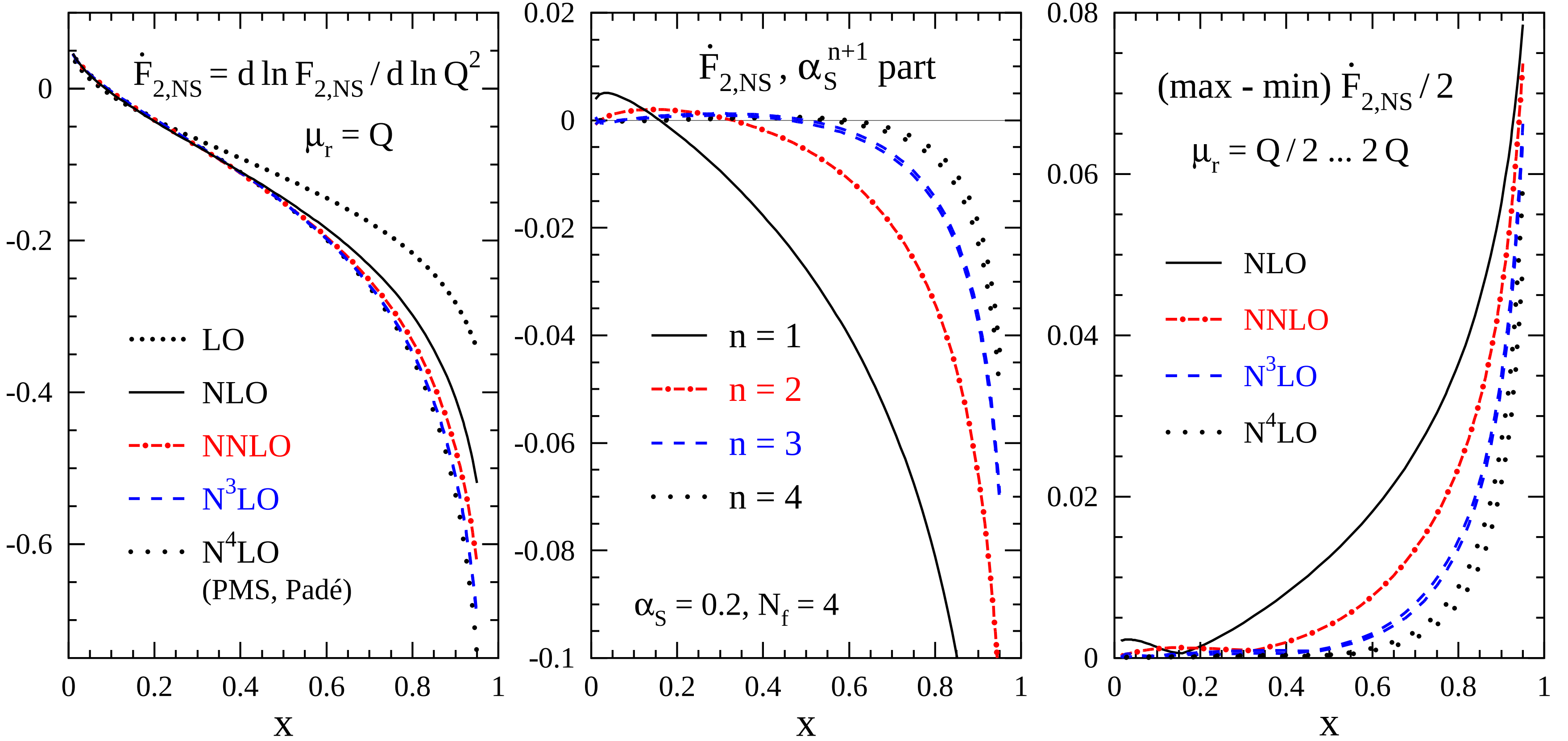
$$xq^{+}_{\rm ns}(x,\mu_0^2) \;=\; x^{\,0.5}(1-x)^3$$

- NLO, NNLO and N$^3$LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



# *The structure function $F_2$ (non-singlet)*

- Large-$x$ convergence of perturbative series
  - approx. N$^3$LO structure functions S.M., Vermaseren, Vogt '05




- Potential for 'gold-plated' determination of $\alpha_s$
  - theory uncertainty $\Delta_{\text{pert.}}\alpha_s < 1\%$

*Singlet*

# Color factors of $SU(n_c)$

- Quadratic Casimir factors $C_r \delta^{ab} \equiv \mathrm{Tr}\,(\, T_r^a T_r^b \,)$
  - fundamental representation $C_F = (n_c^2 - 1)/(2n_c)$;
    adjoint representation $C_A = n_c$
- Quartic Casimir invariants occur for the first time at four loops
  - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels $x, y$ with generators $T_r^a$

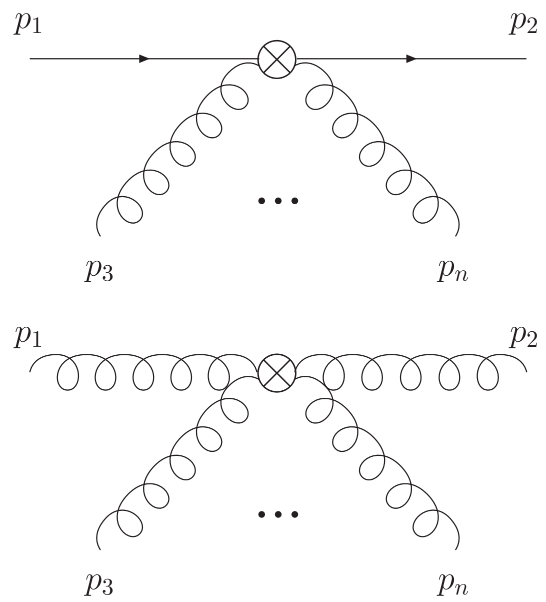
$$d_r^{abcd} = \frac{1}{6}\,\mathrm{Tr}\,(\, T_r^a T_r^b T_r^c T_r^d + \text{ five } bcd \text{ permutations} \,)$$

- SU($n_c$) with fermions in fundamental representation

$$\begin{aligned} d_{AA}^{(4)}/n_A &= \frac{1}{24}\, n_c^2 (n_c^2 + 36)\ , \\ d_{FA}^{(4)}/n_A &= \frac{1}{48}\, n_c (n_c^2 + 6)\ , \\ d_{FF}^{(4)}/n_A &= \frac{1}{96}\, (n_c^2 - 6 + 18 n_c^{-2}) \end{aligned}$$

  - trace-normalized with $T_F = \frac{1}{2}$
  - dimensions of representations $n_F = n_c$ and $n_A = (n_c^2 - 1)$

# *Operator matrix elements*




- Singlet operators of spin-$N$ and twist two

$$O^{\mathrm{q}}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi\ ,$$
$$O^{\mathrm{g}}_{\{\mu_1,\dots,\mu_N\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N\}}{}^{\nu}$$

- Quartic Casimir terms at four loops are effectively 'leading-order'
  - anomalous dimensions fulfil relation for $\mathcal{N}=1$ supersymmetry

$$\gamma^{(3)}_{\mathrm{qq}}(N) + \gamma^{(3)}_{\mathrm{gq}}(N) - \gamma^{(3)}_{\mathrm{qg}}(N) - \gamma^{(3)}_{\mathrm{gg}}(N) \stackrel{Q}{=} 0$$

- color-factor substitutions for $n_f = 1$ Majorana fermions (factor $2n_f$)

$$(2n_f)^2 \frac{d^{(4)}_{FF}}{n_A} = 2n_f \frac{d^{(4)}_{FA}}{n_A} = 2n_f \frac{d^{(4)}_{FF}}{n_F} = \frac{d^{(4)}_{FA}}{n_F} = \frac{d^{(4)}_{AA}}{n_A}$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer '99

$$\gamma^{(0)}_{\mathrm{qg}}(N)\,\gamma^{(3)}_{\mathrm{gq}}(N) \stackrel{Q}{=} \gamma^{(0)}_{\mathrm{gq}}(N)\,\gamma^{(3)}_{\mathrm{qg}}(N)$$

# *Calculation and results*

- Computation of Mellin moments up to $N = 16$ for anomalous dimensions feasible
- Numerical approximaton $x$-space expressions
  - used for large-$x$ limit

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_{n,\mathrm{i}}}{(1-x)_+} + B_{n,\mathrm{i}}\,\delta(1-x) + C_{n,\mathrm{i}}\,\ln(1-x) + D_{n,\mathrm{i}}$$

- Cusp anomalous dimensions related by Casimir scaling up to three loops

$A_{n,\mathrm{g}} = \frac{C_A}{C_F} A_{n,\mathrm{q}}$ for $n \leq 3$

- Casimir scaling at four loops broken due to new color factors
  - $A_{n,\mathrm{q}}$ contains $\frac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_F}$
  - $A_{n,\mathrm{g}}$ contains $\frac{d_A^{abcd} d_A^{abcd}}{n_A}$, $\frac{d_F^{abcd} d_A^{abcd}}{n_A}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_A}$
- Large $n_c$-limit at four loops restores Casimir scaling — Dixon '17

$$A_{4,\mathrm{g}}\big|_{\mathrm{large-n_c}} = \frac{C_A}{C_F}\, A_{4,\mathrm{q}}\big|_{\mathrm{large-n_c}}$$

# *Quark and gluon cusp anomalous dimensions*

- Large-$n_c$ limit of quark cusp anomalous dimension (agrees with Henn, Lee, Smirnov, Smirnov, Steinhauser '16)

$$
\begin{aligned}
A_{4,\mathrm{q}}\big|_{\mathrm{large-n_c}} \;=\; & C_F n_c^3 \left( \frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\zeta_5 \right. \\
& \qquad \left. -32\zeta_3^2 - 876\,\zeta_6 \right) \\
& - C_F n_c^2\, n_f \left( \frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \right) \\
& + C_F n_c\, n_f^2 \left( \frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \right) - C_F n_f^3 \left( \frac{32}{81} - \frac{64}{27}\,\zeta_3 \right)
\end{aligned}
$$

- Result includes non-vanishing coefficients of quartic Casimir contributions $\dfrac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\dfrac{d_F^{abcd} d_F^{abcd}}{n_F}$

# Generalized 'Casimir scaling'

| quark | gluon | $A_{4,q}$ | $A_{4,g}$ |
|---|---|---|---|
| $C_F^4$ | – | $0$ | – |
| $C_F^3 C_A$ | – | $0$ | – |
| $C_F^2 C_A^2$ | – | $0$ | – |
| $C_F C_A^3$ | $C_A^4$ | $610.25 \pm 0.1$ | |
| $d_{FA}^{(4)}/n_F$ | $d_{AA}^{(4)}/n_A$ | $-507.0 \pm 2.0$ | $-507.0 \pm 5.0$ |
| $n_f\, C_F^3$ | $n_f\, C_F^2 C_A$ | $-31.00554$ | |
| $n_f\, C_F^2 C_A$ | $n_f\, C_F C_A^2$ | $38.75 \pm 0.2$ | |
| $n_f\, C_F C_A^2$ | $n_f C_A^3$ | $-440.65 \pm 0.2$ | |
| $n_f\, d_{FF}^{(4)}/n_F$ | $n_f\, d_{FA}^{(4)}/n_A$ | $-123.90 \pm 0.2$ | $-124.0 \pm 0.6$ |
| $n_f^2\, C_F^2$ | $n_f^2\, C_F C_A$ | $-21.31439$ | |
| $n_f^2\, C_F C_A$ | $n_f^2\, C_A^2$ | $58.36737$ | |
| – | $n_f^2\, d_{FF}^{(4)}/n_A$ | – | $0.0 \pm 0.1$ |
| $n_f^3\, C_F$ | $n_f^3\, C_A$ | $2.454258$ | $2.454258$ |

- Numerical value for $n_f\, C_F^3$:
  - approximation: $-31.00 \pm 0.4$ from S.M., Ruijl, Ueda, Vermaseren, Vogt '17
  - exact result: $-\, 31.00554$ (rounded to seven digits) by Grozin '18

# Numerical implications

- Numerical results (expansion in powers of $\alpha_s/(4\pi)$)

$$A_{4,\mathrm{q}} = 20702(2) - 5171.9(2)\, n_f + 195.5772\, n_f^2 + 3.272344\, n_f^3 \,,$$
$$A_{4,\mathrm{g}} = 40880(30) - 11714(2)\, n_f + 440.0488\, n_f^2 + 7.362774\, n_f^3$$

  - Casimir scaling between $A_{4,\mathrm{g}}$ and $A_{4,\mathrm{q}}$ broken by almost 15% in $n_f^0$
  - non-leading large-$n_c$ part of quartic-Casimir term (factors '36' and '6' in $A_{4,\mathrm{g}}$ and $A_{4,\mathrm{q}}$)

- Perturbative expansion very benign for quark

$$A_{\mathrm{q}}(\alpha_s, n_f = 3) = 0.42441\, \alpha_s \,[\, 1 + 0.72657\, \alpha_s + 0.73405\, \alpha_s^2 + 0.6647(2)\, \alpha_s^3 + \ldots ]$$
$$A_{\mathrm{q}}(\alpha_s, n_f = 4) = 0.42441\, \alpha_s \,[\, 1 + 0.63815\, \alpha_s + 0.50998\, \alpha_s^2 + 0.3168(2)\, \alpha_s^3 + \ldots ]$$
$$A_{\mathrm{q}}(\alpha_s, n_f = 5) = 0.42441\, \alpha_s \,[\, 1 + 0.54973\, \alpha_s + 0.28403\, \alpha_s^2 + 0.0133(3)\, \alpha_s^3 + \ldots ]$$

and gluon

$$A_{\mathrm{g}}(\alpha_s, n_f = 3) = 0.95493\, \alpha_s \,[\, 1 + 0.72657\, \alpha_s + 0.73405\, \alpha_s^2 + 0.415(2)\, \alpha_s^3 + \ldots ]$$
$$A_{\mathrm{g}}(\alpha_s, n_f = 4) = 0.95493\, \alpha_s \,[\, 1 + 0.63815\, \alpha_s + 0.50998\, \alpha_s^2 + 0.064(2)\, \alpha_s^3 + \ldots ]$$
$$A_{\mathrm{g}}(\alpha_s, n_f = 5) = 0.95493\, \alpha_s \,[\, 1 + 0.54973\, \alpha_s + 0.28403\, \alpha_s^2 - 0.243(2)\, \alpha_s^3 + \ldots ]$$

# *Analytic results*

- Reconstruction of analytic all-$N$ expressions for $\zeta_5$ terms from solution of Diophantine equations
  - example for $\gamma_{\rm gg}^{(3)}$ with $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\gamma_{\rm gg}^{(3)}(N)\Big|_{\zeta_5\, d_{AA}^{(4)}/n_A} = \frac{64}{3}\Big(30\,(12\,\eta^2 - 4\,\nu^2 - S_1(4\,S_1 + 8\,\eta - 8\,\nu - 11) - 7\nu) + 188\,\eta - \frac{751}{3} - \frac{1}{6}\,N\,(N+1)\Big)$$

- Recall large-$N$ limit of anomalous dimensions

$$\gamma_{\rm ii}^{(k)}(N)\Big|_{N\to\infty} = A_{n,{\rm i}}\,\ln(N) + \mathcal{O}(\mathrm{const}_N)$$

- Terms $S_1(N)^2 \sim \ln(N)^2$ and $N(N+1)$ proportional to $\zeta_5$ must be compensated in large-$N$ limit

# $N = 4$ *Super Yang-Mills theory*

# $N = 4$ *Super Yang-Mills theory*

- The spectral problem anomalous dimension of local operators:
  $\mathcal{O}_n = \mathrm{Tr}(\mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_n)$ and $\mathcal{W}_i \in \{\mathcal{D}\Phi, \mathcal{D}\Psi, \mathcal{D}F\}$

$$\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \rangle = \frac{\delta_{ab}}{(x_1 - x_2)^{2\Delta_a(\lambda)}}$$

- Spectrum of scaling dimensions: $\Delta(\lambda) = \Delta_0 + \gamma(\lambda)$
  - weak coupling expansion in limit: $n_c \to \infty$ and $\lambda = g^2 n_c$ fixed
- Dilatation operator corresponds to Heisenberg spin chain
  - solution with asymptotic Bethe ansatz Beisert, Staudacher '03
  - wrapping corrections when loop order $l \geq 2L$ Beisert, Eden, Staudacher '06

## *Correspondence with QCD*

- QCD results carry over to $N = 4$ SYM after substitution of color factors
  $C_A = C_F = n_c$ and so on
  - principle of "leading transcendentality"
    (keep only highest weight in $\zeta$-function / harmonic sums)
  - at loops $l$-loops harmonic sums of weight $w = 2l - 1$

# *Universal anomalous dimension*

- Universal anomalous dimension $\gamma_{\rm uni}$ in $N = 4$ SYM to three loops
  Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - One-loop example: $\gamma^{(0)}_{\rm uni}(N) = 4 n_c S_1$ emerges from

$$\gamma^{(0)}_{\rm qq}(N) = C_F \left( -3 + 2\frac{1}{N+1} - 2\frac{1}{N} + 4S_1 \right) \quad \text{or}$$

$$\gamma^{(0)}_{\rm gg}(N) = C_A \left( -\frac{11}{3} - \frac{4}{N-1} - \frac{4}{N+1} + \frac{4}{N+2} + \frac{4}{N} + 4S_1 \right) + \frac{2}{3} n_f$$

- Starting at four loops wrapping corrections to complement asymptotic Bethe ansatz
  - control high-energy behaviour ($\sim 1/N^l$)
  - four-loop Bajnok, Janik, Lukowski '08, five-loop Lukowski, Rej, Velizhanin '09, six-loop [...], ...
- $\gamma(N)^{\rm wrap,(4)} \simeq S_1(N)^2 \, f^{\rm wrap}(N)$

$$f^{\rm wrap}(N) \;=\; 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$

- Three-loop Wilson coefficient $c^{(3)}_{\rm ns}(N)$ S.M., Vermaseren, Vogt '05
  - $c^{(3)}_{\rm ns}(N) \simeq C_F \left( C_F - \frac{C_A}{2} \right)^2 \{ N(N+1)\, f^{\rm wrap}(N) \}$

## *Off-forward kinematics*

# *Operator matrix elements*

- QCD applications to hard processes use nonlocal operators of partons at light-like separation

$$\mathcal{O}_\mu(x; z_1, z_2) = \bar{\psi}(x + z_1)\gamma_\mu[z_1, z_2]\psi(x + z_2)$$

  - quark and anti-quark fields joined by Wilson line along '$+$'-direction

$$[z_1, z_2] = \text{Pexp}\left[ig \int_{z_2}^{z_1} dt A_+(t)\right]$$


- Expansion of $\mathcal{O}_\mu(x; z_1, z_2)$ at short distances leads to local operators
  - (anti-)quark fields with covariant derivatives $D_\mu = \partial_+ - igA_+$

    $\bar{\psi}(x)(\overleftarrow{D}_+)^m \gamma_\mu (\overrightarrow{D}_+)^k \psi(x)$

## *Applications*

- (Generalized) parton distributions: PDFs and GPDs
- Hard exclusive reactions with identified hadrons $N(p)$ and $N(p')$ in initial and final state: $\gamma^* N(p) \to \gamma N(p')$ (DVCS)
- Meson-photon transition form factors $\gamma^* \to \gamma\pi$

# Light-ray operators

- Short-distance expansion yields light-ray operators $\mathcal{O}_\mu(x; z_1, z_2)$ with light-like direction $n$

$$[\mathcal{O}](x; z_1, z_2) \quad \equiv \quad \sum_{m,k} \frac{z_1^m z_2^k}{m!k!} \left[ \bar{\psi}(x)(\overleftarrow{D}\cdot n)^m \not{n} (n\cdot \overrightarrow{D})^k \psi(x) \right]$$

- multiplicative renormalization $[\mathcal{O}] = Z\mathcal{O}$
- Light-ray operators satisfy renormalization group equation Balitsky, Braun '87

$$\Big( \mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a) \Big) [\mathcal{O}](x; z_1, z_2) = 0$$

- Integral operator $\mathbb{H}(a)$ acts on light-cone coordinates of fields $z_{12}^\alpha = z_1(1-\alpha) + z_2\alpha$
  - evolution kernel $h(\alpha, \beta)$

$$\mathbb{H}(a)[\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta \, h(\alpha, \beta) \, [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

- Powers $[\mathcal{O}](z_1, z_2) \mapsto (z_1 - z_2)^N$ are eigenfunctions of $\mathbb{H}$
- Eigenvalues $\gamma(N) = \int d\alpha d\beta \, h(\alpha, \beta)(1 - \alpha - \beta)^N$ are anomalous dimensions of leading-twist local operators with $N = m + k$ derivatives

# Evolution equations

- Leading-order result for evolution kernel

$$\mathbb{H}^{(1)} f(z_1, z_2) \quad = 4C_F \Big\{ \int_0^1 d\alpha \, \frac{\bar{\alpha}}{\alpha} \Big[ 2 f(z_1, z_2) - f(z_{12}^{\alpha}, z_2) - f(z_1, z_{21}^{\alpha}) \Big]$$
$$- \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, f(z_{12}^{\alpha}, z_{21}^{\beta}) + \tfrac{1}{2} f(z_1, z_2) \Big\}$$

- Expression comprises all classical leading-order QCD evolution equations
  - PDFs Altarelli, Parisi; $\gamma^{(0)}(N)$
  - meson light-cone distribution amplitudes
    Efremov, Radyushkin, Brodsky, Lepage
  - general evolution equation for GPDs Belitsky, Müller; $h^{(1)}(\alpha, \beta)$

## Task

- Push accuracy of evolution equations to NNLO and beyond
- Computation of $h(\alpha, \beta)$ to three loops and $\gamma(N)$ to four loops

# *Conformal symmetry*

- Full conformal algebra in 4 dimensions includes fifteen generators
  Mack, Salam '69; Treiman, Jackiw, Gross '72

  $\mathbf{P}_\mu$ (4 translations) $\mathbf{M}_{\mu\nu}$ (6 Lorentz rotations)
  $\mathbf{D}$ (dilatation) $\mathbf{K}_\mu$ (4 special conformal transformations)

## *Collinear subgroup* $SL(2,\mathbb{R})$

- Leading order evolution operator $\mathbb{H}^{(1)}$ commutes with (canonical) generators of collinear conformal transformations
  - Special conformal transformation $x_- \to x' = \dfrac{x_-}{1+2ax_-}$
  - Translations $x_- \to x' = x_- + c$ and dilatations $x_- \to x' = \lambda x_-$
- Evolution kernel $h^{(1)}(\alpha,\beta) = \bar{h}(\tau)$ effectively function of one variable
  $\tau = \dfrac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$ (conformal ratio) Braun, Derkachov, Korchemsky, Manashov '99

$$h^{(1)}(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right],$$

# *Higher orders*

- Conformal symmetry is broken in any realistic four-dimensional QFT
  - $\beta(a) \neq 0$
- Leading order generators of $SL(2,\mathbb{R})$ commute with $\mathbb{H}^{(1)}$ satisfy the usual $SL(2,\mathbb{R})$ algebra $[S_0, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_0$

$$
\begin{aligned}
S_-^{(0)} &= -\partial_{z_1} - \partial_{z_2}\,, \\
S_0^{(0)} &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2, \\
S_+^{(0)} &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2)
\end{aligned}
$$

## *Idea*

- Instead of considering consequences of broken conformal symmetry in QCD make use of exact conformal symmetry of modified theory
  - $\beta(a) = 2a(-\epsilon - \beta_0 a - \beta_1 a^2 - \dots)$ with $a = \dfrac{\alpha_s}{4\pi}$
  - large-$n_f$ QCD in $4 - 2\epsilon$ dimensions at critical coupling $a_*$ with $\beta(a_*) = 0$ Banks, Zaks '82
- Maintain exact conformal symmetry, but the generators of $SL(2,\mathbb{R})$ are modified by quantum corrections

# *Conformal anomaly (I)*

- Generators in interacting theory (at the critical point $a_*$) are nontrivial

$$S_- = S_-^{(0)}$$

$$S_0 = S_0^{(0)} + \Delta S_0 = S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_*)$$

$$S_+ = S_+^{(0)} + \Delta S_+ = S_+^{(0)} + (z_1 + z_2)\Big( - \epsilon + \frac{1}{2}\mathbb{H}(a_*)\Big) + (z_1 - z_2)\,\Delta_+(a_*)$$

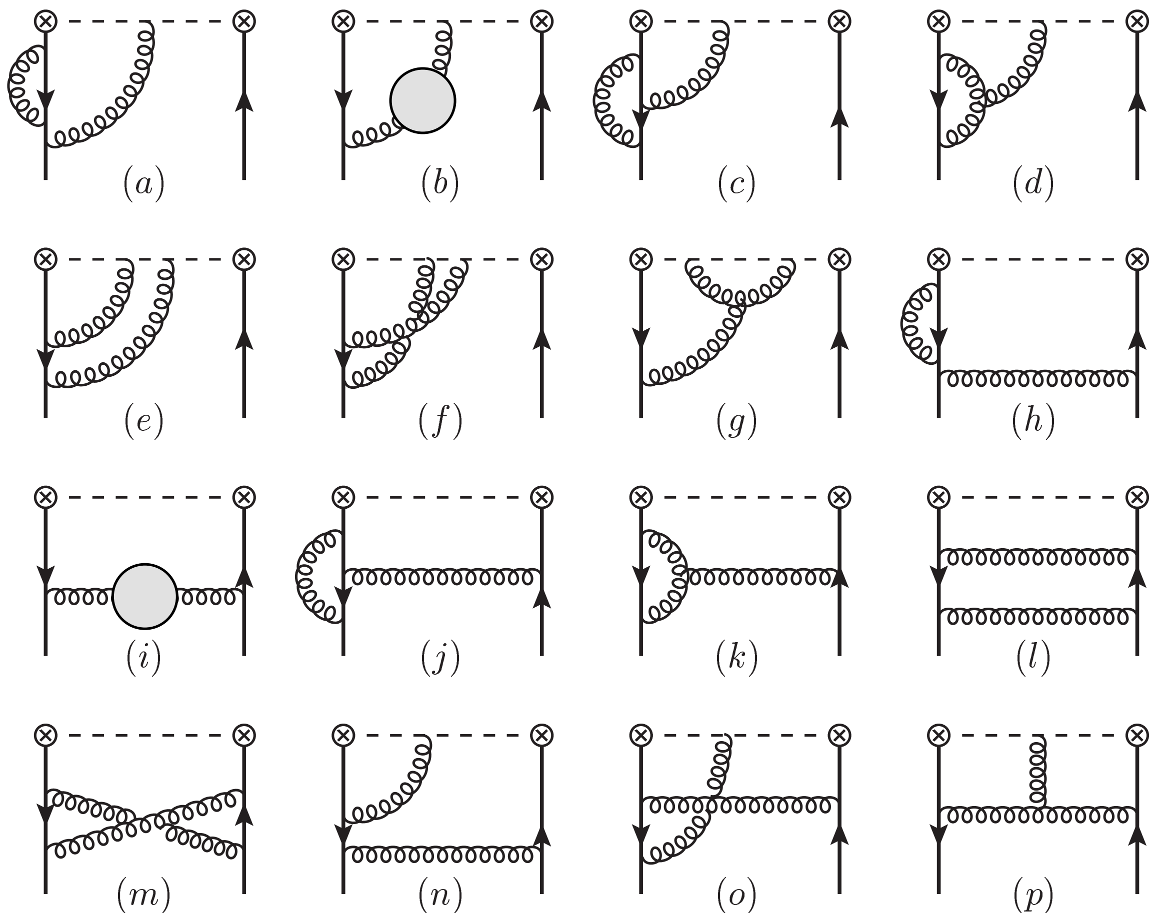
  - generator of special conformal transformations $\Delta S_+$
  - conformal anomaly $\Delta_+(a_*)$ restores conformal Ward identity

- Integral operator $\mathbb{H}(a_*)$ commutes with all generators
  - eigenvalues at order $k$ satisfy $\mathbb{H}^{(k)}(z_1 - z_2)^N = \gamma^{(k)}(N)\,(z_1 - z_2)^N$ where $SL(2,\mathbb{R})$-invariant is function of conformal spin $N(N+1)$
  - full anomalous dimension constrained by 'self-tuning' with universal evolution kernel $\gamma_{\rm u}$ reciprocity-respecting (invariant for $N \to -N-1$)

$$\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(a)\right)$$

# *Conformal anomaly (II)*

- Feynman diagrams for deformation of generator of special conformal transformations $\Delta S_+$




## *Upshot*

- Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics computed with two-loop conformal anomaly and algebraic solution of constraint for conformal Ward identity

# Summary

- Determination of strong coupling $\alpha_s$ at 1% precision requires QCD radiative corrections to evolution equations at N$^3$LO
- Matrix elements of local operators of twist two
  - non-singlet anomalous dimensions $\gamma_{\rm ns}^{(3),\pm,{\rm v}}(N)$ (fixed Mellin moments and exact results for large-$n_c$) at N$^3$LO
  - quartic Casimir contributions to singlet anomalous dimension $\gamma_{ij}^{(3)}(N)$ (fixed Mellin moments and exact results for $\zeta_5$ terms) at N$^3$LO
- Quark and gluon cusp anomalous dimensions
  - generalization of the lower-order 'Casimir scaling'
- Structural similarities of QCD to $N = 4$ Super Yang-Mills theory
- QCD evolution equations for light-ray operators possess a "hidden" conformal symmetry
  - evolution kernel $h(\alpha, \beta)$ at NNLO for off-forward kinematics