Gravity Amplitudes and General Relativity

N. Emil J. Bjerrum-Bohr
Niels Bohr Institute, Copenhagen University

Work together with: P. Damgaard, J. Donoghue,
G. Festuccia, B Holstein, L. Plante, P. Vanhove
(1806.04920; 1609.07477; 1410.4148)

Niels Bohr Institute (NBIA)

- **Senior:**
  - Poul Damgaard
  - Emil Bjerrum-Bohr
  - Jake Bourjaily
  - Paolo Benincasa
  - Mike Trott

  Junior:
  - Andrew McLeod
  - Matt von Hippel
  - Matthias Wilhelm
  - David McGady
  - Humberto Gomez
  - Carlos Cardona
- **PhD Students:**
  - Kays Haddad
  - Matthias Volk

MSc Students :
- David Damgaard
- Solvej Knudsen
- Johannes Sørensen

Outline

- General Relativity as a perturbative effective field theory
- New on-shell toolbox for computations
- New applications for computation of observables in general relativity
  - Scattering angles
  - Light-by-light scattering
- Outlook

General Relativity as an effective field theory

Traditional quantization of gravity

- Known since the 1960ties that a particle version of General Relativity can be derived from the Einstein Hilbert Lagrangian (Feynman, DeWitt)
- Expand Einstein-Hilbert Lagrangian :

$$\mathcal{L}_{\mathrm{E}H} = \int d^4x \left[\sqrt{-g} R \right] \qquad g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

- Derive vertices as in a particle theory - compute amplitudes as Feynman diagrams!

Quantum theory for gravity

- Gravity as a theory with self-interactions

- Non-renormalisable theory! (‘t Hooft and Veltman)

Dimensionful coupling:
$G_N = 1/M^2_{planck}$

- Traditional belief : – no known symmetry can remove all UV-divergences

String theory can by introducing new length scales

Quantum gravity as an effective field theory

- (Weinberg) proposed to view the quantization of general relativity from the viewpoint of effective field theory

$$\mathcal{L} = \sqrt{-g}\left[\frac{2R}{\kappa^2} + \mathcal{L}_{\mathrm{matter}}\right]$$

$$\Downarrow$$

$$\mathcal{L} = \sqrt{-g}\left\{\frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots\right\}$$

- (Donoghue) and (NEJB, Donoghue, Holstein) did the first one-loop concrete computation in such a framework

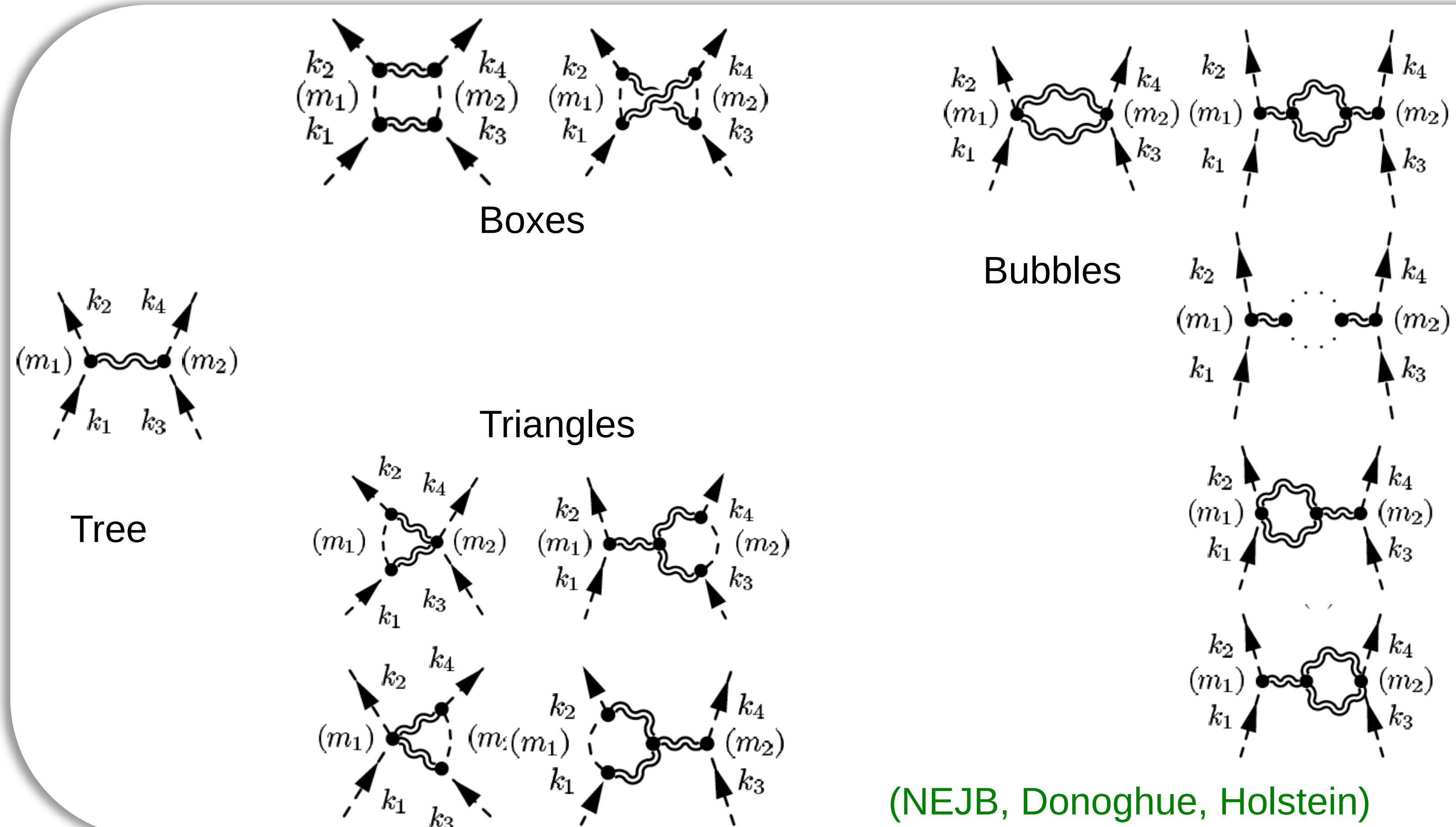
Effective field theory for gravity

- Consistent quantization
  - Working low energy version of quantum gravity

- New point of view:
  - General relativity hbar-> 0 limit of multi-loop expansion
  - Classical pieces comes from loop diagrams!
  - Explanation: contributions appear in loop diagrams feature a cancellation of the loop diagram hbar factor
    - (mass/hbar) expansion.

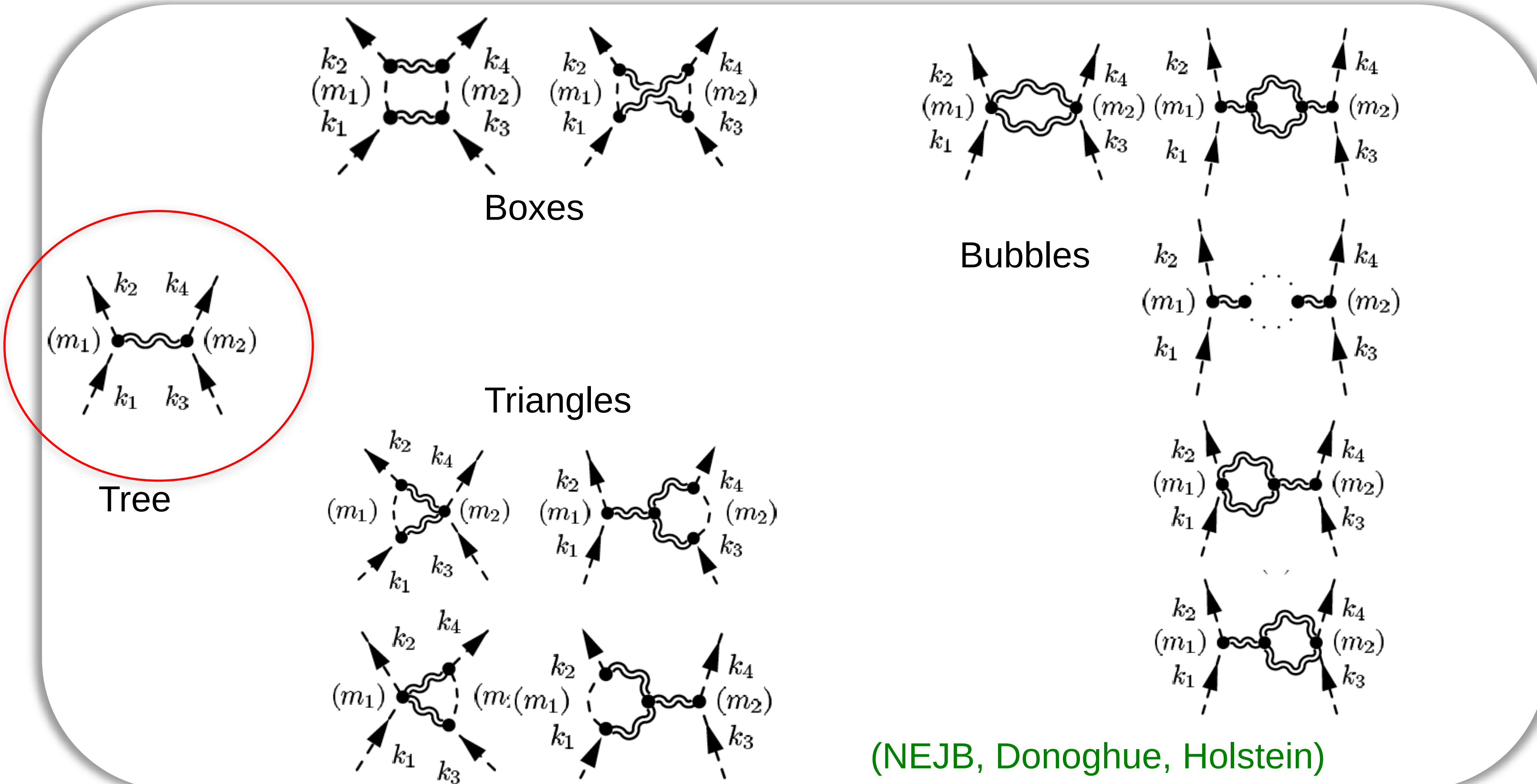
(Donoghue, Holstein)

One-loop (off-shell) gravity computation



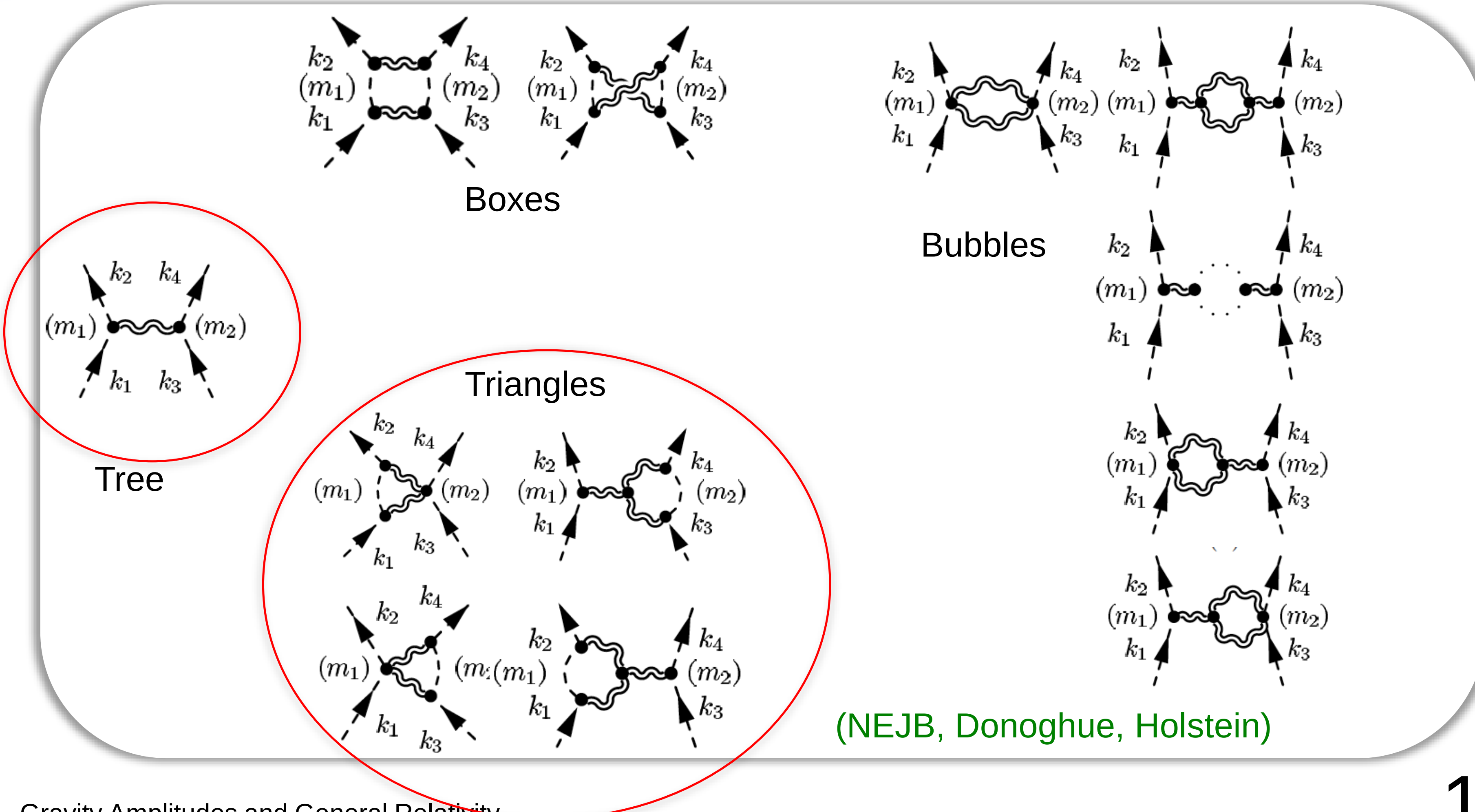
(NEJB, Donoghue, Holstein)

One-loop (off-shell) gravity computation



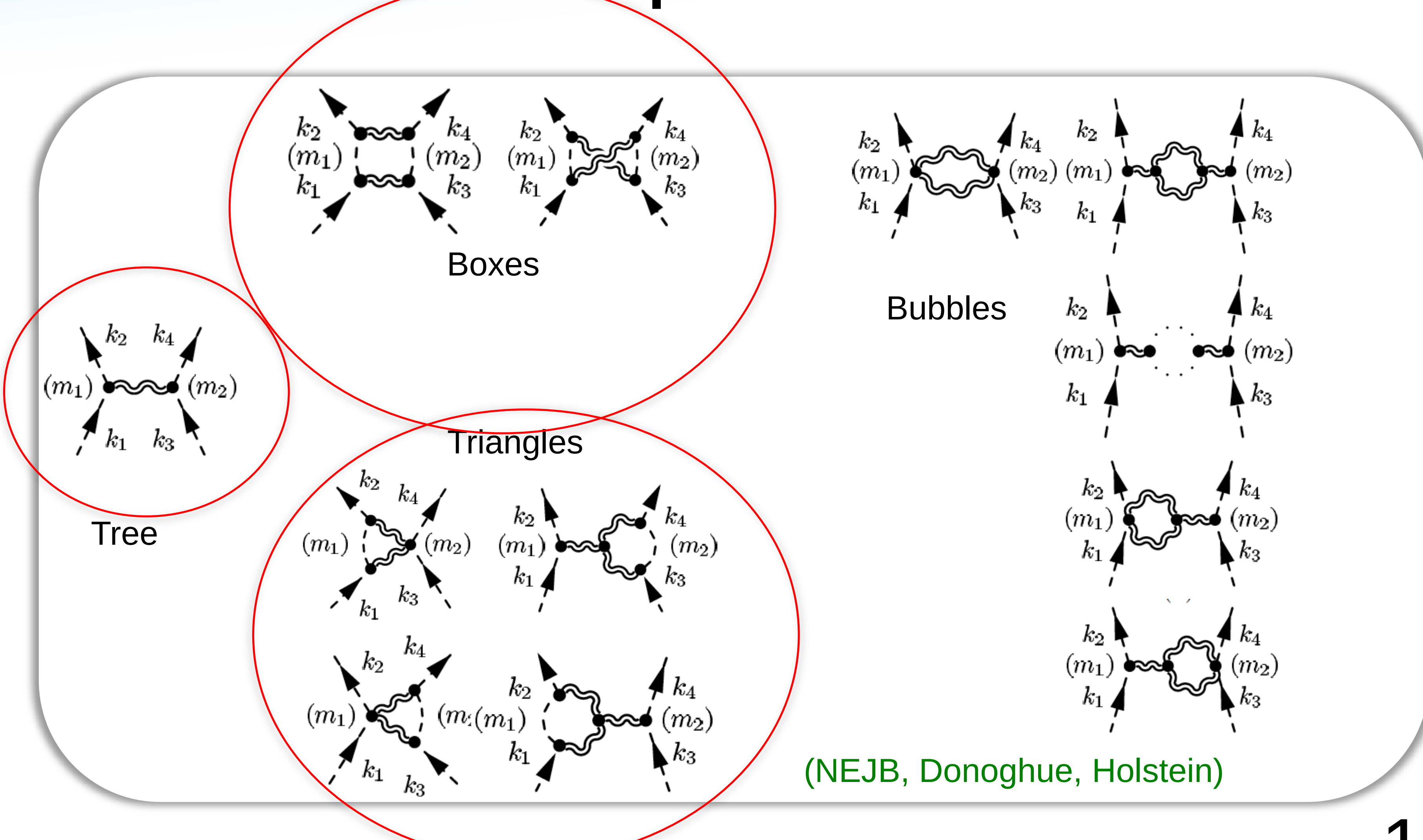
(NEJB, Donoghue, Holstein)

One-loop (off-shell) gravity computation



(NEJB, Donoghue, Holstein)

One-loop (off-shell) gravity computation



One-loop result for gravity

- Four point amplitude can be deduced to take the form

$$\mathcal{M} \sim \left(A + Bq^2 + \ldots + \alpha\kappa^4 \frac{1}{q^2} + \beta_1 \kappa^4 \ln(-q^2) + \beta_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots \right)$$

Short range behaviour

Focus on deriving these ~>

Long-range behavior

(no higher derivative contributions)

(NEJB, Donoghue, Holstein)

One-loop result for gravity

- The result for the amplitude (in coordinate space) after summing all diagrams is (leading in small momentum transfer contribution):

(NEJB, Donoghue, Holstein)

$$-\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}+\frac{41}{10\pi}\frac{G\hbar}{r^2}\right]$$

Post-Newtonian term   New quantum term

Post-Newtonian term in complete accordance with general relativity: (Iwasaki, Holstein and Ross, Neill and Rothstein, NEJB, Damgaard, Festuccia, Plante, Vanhove)

Gravity as an EFT

- Suggest general relativity augmented by higher derivative operators – the most general modified theory
  - Tiny consequences for most observables – since curvature is really small. Interesting connection between observed bounds and theory
- Quantum theory -> classical limit general relativity
  - Post-Newtonian corrections, Hamiltonians for gravitational systems and Post-Minkowskian observables

New on-shell toolbox for computations

Off-shell gravity amplitudes

- Vertices: 3pt, 4pt, 5pt,..n-pt
- Complicated expressions
- Expand Lagrangian, tedious process….

45 terms + sym

$$
\begin{aligned}
V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\
& + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\
& - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\
& + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],
\end{aligned}
$$

(DeWitt;Sannan)

Concrete computation gravity

Featuring a number of unpleasant features

- Complicated Feynman rules (infinitely many vertices)
- Numerous double contractions
- Factorial growth in number of legs
- Feynman diagram topologies: no ordering!
- Loop order: complicated tensor integrals

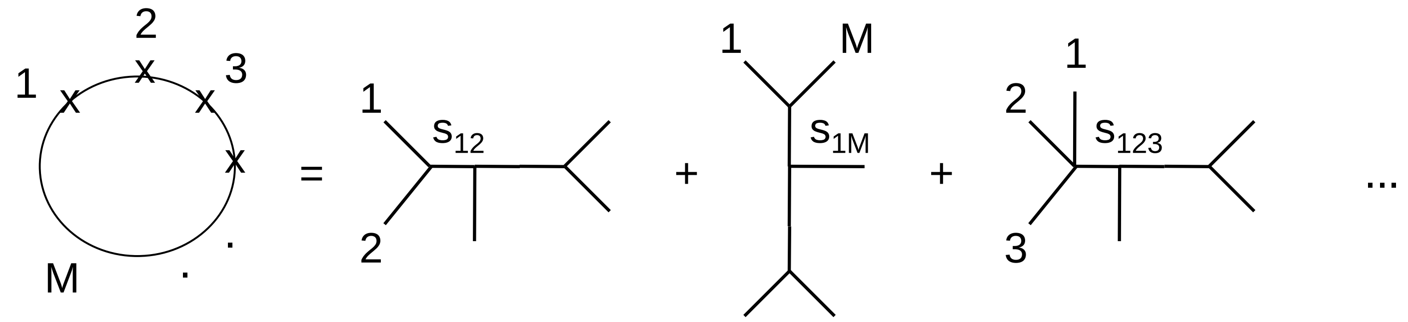
Key: String theory inspiration

Different form for amplitude

String theory adds channels up..

<->

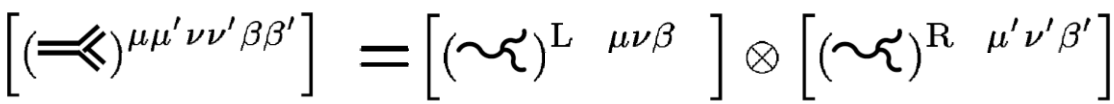
Feynman diagrams sums separate kinematic poles



Gravity Amplitudes

KLT relationship (Kawai, Lewellen and Tye) relates open and closed strings (Bern, Dixon, Dunbar, Perelstein, Rozowsky)

$$A^{M}_{\text{closed}} \sim \sum_{\Pi,\tilde{\Pi}} e^{i\pi\Phi(\Pi,\tilde{\Pi})} A_M^{\text{left open}}(\Pi) A_M^{\text{right open}}(\tilde{\Pi})$$



KLT not manifestly crossing symmetric – explicit representation :

$$M_3^{\text{tree}}(1,2,3) = -iA_3^{\text{tree}}(1,2,3)A_3^{\text{tree}}(1,2,3)\,,$$

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)A_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}\,A_5^{\text{tree}}(1,2,3,4,5)A_5^{\text{tree}}(2,1,4,3,5) + \\ +is_{13}s_{24}\,A_5^{\text{tree}}(1,3,2,4,5)A_5^{\text{tree}}(3,1,4,2,5)\,.$$

Momentum prefactors cancel double poles

Key: on-shell states formalism

Spinor products :

$$\langle i\ j\rangle = \epsilon^{mn}\lambda^i_m\lambda^j_n \quad [i\ j] = \epsilon^{\dot m\dot n}\tilde\lambda^i_{\dot m}\tilde\lambda^j_{\dot n}$$

Different representations of the Lorentz group

$$p_{a\dot a} = \sigma^\mu_{a\dot a}p_\mu$$

$$p^\mu p_\mu = 0 \qquad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a}$$

Momentum parts of amplitudes:

$$q_{a\dot a}=\mu_a\tilde\mu_{\dot a} \quad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a} \quad 2(p\cdot q) = s_{ij} = -\langle\lambda,\mu\rangle[\tilde\lambda,\tilde\mu]$$

Spin-2 polarisation tensors in terms of helicities, (squares of those of YM):

(Xu, Zhang, Chang)

$$\varepsilon^-_{a\dot a} = \frac{\lambda_a\tilde\mu_{\dot a}}{[\tilde\lambda,\tilde\mu]} \qquad \tilde\varepsilon^+_{a\dot a} = \frac{\mu_a\tilde\lambda_{\dot a}}{\langle\mu,\lambda\rangle}$$

$$\varepsilon^-\ \varepsilon^-$$
$$\tilde\varepsilon^+\ \tilde\varepsilon^+$$

Yang-Mills MHV-amplitudes

(n) same helicities vanishes

$$A^{tree}(1^+,2^+,3^+,4^+,..) = 0$$

(n-1) same helicities vanishes

$$A^{tree}(1^+,2^+,...,j^-,..) = 0$$

(n-2) same helicities:

$$A^{tree}(1^+,2^+,...,j^-,...,k^-,..)$$

$A^{tree\ MHV}$ Given by the formula (Parke and Taylor) and proven by (Berends and Giele)

First non-trivial example,

(M)aximally

(H)elicity (V)iolating (MHV) amplitudes

One single term!!

$$i\,\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

Simplifications from Spinor-Helicity

$$s_{ij} = -\langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$$

Huge simplifications

$$\begin{aligned} V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\, \mathrm{sym}\Big[& -\frac{1}{2} P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2} P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\ & + \frac{1}{2} P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\ & - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\ & + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big], \end{aligned}$$

45 terms
+ sym

Vanish in spinor helicity formalism

Contractions

$$\varepsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \qquad \tilde{\varepsilon}^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}$$

Gravity: $A_3(1^-, 2^-, 3^+)$

$$\varepsilon^- \; \varepsilon^-$$
$$\tilde{\varepsilon}^+ \; \tilde{\varepsilon}^+$$

$$A_3(1^-, 2^-, 3^+) = -i \frac{\langle 1\,2 \rangle^6}{\langle 2\,3 \rangle \langle 3\,1 \rangle}$$

Gravity MHV amplitudes

Can be generated from KLT via YM MHV amplitudes.

$$M_4^{\text{tree}}(1^-,2^-,3^+,4^+) = i\,\langle 1\,2\rangle^8 \frac{[1\,2]}{\langle 3\,4\rangle\ N(4)}$$

$$M_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) = i\,\langle 1\,2\rangle^8 \frac{\varepsilon(1,2,3,4)}{N(5)}$$

Anti holomorphic
Contributions
– feature in gravity

(Berends-Giele-Kuijf) recursion formula

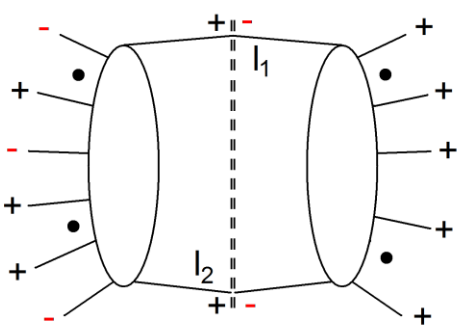
$$M_n^{\text{tree}}(1^-,2^-,3^+,\cdots,n^+)$$
$$= -i\,\langle 1\,2\rangle^8 \times \left[\frac{[1\,2]\,[n-2\,n-1]}{\langle 1\,n-1\rangle\,N(n)}\left(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle i\,j\rangle\right)\prod_{l=3}^{n-3}(-[n|K_{l+1,n-1}|l\rangle) + \mathcal{P}(2,3,\cdots,n-2)\right]$$

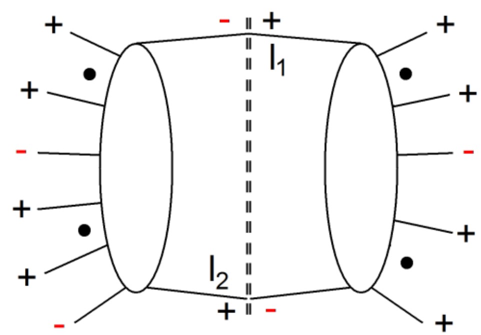
Unitarity cuts

Helicity formalism require unitarity methods

$$C_{i,\dots,j} \;=\; \mathrm{Im}_{K_{i,\dots,j}>0}\, M^{\text{1-loop}}$$

Singlet   Non-Singlet

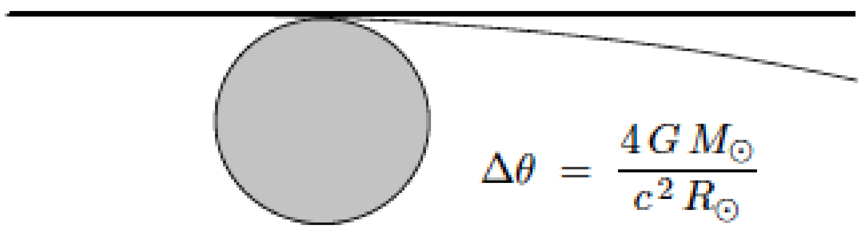




$$C_{i,\dots,j} \equiv \frac{i}{2}\int d\mathrm{LIPS}\Big[M^{\text{tree}}(\ell_1, i, i+1,\dots,j,\ell_2)\times$$
$$\times\, M^{\text{tree}}(-\ell_2, j+1, j+2,\dots,i-1,-\ell_1)\Big]$$

New results: massless matter

- As an example we will consider scattering of massless matter



- Bending of light/massless matter around the Sun
- New features: mass-less external fields ~> IR singularities
- New test of universality of matter

Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

where

$$\mathcal{S}_{\text{scalar}} = \int d^4x \sqrt{-g} \left(-\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}\Big((\partial_\mu \Phi)^2 - M^2\Phi^2 \Big) \right)$$

$$\mathcal{S}_{\text{fermion}} = \frac{i}{2} \int d^4x \sqrt{-g}\, \bar{\chi} \not{D} \chi \,,$$

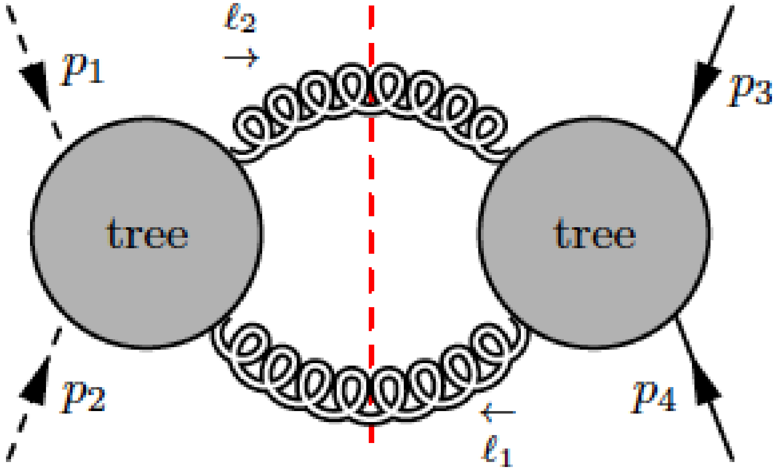
$$\mathcal{S}_{\text{QED}} = -\frac{1}{4} \int d^4x \sqrt{-g}\, (\nabla_\mu A_\nu - \nabla_\nu A_\mu)^2$$

Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

We want to compute the cut



Trees and the cut

- We have the Lagrangian

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right]$$

We want to compute the cut

$$\mathcal{M}_X^{(2)}(p_1, p_2, p_3, p_4)\Big|_{\text{disc}} := \frac{1}{2!\, i} \mu^{2\epsilon} \int d\text{LIPS}(\ell_1, -\ell_2)\, (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$
$$\times \sum_{\lambda_1, \lambda_2} \mathcal{M}^{(1)}_{X^2 G^2}(p_1, \ell_1, p_2 - \ell_2) \times \mathcal{M}^{(1)}_{\phi^2 G^2}(p_3, \ell_2, p_4, -\ell_1)^\dagger$$

Photons and scalars

For photons we have

$$i\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^-(k_2)]}_{[\gamma^+(p_1)\gamma^-(p_2)]} = \frac{\kappa^2}{4}\frac{[p_1\, k_1]^2\, \langle p_2\, k_2\rangle^2\, \langle k_2|p_1|k_1]^2}{(p_1\cdot p_2)(p_1\cdot k_1)(p_1\cdot k_2)}$$

While for scalars

$$i\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} = \frac{\kappa^2}{4}\frac{M^4\,[k_1\, k_2]^4}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)}$$

$$i\overset{0}{\mathcal{M}}{}^{[h^-(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} = \frac{\kappa^2}{4}\frac{\langle k_1|p_1|k_2]^2\,\langle k_1|p_2|k_2]^2}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)}$$

Super compact compared to Feynman diagram results

Result for the amplitude

We can rewrite

$$\mathcal{M}_{\varphi}^{(2)}(p_1, p_2, p_3, p_4) = -\frac{\kappa^4}{32t^2\, i} \sum_{i=1}^{2} \sum_{j=3}^{4} \int \frac{d^D\ell\, \mu^{2\epsilon}}{(2\pi)^D} \frac{\mathcal{N}^S}{\ell_1^2 \ell_2^2 (p_i \cdot \ell_1)(p_j \cdot \ell_1)}$$

where

$$\mathcal{N}^{0}_{\mathrm{non-singlet}} = \frac{1}{2}\left[\left(\mathrm{tr}_-(\ell_1 p_1 \ell_2 p_3)\right)^4 + \left(\mathrm{tr}_-(\ell_2 p_1 \ell_1 p_3)\right)^4\right]$$

Scalar case

$$\mathcal{N}^{\frac{1}{2}+-}_{\mathrm{non-singlet}} = \frac{\left(\mathrm{tr}_-(\ell_1 p_1 \ell_2 p_3)^3 \mathrm{tr}_+(p_1 p_3 p_2 \ell_1 p_3 \ell_2)\right) - (\ell_1 \leftrightarrow \ell_2)}{\langle p_2 | p_3 | p_1]}$$

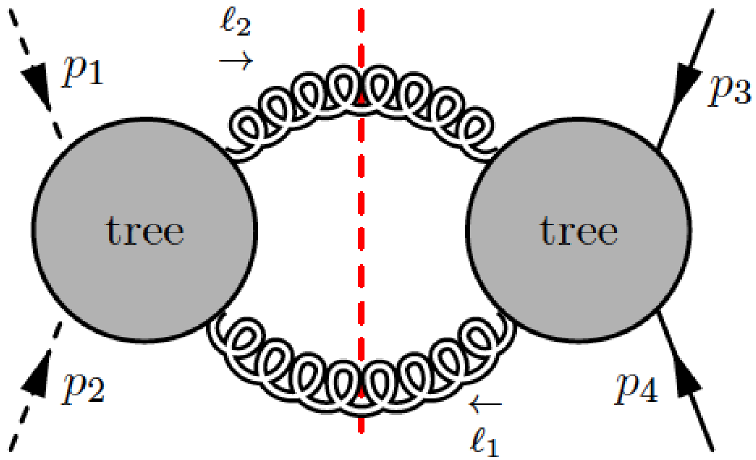
Fermion case

$$\mathcal{N}^{1+-}_{\mathrm{non-singlet}} = \frac{\left(\mathrm{tr}_-(\ell_2 p_1 \ell_1 p_3)\mathrm{tr}_+(\ell_2 p_3 \ell_1 p_1 p_3 p_2)\right)^2 + (\ell_1 \leftrightarrow \ell_2)}{\langle p_1 | p_3 | p_2]^2}$$

Photon case

Combine spinor expressions into traces

Result for the amplitude



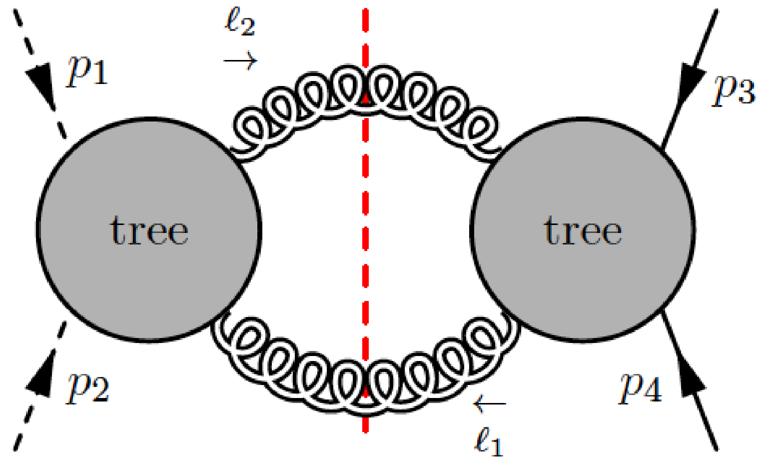
1) Expand out traces
2) Reduce to scalar basis of integrals
3) Isolate coefficients

(Bern, Dixon, Dunbar, Kosower)

$$bo^S(t,s)\,I_4(t,s) + bo^S(t,u)\,I_4(t,u)$$

$$+\, t_{12}^S(t)\,I_3(t,0) + t_{34}^S(t)\,I_3(t,M^2) + bu^S(t,0)\,I_2(t,0)$$

Result for the amplitude

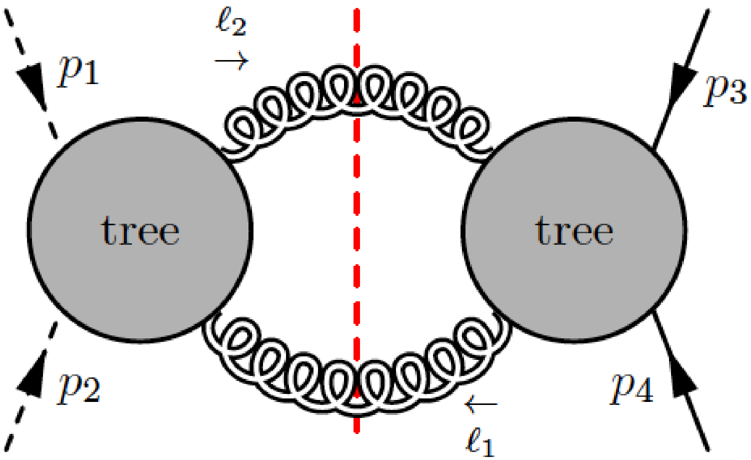


$$bu^{\varphi} = \frac{3}{40} \quad bu^{\gamma} = -\frac{161}{120}$$

$$bu^{X} = -\frac{31}{30}$$

$$-\frac{\mathcal{N}^X}{\hbar}\left[\hbar\frac{\kappa^4}{4}\left(4(M\omega)^4(I_4(t,u)+I_4(t,s))+3(M\omega)^2 tI_3(t)\right.\right.$$
$$\left.\left.-15(M^2\omega)^2 I_3(t,M)+bu^X(M\omega)^2 I_2(t)\right)\right]$$

Result for the amplitude



$$bu^{\varphi} = \frac{3}{40} \quad bu^{\gamma} = -\frac{161}{120}$$

$$bu^{X} = -\frac{31}{30}$$

Taking the post-Newtonian non-relativistic low energy limit

$$\frac{\mathcal{N}^X}{\hbar}(M\omega)^2 \Big[-\kappa^4 \frac{15}{512}\frac{M}{|\boldsymbol{q}|} - \hbar\kappa^4 \frac{15}{512\pi^2} \log\left(\frac{\boldsymbol{q}^2}{M^2}\right) + \hbar\kappa^4 \frac{bu^X}{(8\pi)^2} \log\left(\frac{\boldsymbol{q}^2}{\mu^2}\right)$$
$$- \hbar\kappa^4 \frac{3}{128\pi^2} \log^2\left(\frac{\boldsymbol{q}^2}{\mu^2}\right) + \kappa^4 \frac{M\omega}{8\pi}\frac{i}{\boldsymbol{q}^2} \log\left(\frac{\boldsymbol{q}^2}{M^2}\right) \Big]$$

(NEJB, Donoghue, Holstein, Plante, Vanhove)

Making connection to general relativity

General metric

$$ds^2 = A(r)dt^2 - B(r)^2 dr^2 - r^2 d\Omega^2$$

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2G_N M}{r}$$ Schwarzschild

$$\theta = \frac{4G_N M}{R} + \frac{4G_N^2 M^2}{R^2}\left(\frac{15\pi}{16} - 1\right) + \ldots$$

Can we reproduce?

Stationary phase method

We apply a Fourier transformation to impact parameter space and exponentiate into eikonal phases, so that a stationary phase method can be applied.

(See e.g. Akhoury, Saotome and Sterman)

$$\mathcal{M}(\boldsymbol{q}) = \mathcal{M}_1^{(1)}(\boldsymbol{q}) + \mathcal{M}^{(2)}(\boldsymbol{q})$$

$$\begin{aligned}\mathcal{M}(\boldsymbol{b}) &= 2(s - M^2)\left[(1 + i\chi_2)e^{i\chi_1} - 1\right] \\ &\simeq 2(s - M^2)\left[e^{i(\chi_1+\chi_2)} - 1\right]\end{aligned}$$

Stationary phase method

Now we can compute

$$\chi_1(\boldsymbol{b}) = \frac{\kappa^2 M E}{4} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \frac{1}{\boldsymbol{q}^2}$$

$$\simeq 4 G_N M E \left[\frac{1}{d-2} - \log(b/2) - \gamma_E\right]$$

$$\chi_2(\boldsymbol{b}) = G_N^2 M^2 E \frac{15\pi}{4b} + \frac{G_N^2 M^2 E}{2\pi b^2}\left(8 b u^\eta - 15 + 48 \log\frac{2b_0}{b}\right)$$

Stationary phase method

Leading to static phase when:

$$\frac{\partial}{\partial b}\left(q\, b + \chi_1(b) + \chi_2(b) + \cdots\right) = 0$$

Using that $q = 2E\sin(\theta/2)$

We arrive at:

$$2\sin\frac{\theta}{2} \simeq \theta = -\frac{1}{E}\frac{\partial}{\partial b}\left(\chi_1(b) + \chi_2(b)\right)$$

Stationary phase method

Leading to static phase when:

$$\frac{\partial}{\partial b}\left(q\,b + \chi_1(b) + \chi_2(b) + \cdots\right) = 0$$

Using that $q = 2E\sin(\theta/2)$

Or:

$$\theta \simeq \frac{4G_N M}{b} + \frac{15}{4}\frac{G_N^2 M^2 \pi}{b^2} + \left(8bu^S + 9 - 48\log\frac{b}{2b_0}\right)\frac{\hbar G_N^2 M}{\pi b^3} + \dots$$

Bending of light

Interpreted as a bending angle (eikonal approximation) we have:

$$\theta_\eta \simeq \frac{4GM}{b} + \frac{15}{4}\frac{G^2M^2\pi}{b^2}$$

plus a quantum effect of the order of magnitude:

$$+\frac{8bu^\eta + 9 + 48\log\frac{b}{2r_o}}{\pi}\frac{G^2\hbar M}{b^3}$$

We see that we have universality between scalars, fermions and photons only for the ‘Newton’ and ‘post-Newtonian’ contributions

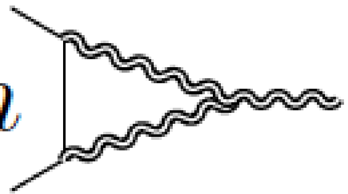
New applications for computation of observables in general relativity

Classical contributions from perturbative computations

- Use of perturbative framework to compute observables in general relativity
- Truncation to only classical terms
- Only non-analytical piece corresponding to long-distance interactions -> Unitarity cuts useful
- Gravitational wave applications: (Blanchet review)
  - Some modern type amplitude computations of post-Newtonian potentials (NEJB, Donoghue, Holstein; Holstein and Ross; Holstein; Neill and Rothstein; NEJB, Donoghue, Vanhove) (Guevara and Cachazo; Guevara; Damour; NEJB, Damgaard, Festuccia, Plante, Vanhove; Cheung, Rothstein, Solon)
  - Some modern approaches to the scattering angle in post-Minkowskian formalism (Westpfahl; Damour; Vines; NEJB, Damgaard, Festuccia, Plante, Vanhove)

Classical contributions from perturbative computations

- In classical gravity the long-distance terms that are related to the post-Newtonian effects are triangle diagrams (at one-loop)

m  $\sim \quad m/\sqrt{-q^2}$

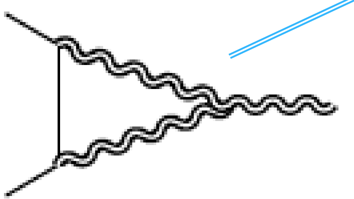
- Such contributions have cancellations of $\hbar$ d lead to purely classical terms

General relativity from loops

New derivation

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon}\frac{1}{(\ell+q)^2+i\epsilon}\frac{1}{(\ell+p_1)^2-m_1^2+i\epsilon}$$

$$(\ell+p_1)^2-m_1^2=\ell^2+2\ell\cdot p_1\simeq 2m_1\ell_0$$



$$\frac{1}{2m_1}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon}\frac{1}{(\ell+q)^2+i\epsilon}\frac{1}{\ell_0+i\epsilon}$$

General relativity from loops

$$\frac{1}{2m_1}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon}\frac{1}{(\ell+q)^2+i\epsilon}\frac{1}{\ell_0+i\epsilon}$$

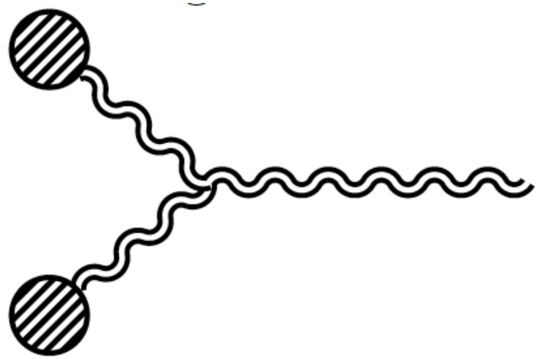
Close contour

$$\int_{|\vec{\ell}|\ll m}\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{i}{4m}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+q)^2}=-\frac{i}{32m|\vec{q}|}$$

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Interpretation

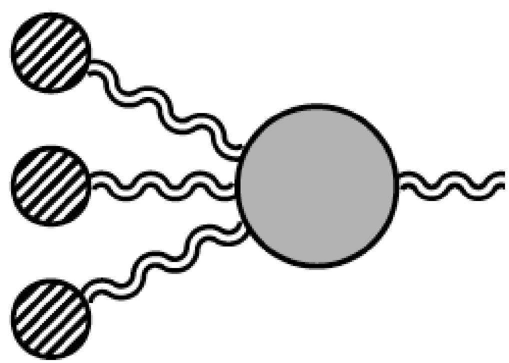
$$\int \frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+q)^2} \longleftrightarrow$$



Integration of classical sources on tree graphs – no loops!

Picture extends to higher loops

$$I_{\triangleright\triangleright(1)}(p_1,q), I_{\triangleright\triangleright(2)}(p_1,q) \leftrightarrow$$

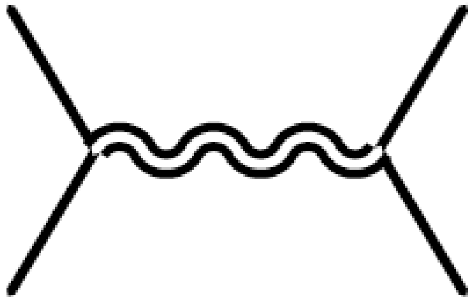


Explains the metric computation by (Duff)

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Scalar interaction potentials (tree)

Tree level

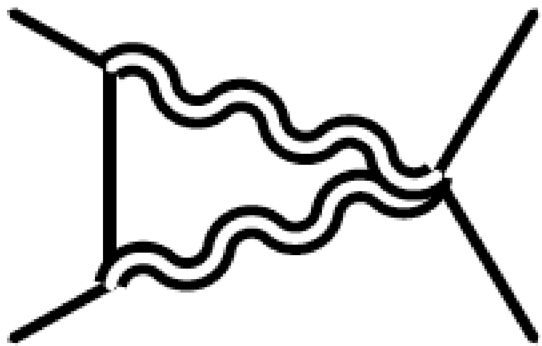
$\mathcal{M}_1 =$ 

$$= -\frac{16\pi G}{q^2}\left(m_1^2 m_2^2 - 2(p_1\cdot p_4)^2 - (p_1\cdot p_4)q^2\right)$$

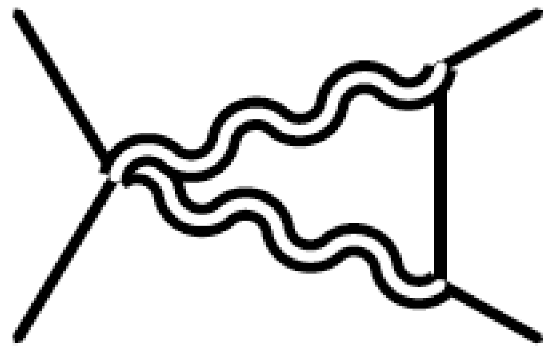
Scalar interaction potentials (one-loop)

One-loop level

$$\mathcal{M}_2 =$$



$$+$$



$$= -i(8\pi G)^2 \left(\frac{c(m_1, m_2) I_{\triangleright}(p_1, q)}{(q^2 - 4m_1^2)^2} + \frac{c(m_2, m_1) I_{\triangleright}(p_4, -q)}{(q^2 - 4m_2^2)^2} \right)$$

Classical contribution from one-loop amplitude

General relativity encoded in triangle coefficients

$$
\begin{aligned}
c(m_1, m_2) = (q^2)^5 + (q^2)^4 \left(6p_1 \cdot p_4 - 10m_1^2\right) \\
+ (q^2)^3 \left(12(p_1 \cdot p_4)^2 - 60m_1^2 p_1 \cdot p_4 - 2m_1^2 m_2^2 + 30m_1^4\right) \\
-(q^2)^2 \left(120m_1^2(p_1 \cdot p_4)^2 - 180m_1^4 p_1 \cdot p_4 - 20m_1^4 m_2^2 + 20m_1^6\right) \\
+q^2 \left(360m_1^4(p_1 \cdot p_4)^4 - 120m_1^6 p_1 \cdot p_4 - 4m_1^6(m_1^2 + 15m_2^2)\right) \\
+ 48m_1^8 m_2^2 - 240m_1^6(p_1 \cdot p_4)^2
\end{aligned}
$$

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Post-Newtonian potentials

Leading order in q

$$\mathcal{M}_2 = \frac{6\pi^2 G^2}{|\vec{q}|}(m_1 + m_2)(5(p_1 \cdot p_4)^2 - m_1^2 m_2^2)$$

All momenta provided at infinity, contractions are done using flat space metric (Minkowski), no reference to coordinates. Gauge invariant expression – to derive potential we have to introduce coordinates, Fourier transform and expand subleading terms in q^0

post-Newtonian interaction potentials

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_4^2}{2m_2} - \frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_4^4}{8m_2^3}$$
$$- \frac{Gm_1m_2}{r} - \frac{G^2m_1m_2(m_1+m_2)}{2r^2}$$
$$- \frac{Gm_1m_2}{2r}\left(\frac{3\vec{p}_1^2}{m_1^2} + \frac{3\vec{p}_4^2}{m_2^2} - \frac{7\vec{p}_1\cdot\vec{p}_4}{m_1m_2} - \frac{(\vec{p}_1\cdot\vec{r})(\vec{p}_4\cdot\vec{r})}{m_1m_2r^2}\right)$$

(Einstein-Infeld-Hoffman)

Subtraction of tree-level Born term to in order to get the correct potential (3 – 7/2 -> -1/2)

post-Minkowskian expansion

Will use similar eikonal setup as for bending of light (extended to massive case):

$$\vec{p}_1 = -\vec{p}_4$$

b orthogonal and

$$b \equiv |\vec{b}|$$

Amplitude computed

$$M(\vec{b}) \equiv \int d^2\vec{q}\, e^{-i\vec{q}\cdot\vec{b}} M(\vec{q})$$

$$M(\vec{b}) = 4p(E_1 + E_2)(e^{i\chi(\vec{b})} - 1)$$

Eikonal phase

post-Minkowskian expansion

Stationary phase condition (leading order in q)

$$2\sin(\theta/2) = \frac{-2M}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{\partial}{\partial b} \left(\chi_1(b) + \chi_2(b)\right)$$

$$\chi_1(b) = 2G \frac{\hat{M}^4 - 2m_1^2 m_2^2}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \left(\frac{1}{d-2} - \log\left(\frac{b}{2}\right) - \gamma_E \right)$$

$$\chi_2(b) = \frac{3\pi G^2}{8\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{m_1 + m_2}{b} (5\hat{M}^4 - 4m_1^2 m_2^2)$$

post-Minkowskian expansion

Final result becomes

$$2\sin\left(\frac{\theta}{2}\right) = \frac{4GM}{b}\Big(\frac{\hat{M}^4 - 2m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2} + \frac{3\pi}{16}\frac{G(m_1+m_2)}{b}\frac{5\hat{M}^4 - 4m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2}\Big)$$

Agrees with (Westpfahl)

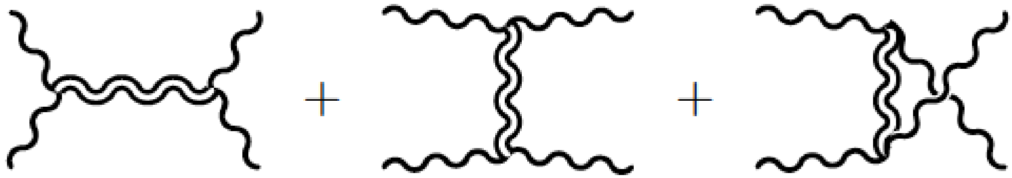
Light-like limit

$$\theta = \frac{4Gm_1}{b} + \frac{15\pi}{4}\frac{G^2m_1^2}{b^2}$$

post-Minkowskian expansion

Exact result light-by-light scattering:

No triangles!

$M_{\gamma\gamma} =$ 

$$
- 8\pi G \frac{2\mathrm{tr}(f_1 f_2 f_3 f_4) + 2\mathrm{tr}(f_1 f_3 f_4 f_2) - \mathrm{tr}(f_1 f_2)\mathrm{tr}(f_3 f_4)}{(p_1 - p_2)^2}
$$

$$
- 8\pi G \frac{2\mathrm{tr}(f_1 f_4 f_3 f_2) + 2\mathrm{tr}(f_1 f_3 f_2 f_4) - \mathrm{tr}(f_1 f_4)\mathrm{tr}(f_2 f_3)}{(p_1 + p_4)^2}
$$

$$
- 8\pi G \frac{2\mathrm{tr}(f_1 f_3 f_4 f_2) + 2\mathrm{tr}(f_1 f_3 f_2 f_4) - \mathrm{tr}(f_1 f_3)\mathrm{tr}(f_2 f_4)}{(p_1 - p_3)^2}
$$

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Outlook

Amplitude toolbox for computations provides efficient new methods.

New applications can be used to increase precision for classical general relativity computations.

Good prospects for further theoretical and practical breakthroughs

- Much progress in short time – practical implementation still lags behind